NONCOMMUTATIVE GEOMETRY
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Abstract

Our geometric concepts evolved first through the discovery of NonEuclidean geometry. The discovery of quantum mechanics in the form of the noncommuting coordinates on the phase space of atomic systems entails an equally drastic evolution. We describe a basic construction which extends the familiar duality between ordinary spaces and commutative algebras to a duality between Quotient spaces and Noncommutative algebras. The basic tools of the theory, K-theory, Cyclic cohomology, Morita equivalence, Operator theoretic index theorems, Hopf algebra symmetry are reviewed. They cover the global aspects of noncommutative spaces, such as the transformation $\theta \to 1/\theta$ for the noncommutative torus $\mathbb{T}^2_\theta$ which are unseen in perturbative expansions in θ such as star or Moyal products. We discuss the foundational problem of "what is a manifold in NCG" and explain the fundamental role of Poincare duality in K-homology which is the basic reason for the spectral point of view. This leads us, when specializing to 4-geometries to a universal algebra called the "Instanton algebra". We describe our joint work with G. Landi which gives noncommutative spheres S^4_θ from representations of the Instanton algebra. We show that any compact Riemannian spin manifold whose isometry group has rank $r \geq 2$ admits isospectral deformations to noncommutative geometries. We give a survey of several recent developments. First our joint work with H. Moscovici on the transverse geometry of foliations which yields a diffeomorphism invariant (rather than the usual covariant one) geometry on the bundle of metrics on a manifold and a natural extension of cyclic cohomology to Hopf algebras. Second, our joint work with D. Kreimer on renormalization and the Riemann-Hilbert problem. Finally we describe the spectral realization of zeros of zeta and L-functions from the noncommutative space of Adele classes on a global field and its relation with the Arthur-Selberg trace formula in the Langlands program. We end with a tentalizing connection between the renormalization group and the missing Galois theory at Archimedean places.

I Introduction

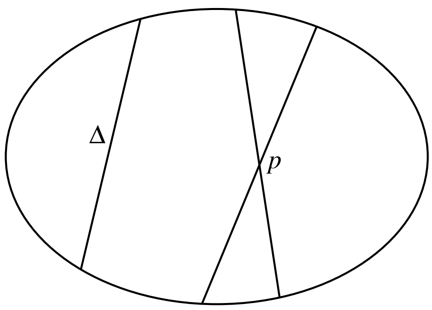
There are two fundamental sources of 'bare' facts for the mathematician. These are, on the one hand the physical world which is the source of *geometry*, and on the other hand the arithmetic of numbers which is the source of *number theory*. Any theory concerning either of these subjects can be tested by performing experiments either in the physical world or with numbers. That is, there are some real things out there to which we can confront our understanding.

If one looks back at the 23 problems of Hilbert then one finds that, fortunately, the twentieth century saw very important discoveries which nobody could have foreseen by 1900. Two of them (of course by no means the only discoveries) involve Hilbert space in a crucial way and will be of particular importance for this talk: The first one is quantum mechanics, and the second, equally important in a sense, is the extension of class field theory to the non-abelian case, thanks to the Langlands program.

In this lecture I'll take both of these discoveries as a pretext and point towards the extension of our familiar geometrical concepts beyond the classical, commutative case. My aim is to discuss the foundation of noncommutative geometry.

II Geometry

Before I do that, let me remind you, using a simple example, of the power of abstraction in mathematics. Around 1800, Mathematicians wondered whether it is true that Euclid's fifth axiom is actually superfluous. For instance Legendre proved that if you have one triangle whose internal angles sum to π then that is enough to guarantee ordinary Euclidean geometry. However, as we all know Euclid's fifth axiom is not superfluous and NonEuclidean Geometry gives a counter-example. The simplest model of NonEuclidean Geometry is probably the Klein model. The points of the geometric space X are the points inside an ellipse,



The lines are the intersections of the ordinary Euclidean lines with X. If you take a point p, outside the line Δ then there are distinct lines which don't meet Δ (*i.e.* are parallel to Δ) but meet each other at p.

At first this was considered as an esoteric example and Gauss didn't publish his discovery, but after some time it became clear that rather than just being a strange counter-example, it was something with remarkable beauty and power. The question then became "what is the source of this beauty and power?" Often in mathematics, understanding comes from generalisation, instead of considering the object *per se* what one tries to find are the concepts which embody the power of the object.

A first generalisation is the *Erlangen* program of Klein and the theory of Lie groups which attributes the beauty of this example to its symmetries, namely the group of projective transformations of the plane which preserve the ellipse.
The second conceptual generalisation is Riemannian geometry as explained in Riemann's inaugural lecture ([26]) in which he reflected on the hypotheses of geometry and introduced two key notions: the concepts of *manifold* and *line element*.
By a manifold Riemann meant 'any space you can think of whose points can vary continuously'. For example, a manifold could be a continuous collection of colours, the parameter space for some mechanical system or, of course, space. In his lecture Riemann explained that it is possible, essentially proceeding by induction, to label the points of such a space by a finite collection of real numbers.
In Riemannian geometry the distance between two points x and y is given by the following *ansatz*:

$$d(x,y) = \mathrm{Inf}\{\int_\gamma ds \,|\gamma \text{ is a path between } x \text{ and } y\} \tag{1}$$

Expanding $d(x,y)$ near the diagonal, after raising it to an even power to ensure smoothness gives a local formula for ds. The first case he considered was the quadratic case (although he explicitly mentionned the quartic case). From the Taylor expansion he obtained, in the quadratic case, the well-known formula for the metric,

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu\,. \tag{2}$$

Riemann's concept of geometry differs greatly from that of Klein because Klein's formulation is based on the idea of rigid motions whereas in Riemannian geometry rigid motions are no longer possible because of the variability of the curvature and the extraordinary freedom in the choice of the components $g_{\mu\nu}$.
The basic notions of ordinary geometry do make sense, for example a straight line is given by the geodesic equation,

$$\frac{d^2\, x^\mu}{dt^2} = -\frac{1}{2}\, g^{\mu\alpha}(g_{\alpha\nu,\rho} + g_{\alpha\rho,\nu} - g_{\nu\rho,\alpha})\,\frac{dx^\nu}{dt}\,\frac{dx^\rho}{dt} \tag{3}$$

but what really vindicated the point of view of Riemann, with respect to that of Klein, was another major discovery of the twentieth century, General Relativity.
One can get a glimpse of this from the following simple fact. If we take the Minkowski metric and perturb it to $dx^2 + dy^2 + dz^2 - (1 + 2V(x,y,z))dt^2$ using the Newtonian potential $V(x,y,z)$, then the geodesic equation can be re-written in the obvious approximation to obtain Newton's law of motion. This makes clear that the variability of the $g_{\mu\nu}$ is precisely necessary in order to get a good geometric model of the physical universe.
It is interesting to note that Riemann was well aware of the limits of his own point of view as is clearly expressed in the last page of his inaugural lecture; ([26])
"Questions about the immeasurably large are idle questions for the explanation of Nature. But the situation is quite different with questions about the immeasurably small. Upon the exactness with which we pursue phenomenon into the infinitely small, does our knowledge of their causal connections essentially depend. The progress of recent centuries in understanding the mechanisms of Nature depends almost entirely on the exactness of construction which has become possible through the invention of

the analysis of the infinite and through the simple principles discovered by Archimedes, Galileo and Newton, which modern physics makes use of. By contrast, in the natural sciences where the simple principles for such constructions are still lacking, to discover causal connections one pursues phenomenon into the spatially small, just so far as the microscope permits. Questions about the metric relations of Space in the immeasurably small are thus not idle ones.

If one assumes that bodies exist independently of position, then the curvature is everywhere constant, and it then follows from astronomical measurements that it cannot be different from zero; or at any rate its reciprocal must be an area in comparison with which the range of our telescopes can be neglected. But if such an independence of bodies from position does not exist, then one cannot draw conclusions about metric relations in the infinitely small from those in the large; at every point the curvature can have arbitrary values in three directions, provided only that the total curvature of every measurable portion of Space is not perceptibly different from zero. Still more complicated relations can occur if the line element cannot be represented, as was presupposed, by the square root of a differential expression of the second degree. Now it seems that the empirical notions on which the metric determinations of Space are based, the concept of a solid body and that of a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of Space in the infinitely small do not conform to the hypotheses of geometry; and in fact one ought to assume this as soon as it permits a simpler way of explaining phenomena.

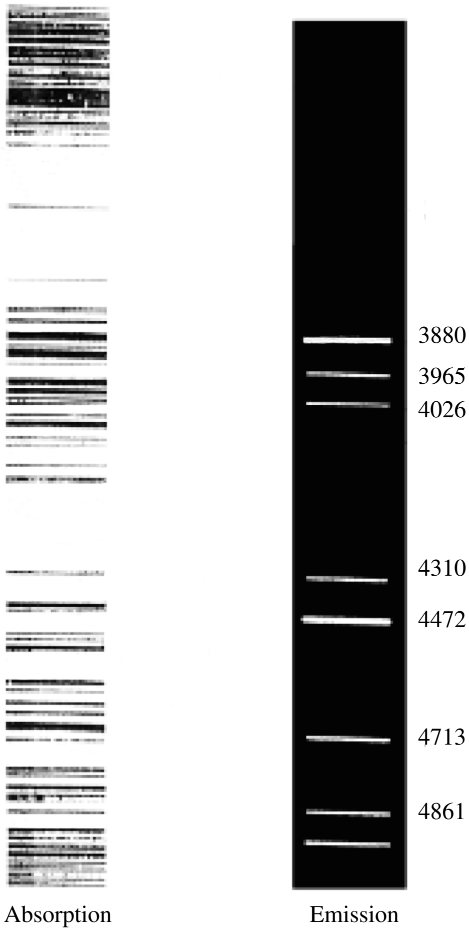
The question of the validity of the hypotheses of geometry in the infinitely small is connected with the question of the basis for the metric relations of space. In connection with this question, which may indeed still be ranked as part of the study of Space, the above remark is applicable, that in a discrete manifold the principle of metric relations is already contained in the concept of the manifold, but in a continuous one it must come from something else. Therefore, either the reality underlying Space must form a discrete manifold, or the basis for the metric relations must be sought outside it, in binding forces acting upon it.

An answer to these questions can be found only by starting from that conception of phenomena which has hitherto been approved by experience, for which Newton laid the foundation, and gradually modifying it under the compulsion of facts which cannot be explained by it. Investigations like the one just made, which begin from general concepts, can serve only to insure that this work is not hindered by too restricted concepts, and that progress in comprehending the connection of things is not obstructed by traditional prejudices.

This leads us away into the domain of another science, the realm of physics, into which the nature of the present occasion does not allow us to enter".

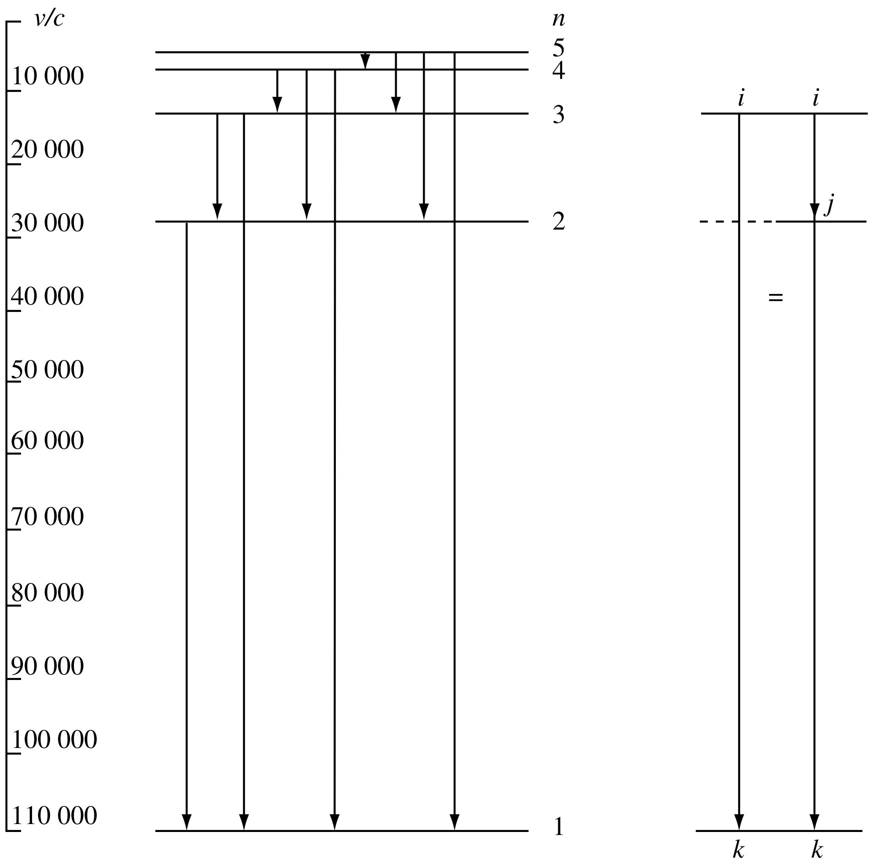
III Quantum mechanics

In fact quantum mechanics showed that indeed the parameter space, or phase space of the mechanical system given by a single atom fails to be a manifold. It is important to convince oneself of this fact and to understand that this conclusion is indeed dictated by the experimental findings of spectroscopy. The information we get from the light coming from distant stars is of spectral nature, the spectral lines are absorption or emission lines



One can infer from this spectral information the chemical composition of the star since the simple elements have recognisable spectra. These spectra obey experimentally discovered laws, the most notable being the Ritz-Rydberg combination principle. The principle can be stated as follows; spectral lines are indexed by pairs of objects. These objects could be numbers, greek letters, or any kind of labels. The statement of the principle then is that certain pairs of spectral lines, when expressed in terms of frequencies, do add up to give another line in the spectrum. Moreover, this happens precisely when the labels are of the form i, j and j, k.

What Heisenberg understood, by analogy with the classical treatment of the interaction of a mechanical system with the electromagnetic field, is that this Ritz-Rydberg combination principle actually dictates an algebraic formula for the product of any two observable physical quantities attached to the atomic system.



Heisenberg wrote down the formula for the product of two observables;

$$(A\,B)_{(i,k)} \;=A_{(i,j)}\,B_{(j,k)} \tag{1}$$

and he noticed of course that this algebra he had found is no longer commutative,

$$A\,B \neq B\,A \tag{2}$$

Now Heisenberg didn't know about matrices, he just worked it out, but he was told later by Born, Jordan and Dirac that the algebra he had worked out was known to mathematicians as the algebra of matrices.
Physicists often tell jokes such as: A physicist walks down the main street of a strange town looking for a laundrette. He sees a shop with signs in the window saying 'bakery' 'grocers' 'laundrette', so he enters. However, the shop is owned by a mathematician and when the physicist asks "when will the washing be ready?" the mathematician replies "we don't clean clothes, we just sell signs!".
In the case of Heisenberg and also that of Einstein who was helped out by Riemann, this was no joke.
However, soon after Heisenberg's discovery, Schrodinger came up with his equation so physicists happily returned to the study of partial differential equations, and the message of Heisenberg was buried to a great extent. Most of my work has been an attempt to take this discovery of Heisenberg seriously. On reflection, this discovery actually clearly displays the limitation of Riemann's formulation of geometry. If we look at the phase space of an atomic system and follow Riemann's procedure to parametrize its points by finitely many real numbers, we first split the manifold into the levels on

which some particular function is constant, but we then need to iterate this process and apply it to the level hypersurfaces. However, according to Heisenberg this doesn't work because as soon as we make the first measurement, we alter the situation drastically. The right way to think about this new phenomenon is to think in terms of a new kind of space in which the coordinates do not commute.
The starting point of noncommutative geometry is to take this new notion of space seriously.

IV Noncommutative geometry

The basis of noncommutative geometry is twofold.
On the one hand there is a wealth of examples of spaces whose coordinate algebra is no longer commutative but which have obvious relevance in physics or mathematics. The first examples came, as we saw above, from phase space in quantum mechanics but there are many others, such as the leaf spaces of foliations, the duals of nonabelian groups, the space of Penrose tilings, the Brillouin zone in solid state physics, the noncommutative tori which appear naturally in string theory and in M-theory compactification, and the Adele class space which as we shall see below provides a natural spectral realisation of zeros of zeta functions. Finally various recent models of space-time itself are interesting examples of noncommutative spaces.
On the other hand the stretching of geometric thinking imposed by passing to noncommutative spaces forces one to rethink about most of our familiar notions. The difficulty is not to add arbitrarily the adjective quantum to our geometric words but to develop far reaching extensions of classical concepts, ranging from the simplest which is measure theory, to the most sophisticated which is geometry itself.
Let us first discuss in greater detail the general principles that allow to construct huge classes of such spaces, it is a vital ingredient indeed since there is no way to build a satisfactory theory without being able to test it on a large variety of examples. We have two principles which allow us to construct examples.
The first is deformation from the commutative to the noncommutative which allows to explore the neighborhood of the comutative world.
The second is a new and very important mathematical principle; the quotient operation. Most of the spaces we are concerned with are not defined by naming every one of their points, but by giving a much bigger set and dividing it by an equivalence relation.
It turns out that there are two ways of extending the geometric - algebraic duality

$$\text{Space} \leftrightarrow \text{Commutative algebra} \tag{1}$$

between a space X and the algebra of functions on that space, when you want to identify two points a and b. The first way which gives the usual algebra of functions associated to the quotient is to restrict oneself to functions which have the same value at the two points.

$$\mathcal{A} = \{f; f(a) = f(b)\}\,. \tag{2}$$

The second way is to keep the two points a and b, but to allow them to 'speak' to each other by using matrices with off-diagonal elements. It consists, instead of taking the subalgebra given by 4-2, to adjoin to the algebra of functions on $\{a, b\}$ the identification of a with b. The obtained algebra is the algebra of two by two matrices

$$\mathcal{B} = \left\{ f = \begin{bmatrix} f_{aa} & f_{ab} \\ f_{ba} & f_{bb} \end{bmatrix} \right\} . \tag{3}$$

When one computes the spectrum of this algebra it turns out that it is composed of only one point, so the two points a and b have been identified. As we shall see this second method is very powerful and allows one to construct thousands of very interesting examples. It allows to refine the above duality of algebraic geometry to,

$$\text{Quotient-Space} \leftrightarrow \text{Noncommutative algebra} \tag{4}$$

in the situation where the space one is contemplating is obtained by the operation of quotient.
At first sight it might seem that, as far as the general theory is concerned, passing from the commutative to the noncommutative situation would just be a matter of cleverly rewriting in algebraic terms our familiar geometric notions without using commutativity anywhere. If noncommutative geometry was just that it would be boring indeed. Fortunately, even at the coarsest level which is measure theory, it became clear at the beginning of the seventies that the noncommutative world is full of beautiful totally unexpected facts which have no commutative counterpart whatsoever. The prototype of such facts is the following

$$\text{Noncommutative measure spaces evolve with time!} \tag{5}$$

In other words there is a 'god-given' one parameter group of automorphisms of the algebra M of measurable coordinates. It is given by the group homomorphism, ([1])

$$\delta : \mathbb{R} \to \mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Int}(M) \tag{6}$$

from the additive group $\mathbb{R}$ to the group of automorphism classes of M modulo inner automorphisms.
I discovered this fact in 1972 when working on the Tomita-Takesaki theory ([2]) and it convinced me that there are amazing features of noncommutative spaces which have no counterpart in the static commutative case.

V A basic example

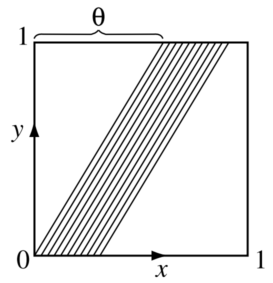
Let us start with a prototype example of quotient space in which the distinction between the quotient operations (4-2) and (4-3) appears clearly, and which played a key role in 1980 at the early stage of the theory ([40]). This example is the following: consider the 2-torus

$$M = \mathbb{R}^2/\mathbb{Z}^2 . \tag{1}$$

The space X which we contemplate is the space of solutions of the differential equation,

$$dx = \theta dy \qquad x, y \in \mathbb{R}/\mathbb{Z} \tag{2}$$

where $\theta \in]0, 1[$ is a fixed irrational number.



Thus the space we are interested in here is just the space of leaves of the foliation defined by the differential equation 5-2. We can label such a leaf by a point of the transversal given by $y = 0$ which is a circle $S^1 = \mathbb{R}/\mathbb{Z}$, but clearly two points of the transversal which differ by an integer multiple of θ give rise to the same leaf. Thus

$$X = S^1/\theta\mathbb{Z} \tag{3}$$

i.e. X is the quotient of S^1 by the equivalence relation which identifies any two points on the orbits of the irrational rotation

$$R_\theta x = x + \theta \quad \text{mod}1\,. \tag{4}$$

When we deal with S^1 as a space in the various categories (smooth, topological, measurable) it is perfectly described by the corresponding algebra of functions,

$$C^\infty(S^1) \subset C(S^1) \subset L^\infty(S^1)\,. \tag{5}$$

When one applies the naive operation (4-2) to pass to the quotient, one finds, irrespective of which category one works with, the trivial answer

$$\mathcal{A} = \mathbb{C}\,. \tag{6}$$

The operation (4-3) however gives very interesting algebras, by no means reduced to $\mathbb{C}$. Elements of the algebra $\mathcal{B}$ associated to the transversal S^1 by the operation (4-3) are just matrices $a(i,j)$ where the indices (i,j) are arbitrary pairs of elements i,j of S^1 which belong to the same leaf, i.e. give the same element of X. The algebraic rules are the same as for ordinary matrices. In the above situation since the equivalence is given by a group action, the construction coincides with the crossed product familiar to algebraist from the theory of central simple algebras.
An element of $\mathcal{B}$ is given by a power series

$$b = \sum_{n\in\mathbb{Z}} b_n U^n \tag{7}$$

where each b_n is an element of the algebra 5-5, while the multiplication rule is given by

$$UhU^{-1} = h \circ R_\theta^{-1}\,. \tag{8}$$

Now the algebra 5-5 is generated by the function V on S^1,

$$V(\alpha) = \exp(2\pi i\alpha) \qquad \alpha \in S^1 \tag{9}$$

and it follows that $\mathcal{B}$ admits the generating system (U, V) with presentation given by the relation

$$VU = \lambda\, UV \qquad \lambda = \exp 2\pi i\theta\,. \tag{10}$$

Thus, if for instance we work in the smooth category a generic element b of $\mathcal{B}$ is given by a power series

$$b = \sum_{\mathbb{Z}^2} b_{nm} U^n V^m \ , \ b \in \mathcal{S}(\mathbb{Z}^2) \tag{11}$$

where $\mathcal{S}(\mathbb{Z}^2)$ is the Schwartz space of sequences of rapid decay on $\mathbb{Z}^2$.
This algebra is by no means trivial and has a very rich and interesting algebraic structure. It is (canonically up to Morita equivalence) associated to the foliation 5-2 and the interplay between the geometry of the foliation and the algebraic structure of $\mathcal{B}$ begins by noticing that to a *closed transversal* T of the foliation corresponds canonically a *finite projective module* over $\mathcal{B}$. Elements of the module associated to the transversal T are rectangular matrices, $\xi(i, j)$ where $(i, j) \in T \times S^1$ while i and j belong to the same leaf, i.e. give the same element of X. The right action of $a(i, j) \in \mathcal{B}$ is by matrix multiplication.
From the transversal $x = 0$, one obtains the following right module over $\mathcal{B}$. The underlying linear space is the usual Schwartz space,

$$\mathcal{S}(\mathbb{R}) = \{\xi,\, \xi(s) \in \mathbb{C} \quad \forall s \in \mathbb{R}\} \tag{12}$$

of smooth functions on the real line all of whose derivatives are of rapid decay.
The right module structure is given by the action of the generators U, V

$$(\xi U)(s) = \xi(s + \theta)\ ,\ (\xi V)(s) = e^{2\pi i s}\xi(s) \quad \forall s \in \mathbb{R}\,. \tag{13}$$

One of course checks the relation 5-10, and it is a beautiful fact that as a right module over $\mathcal{B}$ the space $\mathcal{S}(\mathbb{R})$ is *finitely generated* and *projective* (*i.e.* complements to a free module). It follows that it has the correct algebraic atributes to deserve the name of "noncommutative vector bundle" according to the dictionary,

Space	Algebra
Vector bundle	Finite projective module.

The concrete description of the general finite projective modules over $\mathcal{A}_\theta$ is obtained by combining the results of [62, 40, 63]. They are classified up to isomorphism by a pair of integers (p, q) such that $p + q\theta \geq 0$ and the corresponding modules $\mathcal{H}^\theta_{p,q}$ are obtained by the above construction from the transversals given by closed geodesics of the torus M.
The algebraic counterpart of a vector bundle is its space of smooth sections $C^\infty(X, E)$ and one can in particular compute its dimension by computing the trace of the identity endomorphism of E. If one applies this method in the above noncommutative example, one finds

$$\dim_{\mathcal{B}}(\mathcal{S}) = \theta\,. \tag{14}$$

The appearance of non integral dimension is very exciting and displays a basic feature of von Neumann algebras of type II. The dimension of a vector bundle is the only invariant that remains when one looks from the measure theoretic point of view (*i.e.*

when one takes the third algebra in 5-5). The von Neumann algebra which describes the quotient space X from the measure theoretic point of view is the crossed product,

$$R = L^{\infty}(S^1) \rtimes_{R_\theta} \mathbb{Z} \tag{15}$$

and is the well known hyperfinite factor of type II_1. In particular the classification of finite projective modules $\mathcal{E}$ over R is given by a positive real number, the Murray and von Neumann *dimension*,

$$\dim_R(\mathcal{E}) \in \mathbb{R}_+ \,. \tag{16}$$

The next surprise is that even though the *dimension* of the above module is irrational, when we compute the analogue of the first Chern class, *i.e.* of the integral of the curvature of the vector bundle, we obtain an integer. Indeed the two commuting vector fields which span the tangent space for an ordinary (commutative) 2-torus correspond algebraically to two commuting derivations of the algebra of smooth functions. These derivations continue to make sense when the generators U and V of $C^\infty(\mathbb{T}^2)$ no longer commute but satisfy 5-10 so that they generate $\mathcal{B} = C^\infty(\mathbb{T}^2_\theta)$. They are given by the same formulas as in the commutative case,

$$\delta_1 = 2\pi i U \,\frac{\partial}{\partial U} \;, \delta_2 = 2\pi i V \,\frac{\partial}{\partial V} \tag{17}$$

so that $\delta_1 \left(\sum b_{nm} U^n V^m\right) = 2\pi i \sum n b_{nm} U^n V^m$ and similarly for δ_2. One still has of course

$$\delta_1 \delta_2 = \delta_2 \delta_1 \tag{18}$$

and the δ_j are still derivations of the algebra $\mathcal{B} = C^\infty(\mathbb{T}^2_\theta)$,

$$\delta_j(bb') = \delta_j(b)b' + b\delta_j(b') \quad \forall b, b' \in \mathcal{B} \,. \tag{19}$$

The analogues of the notions of connection and curvature of vector bundles are straightforward to obtain ([40]) since a connection is just given by the associated covariant differentiation ∇ on the space of smooth sections. Thus here it is given by a pair of linear operators,

$$\nabla_j : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}) \tag{20}$$

such that

$$\nabla_j(\xi b) = (\nabla_j \xi) b + \xi \delta_j(b) \quad \forall \xi \in \mathcal{S} \;, b \in \mathcal{B} \,. \tag{21}$$

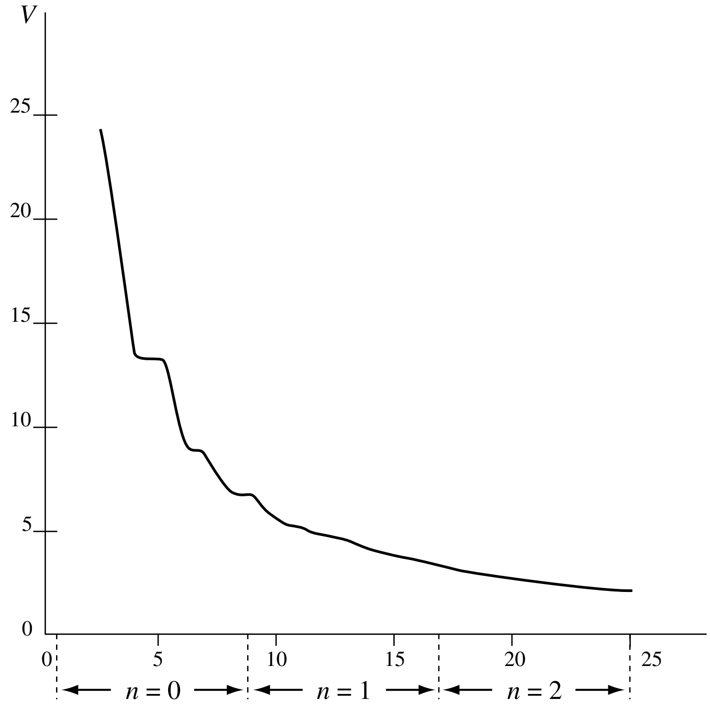
One checks that, as in the usual case, the trace of the curvature $\Omega = \nabla_1 \nabla_2 - \nabla_2 \nabla_1$, is independent of the choice of the connection. Now the remarkable fact here is that (up to the correct powers of $2\pi i$) the total curvature of $\mathcal{S}$ is an integer. In fact for the following choice of connection the curvature Ω is constant, equal to $\frac{1}{\theta}$ so that the irrational number θ disappears in the total curvature, $\theta \times \frac{1}{\theta}$

$$(\nabla_1 \xi)(s) = -\frac{2\pi i s}{\theta}\, \xi(s) \quad (\nabla_2 \xi)(s) = \xi'(s) \,. \tag{22}$$

With this integrality, one could get the wrong impression that the algebra $\mathcal{B} = C^\infty(\mathbb{T}^2_\theta)$ looks very similar to the algebra $C^\infty(\mathbb{T}^2)$ of smooth functions on the 2-torus. A striking difference is obtained by looking at the range of Morse functions. The range of a Morse function on $\mathbb{T}^2$ is of course a connected interval. For the above noncommutative torus $\mathbb{T}^2_\theta$ the range of a Morse function is the spectrum of a real valued function such as

$$h = U + U^* + \mu(V + V^*) \tag{23}$$

and it can be a Cantor set, *i.e.* have infinitely many disconnected pieces. This shows that the one dimensional pictures of our space $\mathbb{T}^2_\theta$ are truly different from what they are in the commutative case. The above noncommutative torus $\mathbb{T}^2_\theta$ is the simplest example of noncommutative manifold, it arises naturally not only from foliations but also from the Brillouin zone in the Quantum Hall effect as understood by J. Bellissard, and in M-theory as we shall see next. In the Quantum Hall effect, the above integrality of the total curvature corresponds to the observed integrality of the Hall conductivity



The analogue of the Yang-Mills action functional and the classification of Yang-Mills connections on the noncommutative tori was developped in [64], with the primary goal of finding a "manifold shadow" for these noncommutative spaces. These moduli spaces turned out indeed to fit this purpose perfectly, allowing for instance to find the usual Riemannian space of gauge equivalence classes of Yang-Mills connections as an invariant of the noncommutative metric.

The next surprise came from the natural occurence (as an unexpected guest) of both the noncommutative tori and the components of the Yang-Mills connections in the classification of the BPS states in M-theory [67].

In the matrix formulation of M-theory the basic equations to obtain periodicity of two of the basic coordinates X_i turn out to be the following,

$$U_i X_j U_i^{-1} = X_j + a\delta_i^j, i = 1, 2 \tag{24}$$

where the U_i are unitary gauge transformations.

The multiplicative commutator $U_1 U_2 U_1^{-1} U_2^{-1}$ is then central and in the irreducible case its scalar value $\lambda = \exp 2\pi i\theta$ brings in the algebra of coordinates on the noncommutative torus. The X_j are then the components of the Yang-Mills connections. It is quite remarkable that the same picture emerged from the other information one has about M-theory concerning its relation with 11 dimensional supergravity and that string theory dualities could be interpreted using Morita equivalence. The latter relates the values of θ on an orbit of $SL(2,\mathbb{Z})$ and simply illustrates that the leaf-space of the original foliation is independent of which transversal is used to parametrize it. This type of relation between for instance θ and $1/\theta$ would be invisible in a purely deformation theoretic perturbative expansion like the one given by the Moyal product.

Nekrasov and Schwarz [74] showed that Yang-Mills gauge theory on noncommutative $\mathbb{R}^4$ gives a conceptual understanding of the nonzero B-field desingularization of the moduli space of instantons obtained by perturbing the ADHM equations.

In [75], Seiberg and Witten exhibited the unexpected relation between the standard gauge theory and the noncommutative one, and clarified the limit in which the entire string dynamics is described by a gauge theory on a noncommutative space.

One should understand from the very start that foliations provide an inexhaustible source of interesting examples of noncommutative spaces. In the above example of $\mathbb{T}^2_\theta$ we could make use of the special vector fields on the torus in order to obtain the analogues of elementary notions of differential geometry. It is quite important to develop the general theory independently of these special features and this is what we shall do in section VII. We shall start by the noncommutative analogues of topology and vector bundles which are necessary preliminary steps.

VI Topology

The development of the topological ideas was prompted by the work of Israel Gel'fand, whose C* algebras give the required framework for noncommutative topology. The two main driving forces were the Novikov conjecture on homotopy invariance of higher signatures of ordinary manifolds as well as the Atiyah-Singer Index theorem. It has led, through the work of Atiyah, Singer, Brown, Douglas, Fillmore, Miscenko and Kasparov [4] [5] [6] [7] [8] to the recognition that not only the Atiyah-Hirzebruch K-theory but more importantly the dual K-homology admit Hilbert space techniques and functional analysis as their natural framework. The cycles in the K-homology group $K_*(X)$ of a compact space X are indeed given by Fredholm representations of the C* algebra A of continuous functions on X. The central tool is the Kasparov bivariant K-theory. A basic example of C* algebra to which the theory applies is the group ring of a discrete group and this makes it clear that restricting oneself to commutative algebras is an undesirable assumption.

For a C^* algebra A, let $K_0(A)$, $K_1(A)$ be its K theory groups. Thus $K_0(A)$ is the algebraic K_0 theory of the ring A and $K_1(A)$ is the algebraic K_0 theory of the ring $A \otimes C_0(\mathbb{R}) = C_0(\mathbb{R}, A)$. If $A \to B$ is a morphism of C^* algebras, then there are induced homomorphisms of abelian groups $K_i(A) \to K_i(B)$. Bott periodicity provides a six term K theory exact sequence for each exact sequence $0 \to J \to A \to B \to 0$ of C^* algebras and excision shows that the K groups involved in the exact sequence only depend on the respective C^* algebras. As an exercice to appreciate the power of this abstract tool one should for instance use the six term K theory exact sequence to give a short proof of the Jordan curve theorem.

Discrete groups, Lie groups, group actions and foliations give rise through their convolution algebra to a canonical C^* algebra, and hence to K theory groups. The analytical meaning of these K theory groups is clear as a receptacle for indices of elliptic operators. However, these groups are difficult to compute. For instance, in the case of semi-simple Lie groups the free abelian group with one generator for each irreducible discrete series representation is contained in $K_0\, C_r^* G$ where $C_r^* G$ is the reduced C^* algebra of G. Thus an explicit determination of the K theory in this case in particular involves an enumeration of the discrete series.
We introduced with P. Baum [9] a geometrically defined K theory which specializes to discrete groups, Lie groups, group actions, and foliations. Its main features are its computability and the simplicity of its definition. In the case of semi-simple Lie groups it elucidates the role of the homogeneous space G/K (K the maximal compact subgroup of G) in the Atiyah-Schmid geometric construction of the discrete series [10]. Using elliptic operators we constructed a natural map μ from our geometrically defined K theory groups to the above analytic (*i.e.* C^* algebra) K theory groups. Much progress has been made in the past years to determine the range of validity of the isomorphism between the geometrically defined K theory groups and the above analytic (*i.e.* C^* algebra) K theory groups. We refer to the three Bourbaki seminars [11], [12], [13] for an update on this topic and for a precise account of the various contributions. Among the most important contributions are those of Kasparov and Higson who showed that the conjectured isomorphism holds for all amenable groups, thus proving the Novikov conjecture for all amenable groups and the Kadison conjecture (i.e. the absence of nontrivial idempotents in the reduced C^*-algebra) for all torsion free amenable groups. The conjectured isomorphism also holds for real semi-simple Lie groups thanks in particular to the work of A. Wassermann. Moreover the recent work of V. Lafforgue crossed the barrier of property T, showing that it holds for cocompact subgroups of rank one Lie groups and also of $SL(3, \mathbb{R})$ or of p-adic Lie groups. He also gave the first general conceptual proof of the isomorphism for real or p-adic semi-simple Lie groups (and as a corollary a direct K-theoretic proof of the construction of all discrete series representations by Dirac-induction). The proof of the isomorphism is certainly accessible for all connected locally compact groups. The proof by G. Yu of the analogue (due to J. Roe) of the conjecture in the context of coarse geometry for metric spaces which are uniformly embeddable in hilbert space, and the work of G. Skandalis J. L. Tu, J. Roe and N. Higson on the groupoid case got very striking consequences such as the injectivity of the map μ for exact $C_r^*(\Gamma)$ due to Kaminker, Guentner and Ozawa, but recent progress due to Gromov, Higson, Lafforgue and Skandalis gives counterexamples to the general conjecture for locally compact groupoids for the simple reason that the functor $G \to K_0(C_r^*(G))$ is not half exact, unlike the functor given by the geometric group. This makes the general problem of computing $K(C_r^*(G))$ really interesting. It shows that besides determining the large class of locally compact groups for which the original conjecture is valid, one should understand how to take homological algebra into account to deal with the correct general formulation.

VII Differential Topology

The development of differential geometric ideas, including de Rham homology, connections and curvature of vector bundles, etc... took place during the eighties thanks to

cyclic cohomology which came from two different horizons ([14] [15] [16] [17] [18]).
In the commutative case, for a compact space X, we have at our disposal in K-theory a tool of great relevance, the Chern character

$$\mathrm{ch} : K^*(X) \to H^*(X, \mathbb{Q}) \tag{1}$$

which relates the K-theory of X to the cohomology of X. When X is a smooth manifold the Chern character may be calculated explicitly by the differential calculus of forms, currents, connections and curvature. More precisely, given a smooth vector bundle E over X, or equivalently the finite projective module, $\mathcal{E} = C^\infty(X, E)$ over $\mathcal{A} = C^\infty(X)$ of smooth sections of E, the Chern character of E

$$\mathrm{ch}(E) \in H^*(X, \mathbb{R}) \tag{2}$$

is represented by the closed differential form:

$$\mathrm{ch}(E) = \mathrm{trace}\,(\exp(\nabla^2/2\pi i)) \tag{3}$$

for any connection ∇ on the vector bundle E. Any closed de Rham current C on the manifold X determines a map φ_C from $K^*(X)$ to $\mathbb{C}$ by the equality

$$\varphi_C(E) = \langle C, \mathrm{ch}(E)\rangle \tag{4}$$

where the pairing between currents and differential forms is the usual one.
One obtains in this way numerical invariants of K-theory classes whose knowledge for arbitrary closed currents C is equivalent to that of $\mathrm{ch}(E)$.
The noncommutative torus gave a striking example where it was obviously worthwhile to adapt the above construction of differential geometry to the noncommutative framework ([40]). As an easy preliminary step towards cyclic cohomology one can reformulate the essential ingredient of the construction without direct reference to derivations in the following way ([17]).

By a cycle of dimension n we mean a triple $(\Omega, d, \int)$ where (Ω, d) is a graded differential algebra, and $\int : \Omega^n \to \mathbb{C}$ is a closed graded trace on Ω.
Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. Then a cycle over $\mathcal{A}$ is given by a cycle $(\Omega, d, \int)$ and a homomorphism $\rho : \mathcal{A} \to \Omega^0$.
Thus a *cycle* over an algebra $\mathcal{A}$ is a way to embed $\mathcal{A}$ as a subalgebra of a differential graded algebra (DGA). We shall see in f) below the role of the graded trace.
The usual notions of connection and curvature extend in a straightforward manner to this context ([17]).
Let $\mathcal{A} \xrightarrow{\rho} \Omega$ be a cycle over $\mathcal{A}$, and $\mathcal{E}$ a finite projective module over $\mathcal{A}$. Then a connection ∇ on $\mathcal{E}$ is a linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1$ such that

$$\nabla(\xi x) = (\nabla\xi)x + \xi \otimes d\rho(x)\ , \quad \forall\, \xi \in \mathcal{E}\ , \quad x \in \mathcal{A}\,. \tag{5}$$

Here $\mathcal{E}$ is a *right* module over $\mathcal{A}$ and Ω^1 is considered as a bimodule over $\mathcal{A}$ using the homomorphism $\rho : \mathcal{A} \to \Omega^0$ and the ring structure of Ω^*. Let us list a number of easy properties ([17]):

a) Let $e \in \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ be an idempotent and ∇ a connection on $\mathcal{E}$; then $\xi \mapsto (e \otimes 1)\nabla\xi$ is a connection on $e\mathcal{E}$.

b) Any finite projective module $\mathcal{E}$ admits a connection.
c)The space of connections is an affine space over the vector space

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1)\,. \tag{6}$$

d) Any connection ∇ extends uniquely to a linear map of $\widetilde{\mathcal{E}} = \mathcal{E} \otimes_{\mathcal{A}} \Omega$ into itself such that

$$\nabla(\xi \otimes \omega) = (\nabla\xi)\omega + \xi \otimes d\omega\ , \quad \forall\, \xi \in \mathcal{E}\ , \quad \omega \in \Omega\,. \tag{7}$$

e) The map $\theta = \nabla^2$ of $\widetilde{\mathcal{E}}$ to $\widetilde{\mathcal{E}}$ is an endomorphism: $\theta \in \mathrm{End}_{\Omega}(\widetilde{\mathcal{E}})$ and with $\delta(T) = \nabla T - (-1)^{degT} T\nabla$, one has $\delta^2(T) = \theta T - T\theta$ for all $T \in \mathrm{End}_{\Omega}(\widetilde{\mathcal{E}})$.
f) For n even, $n = 2m$, the equality

$$\langle[\mathcal{E}], [\tau]\rangle = \frac{1}{m!} \int \theta^m, \tag{8}$$

defines an additive map from the K-group $K_0(\mathcal{A})$ to the scalars.

Of course one can reformulate f) by dualizing the closed graded trace $\int$, i.e. by considering the homology of the quotient $\Omega/[\Omega, \Omega]$ ([60]) and one might be tempted at first sight to assert that a noncommutative algebra often comes naturally equipped with a natural embedding in a DGA which should suffice for the Chern character. This however would be rather naive and would overlook for instance the role of *integral* cycles for which the above additive map only affects *integer* values.
The starting point of cyclic cohomology is the ability to compare different cycles on the same algebra. In fact the invariant of K-theory defined in f) by a given cycle only depends on the multilinear form

$$\varphi(a^0, \ldots, a^n) = \int \rho(a^0)\, d(\rho(a^1))\, d(\rho(a^2)) \ldots d(\rho(a^n)) \qquad \forall\, a^j \in \mathcal{A} \tag{9}$$

(called the character of the cycle) and the functionals thus obtained are exactly those multilinear forms on $\mathcal{A}$ such that

φ is *cyclic* i.e.

$$\varphi(a^0, a^1, \ldots, a^n) = (-1)^n\, \varphi(a^1, a^2, \ldots, a^0) \qquad \forall\, a_j \in \mathcal{A}\,, \tag{10}$$

$b\varphi = 0$ where

$$(b\varphi)(a^0, \ldots, a^{n+1}) = \sum_0^n (-1)^j\, \varphi(a^0, \ldots, a^j a^{j+1}, \ldots, a^{n+1}) + (-1)^{n+1}\, \varphi(a^{n+1}a^0, a^1, \ldots, a^n)\,. \tag{11}$$

This second condition means that φ is a Hochschild cocycle. In particular such a φ admits a Hochschild class

$$I(\varphi) \in H^n(\mathcal{A}, \mathcal{A}^*) \tag{12}$$

for the Hochschild cohomology of $\mathcal{A}$ with coefficients in the bimodule $\mathcal{A}^*$ of linear forms on $\mathcal{A}$.
The n-dimensional *cyclic cohomology* of $\mathcal{A}$ is simply the cohomology $HC^n(\mathcal{A})$ of the *subcomplex* of the Hochschild complex given by cochains which are *cyclic* i.e. fulfill 10.
One has an obvious “forgetful” map

$$HC^n(\mathcal{A}) \xrightarrow{I} H^n(\mathcal{A}, \mathcal{A}^*) \tag{13}$$

but the real story starts with the following long exact sequence which allows in many cases to compute cyclic cohomology from the B operator acting on Hochschild cohomology:

Theorem 1. *The following triangle is exact:*

$$\begin{array}{ccccc} & & H^*(\mathcal{A},\mathcal{A}^*) & & \\ & {}^B\swarrow & & \nwarrow^I & \\ HC^*(\mathcal{A}) & & \stackrel{S}{\longrightarrow} & & HC^*(\mathcal{A}) \end{array}$$

The operator S is obtained by tensoring cycles by the canonical 2-dimensional generator of the cyclic cohomology of $\mathbb{C}$.
The operator B is explicitly defined at the cochain level by the equality

$$B = AB_0\,,\ B_0\,\varphi(a^0,\ldots,a^{n-1}) = \varphi(1,a^0,\ldots,a^{n-1}) - (-1)^n\,\varphi(a^0,\ldots,a^{n-1},1)$$
$$(A\psi)(a^0,\ldots,a^{n-1}) = \sum_0^{n-1}(-1)^{(n-1)j}\psi(a^j,a^{j+1},\ldots,a^{j-1})\,.$$

Its conceptual origin lies in the notion of cobordism of cycles which allows to compare different inclusion of $\mathcal{A}$ in DGA as follows. By a *chain* of dimension $n+1$ we shall mean a quadruple $(\Omega,\partial\Omega,d,\int)$ where Ω and $\partial\Omega$ are differential graded algebras of dimensions $n+1$ and n with a given surjective morphism $r:\Omega\to\partial\Omega$ of degree 0, and where $\int:\Omega^{n+1}\to\mathbb{C}$ is a graded trace such that

$$\int d\omega = 0\ ,\quad \forall\,\omega\in\Omega^n \text{ such that } r(\omega)=0\,. \tag{14}$$

By the *boundary* of such a chain we mean the cycle $(\partial\Omega,d,\int')$ where for $\omega'\in(\partial\Omega)^n$ one takes $\int'\omega'=\int d\omega$ for any $\omega\in\Omega^n$ with $r(\omega)=\omega'$. One easily checks, using the surjectivity of r, that $\int'$ is a graded trace on $\partial\Omega$ and is closed by construction.
We shall say that two cycles $\mathcal{A}\stackrel{\rho}{\longrightarrow}\Omega$ and $\mathcal{A}\stackrel{\rho'}{\longrightarrow}\Omega'$ over $\mathcal{A}$ are *cobordant* if there exists a chain Ω'' with boundary $\Omega\oplus\widetilde{\Omega}'$ (where $\widetilde{\Omega}'$ is obtained from Ω' by changing the sign of $\int$) and a homomorphism $\rho'':\mathcal{A}\to\Omega''$ such that $r\circ\rho''=(\rho,\rho')$.
The conceptual role of the operator B is clarified by the following result,

Theorem 2. *Two cycles over $\mathcal{A}$ are cobordant if and only if their characters $\tau_1,\tau_2\in HC^n(\mathcal{A})$ differ by an element of the image of B, where*

$$B: H^{n+1}(\mathcal{A},\mathcal{A}^*)\to HC^n(\mathcal{A})\,.$$

The operators b,B given as above by

$$(b\varphi)(a^0,\ldots,a^{n+1}) =$$
$$\sum_0^n(-1)^j\,\varphi(a^0,\ldots,a^ja^{j+1},\ldots,a^{n+1}) + (-1)^{n+1}\,\varphi(a^{n+1}a^0,a^1,\ldots,a^n)$$

$$B = AB_0\,,\ \ B_0\,\varphi(a^0,\ldots,a^{n-1}) = \varphi(1,a^0,\ldots,a^{n-1}) - (-1)^n\,\varphi(a^0,\ldots,a^{n-1},1)$$

$$(A\psi)(a^0,\ldots,a^{n-1}) = \sum_0^{n-1} (-1)^{(n-1)j}\,\psi(a^j,a^{j+1},\ldots,a^{j-1})$$

satisfy $b^2 = B^2 = 0$ and $bB = -Bb$ and periodic cyclic cohomology which is the inductive limit of the $HC^n(\mathcal{A})$ under the periodicity map S admits an equivalent description as the cohomology of the (b,B) bicomplex.
With these notations one has the following formula for the Chern character of the class of an idempotent e, up to normalization one has

$$Ch_n(e) = (e - 1/2)\otimes e\otimes e\otimes \ldots \otimes e, \tag{15}$$

where $\otimes$ appears 2n times in the right hand side of the equation.
Both the Hochschild and Cyclic cohomologies of the algebra $\mathcal{A} = C^\infty(V)$ of smooth functions on a manifold V were computed in [16] and [17].

Let V be a smooth compact manifold and $\mathcal{A}$ the locally convex topological algebra $C^\infty(V)$. Then the following map $\varphi \to C_\varphi$ is a canonical isomorphism of the continuous Hochschild cohomology group $H^k(\mathcal{A},\mathcal{A}^*)$ with the space of k-dimensional de Rham currents on V:

$$\langle C_\varphi, f^0\,d\,f^1\wedge\ldots\wedge d\,f^k\rangle = \frac{1}{k!}\sum_{\sigma\in S_k}\varepsilon(\sigma)\,\varphi(f^0,f^{\sigma(1)},\ldots,f^{\sigma(k)})$$

$\forall\, f^0,\ldots,f^k \in C^\infty(V)$.
Under the isomorphism C the operator $I\circ B : H^k(\mathcal{A},\mathcal{A}^*) \to H^{k-1}(\mathcal{A},\mathcal{A}^*)$ is (k times) the de Rham boundary b for currents.

Theorem 3. *Let $\mathcal{A}$ be the locally convex topological algebra $C^\infty(V)$. Then*

1) *For each k, $HC^k(\mathcal{A})$ is canonically isomorphic to the direct sum*

$$\mathrm{Ker}\, b\oplus H_{k-2}(V,\mathbb{C})\oplus H_{k-4}(V,\mathbb{C})\oplus\cdots$$

where $H_q(V,\mathbb{C})$ is the usual de Rham homology of V and b the de Rham boundary.

2) *The periodic cyclic cohomology of $C^\infty(V)$ is canonically isomorphic to the de Rham homology $H_*(V,\mathbb{C})$, with filtration by dimension.*

As soon as we pass to the noncommutative case, more subtle phenomena arise. Thus for instance the filtration of the periodic cyclic homology (dual to periodic cyclic cohomology) together with the lattice $K_0(\mathcal{A})\subset HC_{\rm ev}(\mathcal{A})$, for $\mathcal{A} = C^\infty(\mathbb{T}^2_\theta)$, gives an even analogue of the Jacobian of an elliptic curve. More precisely the filtration of $HC_{\rm ev}$ yields a canonical foliation of the torus $HC_{\rm ev}/K_0$ and one can show that the foliation algebra associated as above to the canonical transversal segment $[0,1]$ is isomorphic to $C^\infty(\mathbb{T}^2_\theta)$.
A simple example of cyclic cocycle on a nonabelian group ring is provided by the following formula. Any *group cocycle* $c\in H^*(B\Gamma) = H^*(\Gamma)$ gives rise to a cyclic cocycle φ_c on the algebra $\mathcal{A} = \mathbb{C}\Gamma$

$$\varphi_c(g_0,g_1,\ldots,g_n) = \begin{cases} 0 & \text{if}\quad g_0\cdots g_n \neq 1\\ c(g_1,\ldots,g_n) & \text{if}\quad g_0\cdots g_n = 1\end{cases}$$

where $c \in Z^n(\Gamma, \mathbb{C})$ is suitably normalized, and the formula is extended by linearity to $\mathbb{C}\Gamma$. The cyclic cohomology of group rings is given by,

Theorem 4. [22] *Let Γ be a discrete group, $\mathcal{A} = \mathbb{C}\Gamma$ its group ring.*

a) *The Hochschild cohomology $H^*(\mathcal{A}, \mathcal{A}^*)$ is canonically isomorphic to the cohomology $H^*((B\Gamma)^{\mathbb{S}^1}, \mathbb{C})$ of the free loop space of the classifying space of Γ.*

b) *The cyclic cohomology $HC^*(\mathcal{A})$ is canonically isomorphic to the $\mathbb{S}^1$-equivariant cohomology $H^*_{\mathbb{S}^1}((B\Gamma)^{\mathbb{S}^1}, \mathbb{C})$.*

The role of the free loop space in this theorem is not accidental and is clarified in general by the equality

$$B\Lambda = BS^1$$

of the classifying space $B\Lambda$ of the *cyclic category* with the classifying space of the compact group S^1. We refer to appendix XVIII for this point.

As we saw in section V the integral curvature of vector bundles on $\mathbb{T}^2_\theta$ was surprisingly giving an integer, in spite of the irrationality of θ. The conceptual understanding of this type of integrality result lies in the existence of a natural lattice of *integral cycles* which we now describe.

Definition. *Let $\mathcal{A}$ be an algebra, a Fredholm module over $\mathcal{A}$ is given by:*

1) *a representation of $\mathcal{A}$ in a Hilbert space $\mathcal{H}$;*

2) *an operator $F = F^*$, $F^2 = 1$, on $\mathcal{H}$ such that*

$$[F, a] \text{ is a compact operator for any } a \in \mathcal{A}\,.$$

Such a Fredholm module will be called *odd*. An *even* Fredholm module is given by an odd Fredholm module $(\mathcal{H}, F)$ as above together with a $\mathbb{Z}/2$ grading γ, $\gamma = \gamma^*$, $\gamma^2 = 1$ of the Hilbert space $\mathcal{H}$ such that:

a) $\gamma a = a\gamma \;\forall\, a \in \mathcal{A}$

b) $\gamma F = -F\gamma$.

The above definition is, up to trivial changes, the same as Atiyah's definition [4] of abstract elliptic operators, and the same as Kasparov's definition [8] for the cycles in K-homology, $KK(A, \mathbb{C})$, when A is a C^*-algebra.

The main point is that a Fredholm module over an algebra $\mathcal{A}$ gives rise in a very simple manner to a DGA containing $\mathcal{A}$. One simply defines Ω^k as the linear span of operators of the form,

$$\omega = a^0\,[F, a^1] \ldots [F, a^k] \qquad a^j \in \mathcal{A}$$

and the differential is given by

$$d\omega = F\omega - (-1)^k\,\omega F \qquad \forall\, \omega \in \Omega^k\,.$$

One easily checks that the ordinary product of operators gives an algebra structure, $\Omega^k\,\Omega^\ell \subset \Omega^{k+\ell}$ and that $d^2 = 0$ owing to $F^2 = 1$.

Moreover if one assumes that the size of the differential $da = [F, a]$ is controlled, i.e. that

$$|da|^{n+1} \quad \text{is trace class,}$$

then one obtains a natural closed graded trace of degree n by the formula,

$$\int \omega = \operatorname{Trace}(\omega)$$

(with the supertrace $\operatorname{Trace}(\gamma\omega)$ in the even case, see [36] for details).
Hence the original Fredholm module gives rise to a *cycle* over $\mathcal{A}$. Such cycles have the remarkable *integrality* property that when we pair them with the K theory of $\mathcal{A}$ we only get *integers* as follows from an elementary index formula ([36]).
We let $Ch_*(\mathcal{H}, F) \in HC^n(\mathcal{A})$ be the character of the cycle associated to a Fredholm module $(\mathcal{H}, F)$ over $\mathcal{A}$. This formula defines the Chern character in K-homology.
Cyclic cohomology got many applications [21], it led for instance to the proof of the Novikov conjecture for hyperbolic groups [19]. Basically, by extending the Chern-Weil characteristic classes to the general framework it allows for many concrete computations of differential geometric nature on noncommutative spaces. It also showed the depth of the relation between the classification of factors and the geometry of foliations.
Von Neumann algebras arise very naturally in geometry from foliated manifolds (V, F). The von Neumann algebra $L^\infty(V, F)$ of a foliated manifold is easy to describe, its elements are random operators $T = (T_f)$, i.e. bounded measurable families of operators T_f parametrized by the leaves f of the foliation. For each leaf f the operator T_f acts in the Hilbert space $L^2(f)$ of square integrable densities on the manifold f. Two random operators are identified if they are equal for almost all leaves f (i.e. a set of leaves whose union in V is negligible). The algebraic operations of sum and product are given by,

$$(T_1 + T_2)_f = (T_1)_f + (T_2)_f\,,\ (T_1\,T_2)_f = (T_1)_f\,(T_2)_f\,, \tag{16}$$

i.e. are effected pointwise.
All types of factors occur from this geometric construction and the continuous dimensions of Murray and von-Neumann play an essential role in the longitudinal index theorem.
Using cyclic cohomology together with the following simple fact,

“A connected group can only act trivially on a homotopy invariant cohomology theory”, (17)

one proves (cf. [20]) that for any codimension one foliation F of a compact manifold V with non vanishing Godbillon-Vey class one has,

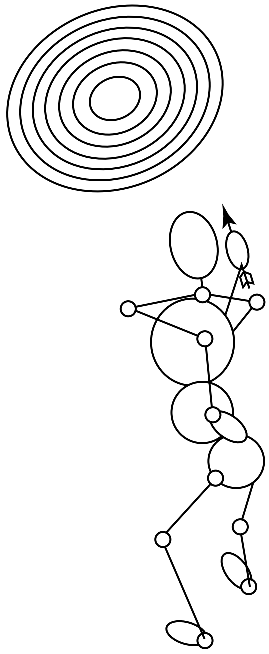
$$\operatorname{Mod}(M) \text{ has finite covolume in } \mathbb{R}_+^*\,, \tag{18}$$

where $\operatorname{Mod}(M)$ is the flow of weights of $M = L^\infty(V, F)$.
In the recent years J. Cuntz and D. Quillen ([23] [24] [25]) have developed a powerful new approach to cyclic cohomology which allowed them to prove excision in full generality.

VIII Calculus and Infinitesimals

The central notion of noncommutative geometry comes from the identification of the noncommutative analogue of the two basic concepts in Riemann's formulation of Geometry, namely those of manifold and of infinitesimal line element. Both of these

noncommutative analogues are of spectral nature and combine to give rise to the notion of spectral triple and spectral manifold, which will be described below. We shall first describe an operator theoretic framework for the calculus of infinitesimals which will provide a natural home for the line element ds.
I first have to make a little excursion, and I want it as naive as possible. I want to turn back to an extremely naive question about what is an infinitesimal. Let me first explain one answer that was proposed for this intuitive idea of infinitesimal and let me explain why this answer is not satisfactory and then give another answer which hopefully is satisfactory. So, I remember quite a long time ago to have seen an answer which was proposed by non standard analysis. The book I was reading [78] was starting from the following problem:
You play a game of throwing darts at some target called Ω



and the question which is asked is: what is the probability $dp(x)$ that actually when you send the dart you land exactly at a given point $x \in \Omega$? Then the following argument was given: certainly this probability $dp(x)$ is smaller than 1/2 because you can cut the target into two equal halves, only one of which contains x. For the same reason $dp(x)$ is smaller than 1/4, and so on and so forth. So what you find out is that $dp(x)$ is smaller than any positive real number ϵ. On the other hand, if you give the answer that $dp(x)$ is 0, this is not really satisfactory, because whenever you send the dart it will land somewhere. So now, if you ask a mathematician about this naive question, he might very well answer: well, $dp(x)$ is a 2-form, or it's a measure, or something like that. But then you can try to ask him more precise questions, for instance "what is the exponential of $-\frac{1}{dp(x)}$ ". And then it will be hard for him to give a satisfactory answer, because you know that the Taylor expansion of the function $f(y) = e^{-\frac{1}{y}}$ is zero at $y = 0$. Now the book I was reading claimed to give an answer, and it was what is called a non standard number. So I worked on this theory for some time, learning some logics, until eventually I realized there was a very bad obstruction preventing one to get concrete answers. It is the following: it's a little lemma that one can easily prove, that if you are given a non standard number you can canonically produce a subset of the interval which is not Lebesgue measurable. Now we know from logic (from results of Paul Cohen and Solovay) that it will forever be impossible to produce explicitely a subset of the real numbers, of the interval $[0, 1]$, say, that is not Lebesgue measurable. So, what this says is that for instance in this example, nobody will actually be able to name a non standard number. A nonstandard number is some sort of chimera which is impossible to grasp and certainly not a concrete object. In fact when you look at nonstandard analysis you find out that except for the use of ultraproducts, which is very efficient, it just shifts the order in logic by one step; it's not doing much more. Now, what I want to explain is that to the above naive question there is a very beautiful and simple answer which is provided by quantum mechanics. This answer will be obtained just by going through the usual dictionary of quantum mechanics, but looking at it more closely. So, let us thus look at the first two lines of the following dictionary which translates classical notions into the language of operators in the Hilbert space $\mathcal{H}$:

Complex variable	Operator in $\mathcal{H}$
Real variable	Selfadjoint operator
Infinitesimal	Compact operator
Infinitesimal of order α	Compact operator with characteristic values μ_n satisfying $\mu_n = O(n^{-\alpha})$, $n \to \infty$
Integral of an infinitesimal of order 1	$⨍ T =$ Coefficient of logarithmic divergence in the trace of T .

The first two lines of the dictionary are familiar from quantum mechanics. The range of a complex variable corresponds to the *spectrum* of an operator. The holomorphic functional calculus gives a meaning to $f(T)$ for all holomorphic functions f on the spectrum of T. It is only holomorphic functions which operate in this generality which reflects the difference between complex and real analysis. When $T = T^*$ is selfadjoint then $f(T)$ has a meaning for all Borel functions f.
The size of the infinitesimal $T \in \mathcal{K}$ is governed by the order of decay of the sequence of characteristic values $\mu_n = \mu_n(T)$ as $n \to \infty$. In particular, for all real positive α the following condition defines infinitesimals of order α:

$$\mu_n(T) = O(n^{-\alpha}) \qquad \text{when } n \to \infty \tag{1}$$

(i.e. there exists $C > 0$ such that $\mu_n(T) \leq C n^{-\alpha} \quad \forall\, n \geq 1$). Infinitesimals of order α also form a two–sided ideal and moreover,

$$T_j \text{ of order } \alpha_j \Rightarrow T_1 T_2 \text{ of order } \alpha_1 + \alpha_2 \,. \tag{2}$$

Hence, apart from commutativity, intuitive properties of the infinitesimal calculus are fulfilled.
Since the size of an infinitesimal is measured by the sequence $\mu_n \downarrow 0$ it might seem that one does not need the operator formalism at all, and that it would be enough to replace the ideal $\mathcal{K}$ in $\mathcal{L}(\mathcal{H})$ by the ideal $c_0(\mathbb{N})$ of sequences converging to zero in the algebra $\ell^\infty(\mathbb{N})$ of bounded sequences. A variable would just be a bounded sequence, and an infinitesimal a sequence $\mu_n, \mu_n \to 0$. However, this commutative version does not allow

for the existence of variables with range a continuum since all elements of $\ell^\infty(\mathbb{N})$ have a point spectrum and a discrete spectral measure. Only *noncommutativity* of $\mathcal{L}(\mathcal{H})$ allows for the coexistence of variables with Lebesgue spectrum together with infinitesimal variables. As we shall see shortly, it is precisely this lack of commutativity between the line element and the coordinates on a space that will provide the measurement of distances.

The integral is obtained by the following analysis, mainly due to Dixmier ([28]), of the logarithmic divergence of the partial traces

$$\mathrm{Trace}_N(T) = \sum_0^{N-1} \mu_n(T)\ ,\ T \geq 0\,. \tag{3}$$

In fact, it is useful to define $\mathrm{Trace}_\Lambda(T)$ for any positive real $\Lambda > 0$ by piecewise affine interpolation for noninteger Λ.

Define for all order 1 operators $T \geq 0$

$$\tau_\Lambda(T) = \frac{1}{\log \Lambda} \int_e^\Lambda \frac{\mathrm{Trace}_\mu(T)}{\log \mu}\ \frac{d\mu}{\mu} \tag{4}$$

which is the Cesaro mean of the function $\frac{\mathrm{Trace}_\mu(T)}{\log \mu}$ over the scaling group $\mathbb{R}_+^*$.

For $T \geq 0$, an infinitesimal of order 1, one has

$$\mathrm{Trace}_\Lambda(T) \leq C\ \log \Lambda \tag{5}$$

so that $\tau_\Lambda(T)$ is bounded. The essential property is the following *asymptotic additivity* of the coefficient $\tau_\Lambda(T)$ of the logarithmic divergence (5):

$$|\tau_\Lambda(T_1 + T_2) - \tau_\Lambda(T_1) - \tau_\Lambda(T_2)| \leq 3C\ \frac{\log(\log \Lambda)}{\log \Lambda} \tag{6}$$

for $T_j \geq 0$.

An easy consequence of (6) is that any limit point τ of the nonlinear functionals τ_Λ for $\Lambda \to \infty$ defines a positive and linear trace on the two–sided ideal of infinitesimals of order 1,

In practice the choice of the limit point τ is irrelevant because in all important examples T is a *measurable* operator, i.e.:

$$\tau_\Lambda(T) \text{ converges when } \ \Lambda \to \infty\,. \tag{7}$$

Thus the value $\tau(T)$ is independent of the choice of the limit point τ and is denoted

$$⨍ T\,. \tag{8}$$

The first interesting example is provided by pseudodifferential operators T on a differentiable manifold M. When T is of order 1 in the above sense, it is measurable and $⨍ T$ is the non-commutative residue of T ([29]). It has a local expression in terms of the distribution kernel $k(x,y)$, $x,y \in M$. For T of order 1 the kernel $k(x,y)$ diverges logarithmically near the diagonal,

$$k(x,y) = -a(x) \log|x-y| + 0(1)\ \text{(for } \ y \to x) \tag{9}$$

where $a(x)$ is a 1–density independent of the choice of Riemannian distance $|x-y|$. Then one has (up to normalization),

$$⨍ T = \int_M a(x). \tag{10}$$

The right hand side of this formula makes sense for all pseudodifferential operators (cf. [29]) since one can see that the kernel of such an operator is asymptotically of the form

$$k(x,y) = \sum a_k(x, x-y) - a(x) \log |x-y| + 0(1) \tag{11}$$

where $a_k(x,\xi)$ is homogeneous of degree $-k$ in ξ, and the 1–density $a(x)$ is defined intrinsically.

The same principle of extension of $⨍$ to infinitesimals of order < 1 works for hypoelliptic operators and more generally as we shall see below, for spectral triples whose dimension spectrum is simple.

We can now go back to our initial naive question about the target and the darts, we find that quantum mechanics gives us an obvious infinitesimal which answers the question: it is the inverse of the Dirichlet Laplacian for the domain Ω. Thus there is now a clear meaning for the exponential of $\frac{-1}{dp}$, that's the well known heat kernel which is an infinitesimal of arbitrarily large order as we expected from the Taylor expansion.

From the H. Weyl theorem on the asymptotic behavior of eigenvalues of Δ it follows that dp is of order 1, and that given a function f on Ω the product $f\, dp$ is measurable, while

$$⨍ f\, dp = \int_\Omega f(x_1, x_2)\, dx_1 \wedge dx_2 \tag{12}$$

gives the ordinary integral of f with respect to the measure given by the area of the target.

IX Spectral triples

In this section we shall come back to the two basic notions introduced by Riemann in the classical framework, those of *manifold* and of *line element*. We shall see that both of these notions adapt remarkably well to the noncommutative framework and this will lead us to the notion of spectral manifold which noncommutative geometry is based on.

In ordinary geometry of course you can give a manifold by a cooking recipe, by charts and local diffeomorphisms, and one could be tempted to propose an analogous cooking recipe in the noncommutative case. This is pretty much what is achieved by the general construction of the algebras of foliations and it is a good test of any general idea that it should at least cover that large class of examples.

But at a more conceptual level, it was recognized long ago by geometors that the main quality of the homotopy type of an oriented manifold is to satisfy Poincaré duality not only in ordinary homology but also in K-homology. Poincaré duality in ordinary homology is not sufficient to describe homotopy type of manifolds [30] but D. Sullivan [31] showed (in the simply connected PL case of dimension ≥ 5 ignoring 2-torsion) that it is sufficient to replace ordinary homology by KO-homology. Moreover the Chern character of the KO-homology fundamental class contains all the rational information on the Pontrjagin classes.

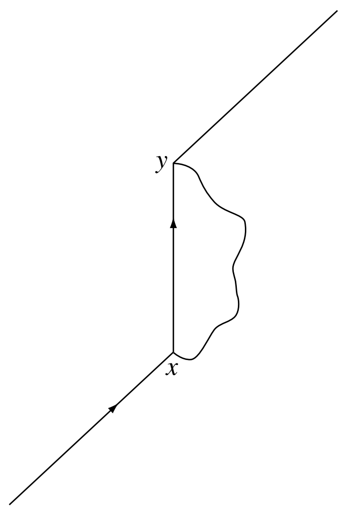
The characteristic property of *differentiable manifolds* which is carried over to the noncommutative case is *Poincaré duality* in KO-homology [31].

Moreover, as we saw above in the discussion of Fredholm modules, K-homology admits a fairly simple definition in terms of Hilbert space and Fredholm representations of algebras.

For an ordinary manifold the choice of the fundamental cycle in K-homology is a refinement of the choice of orientation of the manifold and in its simplest form is a choice of Spin-structure. Of course the role of a spin structure is to allow for the construction of the corresponding Dirac operator which gives a corresponding Fredholm representation of the algebra of smooth functions.

What is rewarding is that this will not only guide us towards the notion of noncommutative manifold but also to a formula, of operator theoretic nature, for the line element ds.

The infinitesimal unit of length "ds" should be an infinitesimal in the sense of section VIII and one way to get an intuitive understanding of the formula for ds is to consider Feynman diagrams which physicist use currently in the computations of quantum field theory. Let us contemplate the diagram



which is involved in the computation of the self-energy of an electron in QED. The two points x and y of space-time at which the photon (the wiggly line) is emitted and reabsorbed are very close by and our ansatz for ds will be at the intuitive level,

$$ds = \times\!\!-\!\!\times\,. \tag{1}$$

The right hand side has good meaning in physics, it is called the Fermion propagator and is given by

$$\times\!\!-\!\!\times = D^{-1} \tag{2}$$

where D is the Dirac operator.

We thus arrive at the following basic ansatz,

$$ds = D^{-1}\,. \tag{3}$$

In some sense it is simpler than the ansatz giving ds^2 as $g_{\mu\nu}\,dx^\mu\,dx^\nu$, the point being that the spin structure allows really to extract the square root of ds^2 (as is well known Dirac found the corresponding operator as a differential square root of a Laplacian).

The first thing we need to do is to check that we are still able to measure distances with our "unit of length" ds. In fact we saw in the discussion of the quantized calculus that variables with continuous range cant commute with "infinitesimals" such as ds and it is thus not very surprising that this lack of commutativity allows to compute, in the classical Riemannian case, the geodesic distance $d(x,y)$ between two points. The precise formula is

$$d(x,y) = \text{Sup}\,\{|f(x) - f(y)|\,;\; f \in \mathcal{A}\,,\; \|[D,f]\| \leq 1\} \tag{4}$$

where $D = ds^{-1}$ as above and $\mathcal{A}$ is the algebra of smooth functions. Note that if ds has the dimension of a length L, then D has dimension L^{-1} and the above expression for $d(x,y)$ also has the dimension of a length.

Thus we see in the classical geometric case that both the fundamental cycle in K-homology and the metric are encoded in the *spectral triple* $(\mathcal{A},\mathcal{H},D)$ where $\mathcal{A}$ is the algebra of functions acting in the Hilbert space $\mathcal{H}$ of spinors, while D is the Dirac operator.
To get familiar with this notion one should check that we recover the volume form of the Riemannian metric by the equality (valid up to a normalization constant [36])

$$⨍ f\,|ds|^n = \int_{M_n} f\,\sqrt{g}\;d^n x \tag{5}$$

but the first interesting point is that besides this coherence with the usual computations there are new simple questions we can ask now such as "what is the two-dimensional measure of a four manifold" in other words "what is its area ?". Thus one should compute

$$⨍ ds^2 \tag{6}$$

It is obvious from invariant theory that this should be proportional to the Hilbert–Einstein action but doing the direct computation is a worthwile exercice (cf. [52] [51]), the exact result being

$$⨍ ds^2 = \frac{-1}{48\pi^2}\int_{M_4} r\,\sqrt{g}\;d^4x \tag{7}$$

where as above $dv = \sqrt{g}\;d^4x$ is the volume form, $ds = D^{-1}$ the length element, *i.e.* the inverse of the Dirac operator and r is the scalar curvature.

In the general framework of Noncommutative Geometry the confluence of the Hilbert space incarnation of the two notions of metric and fundamental class for a manifold led very naturally to define a geometric space as given by a *spectral triple:*

$$(\mathcal{A},\mathcal{H},D) \tag{8}$$

where $\mathcal{A}$ is a concrete algebra of coordinates represented on a Hilbert space $\mathcal{H}$ and the operator D is the inverse of the line element.

$$ds = 1/D. \tag{9}$$

This definition is entirely spectral; the elements of the algebra are operators, the points, if they exist, come from the joint spectrum of operators and the line element is an operator.
The basic properties of such spectral triples are easy to formulate and do not make any reference to the commutativity of the algebra $\mathcal{A}$. They are

$$[D, a] \text{ is bounded for any } a \in \mathcal{A}\,, \tag{10}$$

$$D = D^* \text{ and } (D+\lambda)^{-1} \text{ is a compact operator } \forall\, \lambda \notin \mathbb{C}\,. \tag{11}$$

(Of course D is an *unbounded* operator).
There is no difficulty to adapt the above formula for the distance in the general non-commutative case, one uses the same, the points x and y being replaced by arbitrary states φ and ψ on the algebra $\mathcal{A}$. Recall that a state is a normalized positive linear form on $\mathcal{A}$ such that $\varphi(1) = 1$,

$$\varphi : \bar{\mathcal{A}} \to \mathbb{C}\ ,\ \varphi(a^* a) \geq 0\ , \quad \forall\, a \in \bar{\mathcal{A}}\ ,\ \varphi(1) = 1\,. \tag{12}$$

The distance between two states is given by,

$$d(\varphi, \psi) = \mathrm{Sup}\,\{|\varphi(a) - \psi(a)|\ ;\ a \in \mathcal{A}\ ,\ \|[D, a]\| \leq 1\}\,. \tag{13}$$

The significance of D is two-fold. On the one hand it defines the metric by the above equation, on the other hand its homotopy class represents the K-homology fundamental class of the space under consideration.
It is crucial to understand from the start the tension between the conditions 9-10 and 9-11. The first condition would be trivially fulfilled if D were bounded but condition 9-11 shows that it is unbounded. To understand this tension let us work out a very simple case. We let the algebra $\mathcal{A}$ be generated by a single unitary operator U. Let us show that if the index pairing between U and D, i.e. the index of PUP where P is the orthogonal projection on the positive eigenspace of D, *does not vanish* then the number $N(E)$ of eigenvalues of D whose absolute value is less than E grows at least like E when $E \to \infty$. This means that in the above circumstance $ds = D^{-1}$ is of order one or less.

To prove this we choose a smooth function $f \in C_c^\infty(\mathbb{R})$ identically one near 0, even and with Support $(f) \subset [-1, 1]$. We then let $R(\varepsilon) = f(\varepsilon D)$. One first shows ([36]) that the operator norm of the commutator $[R(\varepsilon), U]$ tends to 0 like ε. It then follows that the trace norm satisfies

$$\|[R(\varepsilon), U]\|_1 \leq C\, \varepsilon\, N(1/\varepsilon) \tag{14}$$

as one sees using the control of the rank of $R(\varepsilon)$ from $N(1/\varepsilon)$. The index pairing is given by $-\frac{1}{2}\,\mathrm{Trace}\,(U^*[F, U])$ where F is the sign of D and one has,

$$\mathrm{Trace}\,(U^*[F, U]) = \lim_{\varepsilon \to 0} \mathrm{Trace}\,(U^*[F, U]\, R(\varepsilon)) = \lim_{\varepsilon \to 0} \mathrm{Trace}\,(U^*\, F[U, R(\varepsilon)])\,. \tag{15}$$

Thus the limit being non zero we get a lower bound on the trace norm of $[U, R(\varepsilon)]$ and hence on $\varepsilon\, N\left(\frac{1}{\varepsilon}\right)$ which shows that $N(E)$ grows at least like E when $E \to \infty$.
This shows that ds cannot be too small (it cannot be of order $\alpha > 1$). In fact when ds is of order 1 one has the following index formula,

$$\mathrm{Index}\,(PUP) = -\frac{1}{2} ⨍ U^{-1}[D, U]\, |ds|\,. \tag{16}$$

The simplest case in which the index pairing between D and U does not vanish, with ds of order 1, is obtained by requiring the further condition,

$$U^{-1}[D,U] = 1\,. \tag{17}$$

It is a simple exercise to compute the geometry on $S^1 = \text{Spectrum}\,(U)$ given by an irreducible representation of condition 17. One obtains the standard circle with length 2π.

The above index formula is a special case of a general result ([36]) which computes the n-dimensional Hochschild class of the Chern character of a spectral triple of dimension n.

Theorem 5. *Let $(\mathcal{H}, F)$ be a Fredholm module over an involutive algebra $\mathcal{A}$. Let D be an unbounded selfadjoint operator in $\mathcal{H}$ such that D^{-1} is of order* $1/n$, $\text{Sign}\,D = F$, *and such that for any $a \in \mathcal{A}$ the operators a and $[D,a]$ are in the domain of all powers of the derivations δ, given by $\delta(x) = [|D|, x]$. Let $\tau_n \in HC^n(\mathcal{A})$ be the Chern character of $(\mathcal{H}, F)$.*

For every n-dimensional Hochschild cycle $c \in Z_n(\mathcal{A}, \mathcal{A})$, $c = \sum a^0 \otimes a^1 \ldots \otimes a^n$, one has $\langle \tau_n, c\rangle = ⨍ \sum a^0\,[D,a^1] \ldots [D,a^n]\,|D|^{-n}$.

We refer to [36] for precise normalization and to [66] for the detailed proof. By construction, this formula is scale invariant, i.e. it remains unchanged if we replace D by λD for $\lambda \in \mathbb{R}_+^*$. The operators T_c of the form

$$T_c = \sum a^0\,[D,a^1] \ldots [D,a^n]\,|D|^{-n} \tag{18}$$

are *measurable* in the sense of section VIII.

The long exact sequence of cyclic cohomology (Section VII) shows that the Hochschild class of τ_n is the obstruction to a better summability of $(\mathcal{H}, F)$, indeed τ_n belongs to the image $S(HC^{n-2}(\mathcal{A}))$ (which is the case if the degree of summability can be improved by 2) if and only if the Hochschild cohomology class $I(\tau_n) \in H^n(\mathcal{A}, \mathcal{A}^*)$ is equal to 0.

In particular, the above theorem implies nonvanishing of residues when the cohomological dimension of $\text{ch}_*(\mathcal{H}, F)$ is not lower than n:

Corollary. *With the hypothesis of Theorem 5 and if the Hochschild class of* $\text{ch}_*(\mathcal{H}, F)$ *pairs nontrivially with $H_n(\mathcal{A}, \mathcal{A})$ one has*

$$⨍ |D|^{-n} \neq 0\,. \tag{19}$$

In other words the residue of the function $\zeta(s) = \text{Trace}\,(|D|^{-s})$ at $s = n$ cannot vanish.

In higher dimension, the Hochschild class of the character suffices to determine the index pairing with the K-theory class of an idempotent e provided the lower dimensional components of $\text{ch}(e)$ vanish. As we saw above these components are given, up to normalization by,

$$\text{ch}_n(e) = \left(e - \frac{1}{2}\right) \otimes e \otimes \cdots \otimes e \tag{20}$$

(with $2n$ tensor signs) and as such cannot vanish. But both Hochschild and cyclic cohomology are Morita invariant, which implies that the class of $\text{ch}(e)$ in the normalized

(b, B) bicomplex (in homology) does not change when we project each of its components $\mathrm{ch}_n(e)$ on the commutant of a matrix algebra $M_q(\mathbb{C}) \subset \mathcal{A}$. The formula for this projection $\langle \mathrm{ch}_n(e) \rangle$ in terms of the matrix components e_{ij},

$$e = [e_{ij}] \, , \qquad e_{ij} \in M_q(\mathbb{C})' \cap \mathcal{A} \tag{21}$$

is the following,

$$\langle \mathrm{ch}_n(e) \rangle = \sum \left(e_{i_0 i_1} - \frac{1}{2} \, \delta_{i_0 i_1} \right) \otimes e_{i_1 i_2} \otimes e_{i_2 i_3} \otimes \cdots \otimes e_{i_{2n} i_0} \tag{22}$$

and there are very interesting situations in which all the lower components $\langle \mathrm{ch}_j(e) \rangle$ actually vanish,

$$\langle \mathrm{ch}_j(e) \rangle = 0 \qquad j < m \, . \tag{23}$$

For $m = 1$ for instance we can take $q = 2$ and the condition $\langle \mathrm{ch}_0(e) \rangle = 0$ means that e is of the form,

$$e = \begin{bmatrix} t & z \\ z^* & (1-t) \end{bmatrix} \, . \tag{24}$$

(The equation $e^2 = e$ then means that $t^2 + z^* \, z = t$, $tz + z(1-t) = z$, $z^* \, t + (1-t) \, z^* = z^*$, $z^* \, z + (1-t)^2 = (1-t)$ which shows that the algebra generated by the components z, z^*, t of e is abelian).

It then follows automatically that $\langle \mathrm{ch}_1(e) \rangle$ is a Hochschild cycle and hence by theorem 5, that if $ds = D^{-1}$ is of order $\frac{1}{2}$ the index pairing is given by,

$$\mathrm{Index} D_e^+ = - ⨍ \gamma \left(e - \frac{1}{2} \right) [D, e]^2 \, ds^2 \, . \tag{25}$$

Exactly as above this shows that ds cannot be of order $\alpha > \frac{1}{2}$ if the index pairing is non zero, and we also get the analogue of equation 9-17 in the form,

$$\left\langle \left(e - \frac{1}{2} \right) [D, e]^2 \right\rangle = \gamma \tag{26}$$

where $\langle \ \rangle$ is simply the projection on the commutant of $M_2(\mathbb{C})$ in $\mathcal{L}(\mathcal{H})$.

This equation together with 25 implies that the area $⨍ ds^2$ is an integer since it is given by a Fredholm index. One can show that the algebra $\mathcal{A}$ generated by the components of e is $C(S^2)$ the algebra of continuous functions on S^2 and that any Riemannian metric g on S^2 with fixed volume form gives a solution to the above equations.

There is a converse to that result ([50]) but it requires the further hypothesis that D is of order one:

$$[[D, e_{ij}], e_{k\ell}] = 0 \tag{27}$$

where the e_{ij} are the components of the idempotent e, i.e. are the generators of the algebra.

This order one condition is the counterpart in our operator theoretic setting of the "quadratic" nature of Riemann's equation $ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu$. It is easier to formulate in terms of the square root which we extracted using the spin structure. We shall come later to the correct formulation of the order one condition when the algebra $\mathcal{A}$ is noncommutative.

To end this section let us move on to the four dimensional case, i.e. $n = 2$. We take $q = 4$, i.e. we deal with $M_4(\mathbb{C})$.
We first determine the C^* algebra generated by $M_4(\mathbb{C})$ and a projection $e = e^*$ such that $\left\langle e - \frac{1}{2}\right\rangle = 0$ as above and whose two by two matrix expression is of the form,

$$[e^{ij}] = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \tag{28}$$

where each q_{ij} is a 2×2 matrix of the form,

$$q = \begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix} . \tag{29}$$

Since $e = e^*$, both q_{11} and q_{22} are selfadjoint, moreover since $\left\langle e - \frac{1}{2}\right\rangle = 0$, we can find $t = t^*$ such that,

$$q_{11} = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} , \ q_{22} = \begin{bmatrix} (1-t) & 0 \\ 0 & (1-t) \end{bmatrix} . \tag{30}$$

We let $q_{12} = \begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix}$, we then get from $e = e^*$,

$$q_{21} = \begin{bmatrix} \alpha^* & -\beta \\ \beta^* & \alpha \end{bmatrix} . \tag{31}$$

We thus see that the commutant $\mathcal{A}$ of $M_4(\mathbb{C})$ is generated by t, α, β and we first need to find the relations imposed by the equality $e^2 = e$.
In terms of $e = \begin{bmatrix} t & q \\ q^* & 1-t \end{bmatrix}$, the equation $e^2 = e$ means that $t^2 - t + qq^* = 0$, $t^2 - t + q^* q = 0$ and $[t, q] = 0$. This shows that t commutes with α, β, α^* and β^* and since $qq^* = q^* q$ is a diagonal matrix

$$\alpha\alpha^* = \alpha^*\alpha\,,\ \alpha\beta = \beta\alpha\,,\ \alpha^*\beta = \beta\alpha^*\,,\ \beta\beta^* = \beta^*\beta \tag{32}$$

so that the C^* algebra $\mathcal{A}$ is abelian, with the only further relation, (besides $t = t^*$),

$$\alpha\alpha^* + \beta\beta^* + t^2 - t = 0\,. \tag{33}$$

This is enough to check that,

$$\mathcal{A} = C(S^4) \tag{34}$$

where S^4 appears naturally as quaternionic projective space,

$$S^4 = P_1(\mathbb{H})\,. \tag{35}$$

The original C^* algebra is thus,

$$B = C(S^4) \otimes M_4(\mathbb{C})\,. \tag{36}$$

We shall now check that the two dimensional component $\langle Ch_1(e)\rangle$ automatically vanishes as an element of the (normalized) (b,B)-bicomplex so that,

$$\langle Ch_n(e)\rangle = 0\,,\ n = 0, 1. \tag{37}$$

With $q = \begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix}$, we get,

$$\begin{aligned} \langle Ch_1(e)\rangle \;\; &= \Big\langle \left(t - \frac{1}{2}\right) (dq\, dq^* - dq^*\, dq) \\ &\quad + q\,(dq^*\, dt - dt\, dq^*) + q^*\,(dt\, dq - dq\, dt) \Big\rangle \end{aligned} \tag{38}$$

where the expectation in the right hand side is relative to $M_2(\mathbb{C})$ and we use the notation dx instead of the tensor notation.
The diagonal elements of $\omega = dq\, dq^*$ are

$$\omega_{11} = d\alpha\, d\alpha^* + d\beta\, d\beta^* \,,\ \ \omega_{22} = d\beta^*\, d\beta + d\alpha^*\, d\alpha$$

while for $\omega' = dq^*\, dq$ we get,

$$\omega'_{11} = d\alpha^*\, d\alpha + d\beta\, d\beta^* \,,\ \ \omega'_{22} = d\beta^*\, d\beta + d\alpha\, d\alpha^* \,.$$

It follows that, since t is diagonal,

$$\Big\langle \left(t - \frac{1}{2}\right) (dq\, dq^* - dq^*\, dq) \Big\rangle = 0 \,. \tag{39}$$

The diagonal elements of $q\, dq^*\, dt = \rho$ are

$$\rho_{11} = \alpha\, d\alpha^*\, dt + \beta\, d\beta^*\, dt \,,\ \ \rho_{22} = \beta^*\, d\beta\, dt + \alpha^*\, d\alpha\, dt$$

while for $\rho' = q^*\, dq\, dt$ they are

$$\rho'_{11} = \alpha^*\, d\alpha\, dt + \beta\, d\beta^*\, dt \,,\ \ \rho'_{22} = \beta^*\, d\beta\, dt + \alpha\, d\alpha^*\, dt \,.$$

Similarly for $\sigma = q\, dt\, dq^*$ and $\sigma' = q^*\, dt\, dq$ one gets the required cancellations so that,

$$\langle Ch_1(e)\rangle = 0 \,, \tag{40}$$

It follows thus that $\langle Ch_2(e)\rangle$ is a Hochschild cycle and that for any $ds = D^{-1}$ of order $\frac{1}{4}$ commuting with $M_4(\mathbb{C})$, the index pairing of D with e is

$$\mathrm{Index} D_e^+ = ⨍ \gamma \left(e - \frac{1}{2}\right) [D, e]^4\, ds^4 \,. \tag{41}$$

Exactly as above this shows that ds cannot be of order $\alpha > \frac{1}{4}$ if the index pairing is non zero, and we also get the analogue of equation 9-17 in the form,

$$\Big\langle \left(e - \frac{1}{2}\right) [D, e]^4 \Big\rangle = \gamma \tag{42}$$

where $\langle\ \rangle$ is simply the projection on the commutant of $M_4(\mathbb{C})$ in $\mathcal{L}(\mathcal{H})$.
This equation together with (41) implies the integrality of the 4-dimensional volume,

$$⨍ ds^4 \in \mathbb{N}, \tag{43}$$

since it is given by a Fredholm index.
One can show that the algebra $\mathcal{A}$ generated by the components of e is $C(S^4)$ the algebra of continuous functions on S^4 and that any Riemannian metric g on S^4 gives a solution to the above equations, provided its volume form is,

$$v = \frac{1}{1-2t}\, d\alpha \wedge d\,\overline{\alpha} \wedge d\beta \wedge d\,\overline{\beta}\,. \tag{44}$$

As in the two dimensional case there is a converse, assuming the order one condition on D.
The next question is how is D to be chosen from within the homotopy class which characterizes its K-homology class? There are two answers to this question. The first uses the naive idea of a formal metric,

$$G = \sum_{\mu,\nu=1}^{d} dx^\mu g_{\mu\nu} (dx^\nu)^* \in \Omega^2_+(\mathcal{A})\,, \tag{45}$$

and the choice of D is performed by minimizing the action functional,

$$A = \sum_{\mu,\nu=1}^{d} ⨍ [D, x^\mu] g_{\mu\nu} ([D, x^\nu])^* |D^{-4}|\,, \tag{46}$$

among the D's which fulfill equation (42) holding G fixed.
The minimum is then given by the Dirac operator associated to the unique Riemannian metric with volume form v in the conformal class of $g_{\mu\nu} dx^\mu dx^\nu$.
The second way to select D from within its K-homology class is to use an action functional with the largest possible invariance group which is the unitary group of Hilbert space. The corresponding action is then spectral and only depends upon the eigenvalues of D. The simplest such action is of the form, [58]

$$S(D) = \mathrm{Trace}(f(D)). \tag{47}$$

where f is an even function vanishing at ∞. If we take for f a step function equal to 1 in $[-\Lambda, \Lambda]$, the value of $S(D)$ is,

$$N(\Lambda) = \#\text{ eigenvalues of } D \text{ in } [-\Lambda, \Lambda]. \tag{48}$$

This step function $N(\Lambda)$ is the superposition of two terms,

$$N(\Lambda) = \langle N(\Lambda) \rangle + N_{\mathrm{osc}}(\Lambda).$$

The oscillatory part $N_{\mathrm{osc}}(\Lambda)$ is the same as for a random matrix, governed by the statistic dictated by the symmetries of the system and does not concern us here. The average part $\langle N(\Lambda)\rangle$ is computed by a semiclassical approximation and the leading term in the asymptotic expansion is,

$$\frac{\Lambda^4}{2} ⨍ ds^4 \tag{49}$$

which by (43) is independent of the choice of D in its K-homology class.

If we restrict ourselves to solutions given by ordinary Riemannian metrics the next term in the asymptotic expansion is the Hilbert–Einstein action functional for the Riemannian metric,

$$\frac{-\Lambda^2}{96\pi^2} \int_{S_4} r \sqrt{g}\, d^4x \tag{50}$$

Other nonzero terms in the asymptotic expansion are cosmological, Weyl gravity and topological terms.

X Noncommutative 4-manifolds and the Instanton algebra

In this section, based on our collaboration with G. Landi ([65]), we shall show that the basic equation for an instanton in dimension 4, namely

$$e = e^2 = e^* \tag{1}$$

and

$$\langle \mathrm{ch}_0(e) \rangle = 0\ , \quad \langle \mathrm{ch}_1(e) \rangle = 0 \tag{2}$$

(where ch_n are the components of the Chern character,

$$\mathrm{ch}_n(e) = \left(e - \frac{1}{2}\right) \otimes e \otimes \ldots \otimes e \tag{3}$$

and $\langle\ \rangle$ is the projection onto the commutant of a 4×4 matrix algebra) do admit noncommutative solutions. In other words the algebra generated by the 16 components of the 4×4 matrix,

$$e = [e_{ij}] \tag{4}$$

will be noncommutative.

In fact this prompts us to introduce, a priori, the algebra $\mathcal{A}$ with 16 generators e_{ij} and whose presentation is given by the relations (1) and (2). The relation $\langle \mathrm{ch}_0(e) \rangle = 0$ just means that

$$e_{11} + e_{22} + e_{33} + e_{44} = 2 \tag{5}$$

and the equation $e = e^*$ defines the involution in $\mathcal{A}$. The relation $e^2 = e$ is easy to comprehend as a quadratic relation between the generators.

The relation $\langle \mathrm{ch}_1(e) \rangle = 0$ is more delicate to understand since it involves tensors and the simplest way to think about it is to represent the e_{ij} as operators in Hilbert space $\mathcal{H}$. What we ask then is that,

$$\sum \left(e_{ij} - \frac{1}{2}\, \delta_{ij}\right) \otimes \widetilde{e}_{jk} \otimes \widetilde{e}_{ki} = 0 \tag{6}$$

where the $\sim$ means that we take the class modulo the scalar multiples of 1.

This allows to define what is a unitary representation π of the algebra $\mathcal{A}$ and we can endow its elements, i.e. polynomials in the noncommuting generators e_{ij}, with the C^*-norm,

$$\|x\| = \sup_{\pi} \|\pi(x)\| \tag{7}$$

where π ranges through all unitary representations. It is easy to show that for $x \in \mathcal{A}$ the supremum is finite since in any unitary representation, the e_{ij} satisfy,

$$\|\pi(e_{ij})\| \leq 1 \tag{8}$$

as matrix elements of a selfadjoint idempotent.

Definition. *We let $C(\mathrm{Gr})$ be the C^* completion of $\mathcal{A}$ and $C^\infty(\mathrm{Gr})$ the smooth closure of $\mathcal{A}$ in $C(\mathrm{Gr})$.*

The letter Gr stands for the Grassmanian but our construction has little to do with the known "noncommutative Grassmanians". The really non-trivial condition is the cubic condition 6. In fact as we saw above the same construction in dimension 2 does give a *commutative* answer namely $P_1(\mathbb{C})$.

One should observe from the outset that the compact Lie group $SU(4)$ acts by automorphisms,

$$PSU(4) \subset \mathrm{Aut}\,(C^\infty(\mathrm{Gr})) \tag{9}$$

by the following operation,

$$e \to U\,e\,U^* \tag{10}$$

where $U \in SU(4)$ is viewed as a 4×4 matrix and $e = [e_{ij}]$ is as above.

What we saw in section IX is that there is a surjection,

$$C(\mathrm{Gr}) \to C(S^4) \tag{11}$$

while the corresponding symmetry group breaks down to $SO(4)$, the isometry group of the 3-sphere from which S^4 is obtained by suspension. We shall now show that the algebra $C(\mathrm{Gr})$ is noncommutative by constructing explicit surjections,

$$C(\mathrm{Gr}) \to C(S^4_\theta) \tag{12}$$

whose form is dictated by natural deformations of the 4-sphere similar in spirit to the above deformation of $\mathbb{T}^2$ to $\mathbb{T}^2_\theta$.

We first determine the C^* algebra generated by $M_4(\mathbb{C})$ and a projection $e = e^*$ such that $\left\langle e - \frac{1}{2} \right\rangle = 0$ as above and whose two by two matrix expression is of the form,

$$[e^{ij}] = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \tag{13}$$

where each q_{ij} is a 2×2 matrix of the form,

$$q = \begin{bmatrix} \alpha & \beta \\ -\lambda\beta^* & \alpha^* \end{bmatrix}. \tag{14}$$

where $\lambda = \exp 2\pi i\theta$ is a complex number of modulus one, different from -1 for convenience. Since $e = e^*$, both q_{11} and q_{22} are selfadjoint, moreover since $\left\langle e - \frac{1}{2} \right\rangle = 0$, we can find $t = t^*$ such that,

$$q_{11} = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}, \; q_{22} = \begin{bmatrix} (1-t) & 0 \\ 0 & (1-t) \end{bmatrix}. \tag{15}$$

We let $q_{12} = \begin{bmatrix} \alpha & \beta \\ -\lambda\beta^* & \alpha^* \end{bmatrix}$, we then get from $e = e^*$,

$$q_{21} = \begin{bmatrix} \alpha^* & -\bar{\lambda}\beta \\ \beta^* & \alpha \end{bmatrix} . \tag{16}$$

We thus see that the commutant $\mathcal{B}_\theta$ of $M_4(\mathbb{C})$ is generated by t, α, β and we first need to find the relations imposed by the equality $e^2 = e$.
In terms of $e = \begin{bmatrix} t & q \\ q^* & 1-t \end{bmatrix}$, the equation $e^2 = e$ means that $t^2 - t + qq^* = 0$, $t^2 - t + q^*q = 0$ and $[t, q] = 0$. This shows that t commutes with α, β, α^* and β^* and since $qq^* = q^*q$ is a diagonal matrix

$$\alpha\alpha^* = \alpha^*\alpha\,,\ \alpha\beta = \lambda\beta\alpha\,,\ \alpha^*\beta = \bar{\lambda}\beta\alpha^*\,,\ \beta\beta^* = \beta^*\beta \tag{17}$$

so that the C^* algebra $\mathcal{B}_\theta$ is not abelian for λ different from 1. The only further relation is, (besides $t = t^*$),

$$\alpha\alpha^* + \beta\beta^* + t^2 - t = 0\,. \tag{18}$$

We denote by S^4_θ the corresponding noncommutative space, so that $C(S^4_\theta) = \mathcal{B}_\theta$. It is by construction the suspension of the noncommutative 3-sphere S^3_θ whose coordinate algebra is generated by α and β as above for the special value $t = 1/2$. This noncommutative 3-sphere is related by analytic continuation of the parameter q to the quantum group $SU(2)_q$ but the usual theory requires q to be real whereas we need a complex number of modulus one which spoils the unitarity of the coproduct.
We shall now check that the two dimensional component $\langle Ch_1(e)\rangle$ automatically vanishes as an element of the (normalized) (b,B)-bicomplex.

$$\langle Ch_n(e)\rangle = 0\,,\ n = 0, 1. \tag{19}$$

With $q = \begin{bmatrix} \alpha & \beta \\ -\lambda\beta^* & \alpha^* \end{bmatrix}$, we get,

$$\begin{aligned} \langle Ch_1(e)\rangle \ &= \Big\langle \left(t - \frac{1}{2}\right)(dq\,dq^* - dq^*\,dq) \\ &\quad + q\,(dq^*\,dt - dt\,dq^*) + q^*\,(dt\,dq - dq\,dt) \Big\rangle \end{aligned} \tag{20}$$

where the expectation in the right hand side is relative to $M_2(\mathbb{C})$ and we use the notation dx instead of the tensor notation.
The diagonal elements of $\omega = dq\,dq^*$ are computed as above,

$$\omega_{11} = d\alpha\,d\alpha^* + d\beta\,d\beta^*\,,\ \omega_{22} = d\beta^*\,d\beta + d\alpha^*\,d\alpha$$

while for $\omega' = dq^*\,dq$ we get,

$$\omega'_{11} = d\alpha^*\,d\alpha + d\beta\,d\beta^*\,,\ \omega'_{22} = d\beta^*\,d\beta + d\alpha\,d\alpha^*\,.$$

It follows that, since t is diagonal,

$$\Big\langle \left(t - \frac{1}{2}\right)(dq\,dq^* - dq^*\,dq)\Big\rangle = 0\,. \tag{21}$$

The diagonal elements of $q\, dq^*\, dt = \rho$ are

$$\rho_{11} = \alpha\, d\alpha^*\, dt + \beta\, d\beta^*\, dt\,,\;\; \rho_{22} = \beta^*\, d\beta\, dt + \alpha^*\, d\alpha\, dt$$

while for $\rho' = q^*\, dq\, dt$ they are

$$\rho'_{11} = \alpha^*\, d\alpha\, dt + \beta\, d\beta^*\, dt\,,\;\; \rho'_{22} = \beta^*\, d\beta\, dt + \alpha\, d\alpha^*\, dt\,.$$

Similarly for $\sigma = q\, dt\, dq^*$ and $\sigma' = q^*\, dt\, dq$ one gets the required cancellations so that,

$$\langle Ch_1(e)\rangle = 0\,, \tag{22}$$

It follows thus that $\langle Ch_2(e)\rangle$ is a Hochschild cycle and that for any $ds = D^{-1}$ of order $\frac{1}{4}$ commuting with $M_4(\mathbb{C})$, the index pairing of D with e is

$$\text{Index}D_e^+ = ⨍ \gamma\left(e - \frac{1}{2}\right)[D,e]^4\, ds^4\,. \tag{23}$$

Exactly as above this shows that ds cannot be of order $\alpha > \frac{1}{4}$ if the index pairing is non zero, and we also get the analogue of equation 9-17 in the form,

$$\left\langle \left(e - \frac{1}{2}\right)[D,e]^4 \right\rangle = \gamma \tag{24}$$

where $\langle\ \rangle$ is simply the projection on the commutant of $M_4(\mathbb{C})$ in $\mathcal{L}(\mathcal{H})$.
This equation together with (23) implies the integrality of the 4-dimensional volume,

$$⨍ ds^4 \in \mathbb{N}, \tag{25}$$

since it is given by a Fredholm index. We shall refer to [65] for the explicit construction of solutions of (24). It should be clear to the reader that this amply justifies the clarification to which we turn next, of the notion of manifold in Noncommutative Geometry.

XI Noncommutative Spectral Manifolds

In our discussion in section IX of the K-homology fundamental class of a manifold we skipped over the nuance between K-homology and KO-homology. This nuance turns out to be essential in the noncommutative case. Thus to describe the fundamental class of a noncommuative space by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, will require an additional "real structure" on the Hilbert space $\mathcal{H}$ given by an antilinear isometry J. The anti-linear isometry J is given in Riemannian geometry by the charge conjugation operator and in the noncommutative case by the Tomita-Takesaki antilinear conjugation operator [2]. The action of $\mathcal{A}$ satisfies the commutation rule, $[a, b^0] = 0 \quad \forall\, a, b \in \mathcal{A}$ where

$$b^0 = Jb^* J^{-1} \qquad \forall b \in \mathcal{A} \tag{1}$$

so $\mathcal{H}$ becomes an $\mathcal{A}$-bimodule using the representation of $\mathcal{A} \otimes \mathcal{A}^0$, where $\mathcal{A}^0$ is the opposite algebra, given by,

$$a \otimes b^0 \to aJb^* J^{-1} \qquad \forall a, b \in \mathcal{A} \tag{2}$$

This allows to overcome the main difficulty of the noncommutative case which is that the diagonal in the square of the space no longer corresponds to an algebra homomorphism (the map $x \otimes y \to xy$ is no longer an algebra homomorphism),
The *fundamental class* of a noncommutative space is a class μ in the KR–homology of the algebra $\mathcal{A} \otimes \mathcal{A}^0$ equipped with the involution

$$\tau(x \otimes y^0) = y^* \otimes (x^*)^0 \qquad \forall\, x, y \in \mathcal{A} \tag{3}$$

where $\mathcal{A}^0$ denotes the algebra opposite to $\mathcal{A}$. The KR-homology cycle representing μ is given by a spectral triple, as above, equipped with an anti-linear isometry J on $\mathcal{H}$ which implements the involution τ,

$$JwJ^{-1} = \tau(w) \qquad \forall\, w \in \mathcal{A} \otimes \mathcal{A}^0\,, \tag{4}$$

KR-homology ([8] [55]) is periodic with period 8 and the dimension modulo 8 is specified by the following commutation rules. One has $J^2 = \varepsilon$, $JD = \varepsilon' DJ$, $J\gamma = \varepsilon'' \gamma J$ where $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ and with n the dimension modulo 8,

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

The class μ specifies only the stable homotopy class of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ equipped with the isometry J (and $\mathbb{Z}/2$–grading γ if n is even). The non-triviality of this homotopy class shows up in the intersection form

$$K_*(\mathcal{A}) \times K_*(\mathcal{A}) \to \mathbb{Z} \tag{5}$$

which is obtained from the Fredholm index of D with coefficients in $K_*(\mathcal{A} \otimes \mathcal{A}^0)$. Note that it is defined without using the diagonal map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, which is not a homomorphism in the noncommutative case. This form is quadratic or symplectic according to the value of n modulo 8.
The Kasparov intersection product [8] allows to formulate the Poincaré duality in terms of the invertibility of μ,

$$\exists\, \beta \in KR_n(\mathcal{A}^0 \otimes \mathcal{A})\ ,\ \beta \otimes_{\mathcal{A}} \mu = \mathrm{id}_{\mathcal{A}^0}\ ,\ \mu \otimes_{\mathcal{A}^0} \beta = \mathrm{id}_{\mathcal{A}}\ . \tag{6}$$

It implies the isomorphism $K_*(\mathcal{A}) \xrightarrow{\cap\mu} K^*(\mathcal{A})$.
The condition that D is an operator of order one becomes

$$[[D, a], b^0] = 0 \qquad \forall\, a, b \in \mathcal{A}\,. \tag{7}$$

(Notice that since a and b^0 commute this condition is equivalent to $[[D, a^0], b] = 0 \quad \forall\, a, b \in \mathcal{A}$.)
One can show that the von Neumann algebra $\mathcal{A}''$ generated by $\mathcal{A}$ in $\mathcal{H}$ is automatically finite and hyperfinite and there is a complete list of such algebras up to isomorphism. The algebra $\mathcal{A}$ is stable under smooth functional calculus in its norm closure $A = \bar{\mathcal{A}}$

so that $K_j(\mathcal{A}) \simeq K_j(A)$, i.e. $K_j(\mathcal{A})$ depends only on the underlying topology (defined by the C^* algebra A). The integer $\chi = \langle \mu, \beta \rangle \in \mathbb{Z}$ gives the Euler characteristic in the form

$$\chi = \text{Rang}\, K_0(\mathcal{A}) - \text{Rang}\, K_1(\mathcal{A}) \tag{8}$$

and the general operator theoretic index formula of section 13 below, gives a local formula for χ.

We gave in [50] the necessary and sufficient conditions that a spectral triple (with real structure J) should fulfill in order to come from an ordinary compact Riemannian spin manifold. These conditions extend in a straightforward manner to the noncommutative case ([50]). To appreciate the richness of examples which fulfill them we shall just quote the following result ([65]),

Theorem 6. *Let M be a compact Riemannian spin manifold. Then if the isometry group of M has rank $r \geq 2$, M admits a non-trivial one parameter isospectral deformation to noncommutative geometries M_θ.*

The group $\text{Aut}^+(\mathcal{A})$ of automorphisms α of the involutive algebra $\mathcal{A}$, which are implemented by a unitary operator U in $\mathcal{H}$ commuting with J,

$$\alpha(x) = U\, x\, U^{-1} \qquad \forall\, x \in \mathcal{A}\ , \tag{9}$$

plays the role of the group $\text{Diff}^+(M)$ of diffeomorphisms preserving the K-homology fundamental class for a manifold M.

In the general noncommutative case, parallel to the normal subgroup $\text{Int}\,\mathcal{A} \subset \text{Aut}\,\mathcal{A}$ of inner automorphisms of $\mathcal{A}$,

$$\alpha(f) = u f u^* \qquad \forall\, f \in \mathcal{A} \tag{10}$$

where u is a unitary element of $\mathcal{A}$ (i.e. $uu^* = u^*u = 1$), there exists a natural foliation of the space of spectral geometries on $\mathcal{A}$ by equivalence classes of *inner deformations* of a given geometry. To understand how they arise we need to understand how to transfer a given spectral geometry to a Morita equivalent algebra. Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and the Morita equivalence [56] between $\mathcal{A}$ and an algebra $\mathcal{B}$ where

$$\mathcal{B} = \text{End}_{\mathcal{A}}(\mathcal{E}) \tag{11}$$

where $\mathcal{E}$ is a finite, projective, hermitian right $\mathcal{A}$–module, one gets a spectral triple on $\mathcal{B}$ by the choice of a *hermitian connection* on $\mathcal{E}$. Such a connection ∇ is a linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$ satisfying the rules ([36])

$$\nabla(\xi a) = (\nabla \xi) a + \xi \otimes da \qquad \forall\, \xi \in \mathcal{E}\ ,\ a \in \mathcal{A} \tag{12}$$

$$(\xi, \nabla \eta) - (\nabla \xi, \eta) = d(\xi, \eta) \qquad \forall\, \xi, \eta \in \mathcal{E} \tag{13}$$

where $da = [D, a]$ and where $\Omega^1_D \subset \mathcal{L}(\mathcal{H})$ is the $\mathcal{A}$–bimodule of operators of the form

$$A = \Sigma\, a_i [D, b_i]\ ,\ a_i, b_i \in \mathcal{A}\,. \tag{14}$$

Any algebra $\mathcal{A}$ is Morita equivalent to itself (with $\mathcal{E} = \mathcal{A}$) and when one applies the above construction in the above context one gets the inner deformations of the spectral geometry.

Such a deformation is obtained by the following formula (with suitable signs depending on the dimension mod 8) without modifying neither the representation of $\mathcal{A}$ in $\mathcal{H}$ nor the anti-linear isometry J

$$D \to D + A + JAJ^{-1} \tag{15}$$

where $A = A^*$ is an arbitrary selfadjoint operator of the form 14. The action of the group $\mathrm{Int}(\mathcal{A})$ on the spectral geometries is simply the following gauge transformation of A

$$\gamma_u(A) = u[D, u^*] + uAu^* \,. \tag{16}$$

The required unitary equivalence is implemented by the following representation of the unitary group of $\mathcal{A}$ in $\mathcal{H}$,

$$u \to uJuJ^{-1} = u(u^*)^0 \,. \tag{17}$$

The transformation (15) is the identity in the usual Riemannian case. To get a non-trivial example it suffices to consider the product of a Riemannian triple by the unique spectral geometry on the finite-dimensional algebra $\mathcal{A}_F = M_N(\mathbb{C})$ of $N \times N$ matrices on $\mathbb{C}$, $N \geq 2$. One then has $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$, $\mathrm{Int}(\mathcal{A}) = C^\infty(M, PSU(N))$ and inner deformations of the geometry are parameterized by the gauge potentials for the gauge theory of the group $SU(N)$. The space of pure states of the algebra $\mathcal{A}$, $P(\mathcal{A})$, is the product $P = M \times P_{N-1}(\mathbb{C})$ and the metric on $P(\mathcal{A})$ determined by the formula (9.13) depends on the gauge potential A. It coincides with the Carnot metric [57] on P defined by the horizontal distribution given by the connection associated to A. The group $\mathrm{Aut}(\mathcal{A})$ of automorphisms of $\mathcal{A}$ is the following semi–direct product

$$\mathrm{Aut}(\mathcal{A}) = \mathcal{U} \rtimes \mathrm{Diff}^+(M) \tag{18}$$

of the local gauge transformation group $\mathrm{Int}(\mathcal{A})$ by the group of diffeomorphisms.

XII Test with space-time

What we have done so far is to stretch the usual framework of ordinary geometry beyond its commutative restrictions (set theoretic restrictions) and of course now it's not perhaps a bad idea to test it with what we know about physics and to try to find a better model of space-time within this new framework. The best way is to start with the hard core information one has from physics and that can be summarized by a Lagrangian. This Lagrangian is the Einstein Lagrangian plus the standard model Lagrangian. I am not going to write it down, it's a very complicated expression since just the standard model Lagrangian comprises five types of terms. But one can start understanding something by looking at the symmetry group of this Lagrangian. Now, if it were just the Einstein theory, the symmetry group of the Lagrangian would just be, by the equivalence principle, the diffeomorphism group of the space-time manifold. But because of the standard model piece the symmetry group of this Lagrangian is not just the diffeomorphism group, because the gauge theory has another huge symmetry group which is the group of maps from the manifold to the small gauge group, namely $U_1 \times SU_2 \times SU_3$ as far as we know. Thus, the symmetry group G of the full Lagrangian is neither the diffeomorphism group nor the group of gauge transformations of second kind nor their product, but it is their semi-direct product. It is exactly like what happens with the Poincaré group where you have translations and Lorentz transformations, so

it is the semi-direct product of these two subgroups. Now we can ask a very simple question: would there be some space X so that this group G would be equal to $\text{Diff}(X)$? If such a space would exist, then we would have some chance to actually geometrize completely the theory, namely to be able to say that it's pure gravity on the space X. Now, if you look for the space X among ordinary manifolds, you have no chance since by a result of John Mather the diffeomorphism group of a (connected) manifold is a simple group. A simple group cannot have a nontrivial normal subgroup, so you cannot have this structure of semi-direct product.
However, we can use our dictionary, and in this dictionary if we browse through it, we find that what corresponds to diffeomorphisms for a non commutative space is just the group $\text{Aut}^+(\mathcal{A})$ of automorphisms of the algebra of coordinates $\mathcal{A}$, which preserve the fundamental class in K-homology, as described above in section XI.
Now there is a beautiful fact which is that when an algebra is not commutative, then among its automorphisms there are very trivial ones, there are automorphisms which are there for free, I mean the inner ones, which associate to an element x of the algebra the element uxu^{-1}. Of course uxu^{-1} is not, in general equal to x because the algebra is not commutative, and these automorphisms form a normal subgroup of the group of automorphisms. Thus you see that the group $\text{Aut}^+(\mathcal{A})$ has the same type of structure, namely it has a normal subgroup of internal automorphisms and it has a quotient. Now it turns out that there is one very natural non commutative algebra $\mathcal{A}$ whose group of internal automorphisms corresponds to the group of gauge transformations and the quotient $\text{Aut}^+(\mathcal{A})/\text{Int}(\mathcal{A})$ corresponds exactly to diffeomorphisms [54]. It is amusing that the physics vocabulary is actually the same as the mathematical vocabulary. Namely in physics you talk about internal symmetries and in mathematics you talk about inner automorphisms, you could call them internal automorphisms. Now the corresponding space is a product $M \times F$ of an ordinary manifold M by a finite noncommutative space F. The corresponding algebra $\mathcal{A}_F$ is the direct sum of the algebras $\mathbb{C}$, $\mathbb{H}$ (the quaternions), and $M_3(\mathbb{C})$ of 3×3 complex matrices.
The algebra $\mathcal{A}_F$ corresponds to a *finite* space where the standard model fermions and the Yukawa parameters (masses of fermions and mixing matrix of Kobayashi Maskawa) determine the spectral geometry in the following manner. The Hilbert space is finite-dimensional and admits the set of elementary fermions as a basis. For example for the first generation of quarks, this set is

$$u_L, u_R, d_L, d_R, \bar{u}_L, \bar{u}_R, \bar{d}_L, \bar{d}_R \,. \tag{1}$$

The algebra $\mathcal{A}_F$ admits a natural representation in $\mathcal{H}_F$ (see [53]) and the Yukawa coupling matrix Y determines the operator D.

The detailed structure of Y (and in particular the fact that color is not broken) allows to check the axioms of noncommutative geometry.

The next step consists in the computation of internal deformations

$$D \to D + A + JAJ^{-1} \tag{2}$$

(cf. section XI), of the product geometry $M \times F$ where M is a 4–dimensional Riemannian spin manifold. The computation gives the standard model gauge bosons $\gamma, W^{\pm}, Z$, the eight gluons and the Higgs fields φ with accurate quantum numbers.
Now the next question that comes about is how do we recover the original action functional which contained both the Einstein-Hilbert term as well as the standard

model ? The answer is very simple: the Fermionic part of this action is there from the start and one recovers the bosonic part as follows. Both the Hilbert–Einstein action functional for the Riemannian metric, the Yang–Mills action for the vector potentials, the self interaction and the minimal coupling for the Higgs fields all appear with the correct signs in the asymptotic expansion for large Λ of the number $N(\Lambda)$ of eigenvalues of D which are $\leq \Lambda$ (cf. [58]),

$$N(\Lambda) = \# \text{ eigenvalues of } D \text{ in } [-\Lambda, \Lambda]. \tag{3}$$

Exactly as above, this step function $N(\Lambda)$ is the superposition of two terms,

$$N(\Lambda) = \langle N(\Lambda) \rangle + N_{\text{osc}}(\Lambda).$$

The oscillatory part $N_{\text{osc}}(\Lambda)$ is the same as for a random matrix, governed by the statistic dictated by the symmetries of the system and does not concern us here. The average part $\langle N(\Lambda) \rangle$ is computed by a semiclassical approximation from local expressions involving the familiar heat equation expansion and delivers the correct terms. We showed above in section IX, that if one studies natural presentations of the algebra generated by $\mathcal{A}$ and D one naturally gets only metrics with a fixed volume form so that the bothering cosmological term does not enter in the variational equations associated to the spectral action $\langle N(\Lambda) \rangle$. It is tempting to speculate that the phenomenological Lagrangian of physics, combining matter and gravity appears from the solution of an extremely simple operator theoretic equation along the lines described above in sections IX and X.

XIII Operator theoretic Index Formula

The power of the general theory comes from deeper general theorems such as the local computation of the analogue of Pontrjagin classes: *i.e.* of the components of the cyclic cocycle which is the Chern character of the K-homology class of D and which make sense in general. This result allows, using the infinitesimal calculus, to go from local to global in the general framework of spectral triples $(\mathcal{A}, \mathcal{H}, D)$.

The Fredholm index of the operator D determines (in the odd case) an additive map $K_1(\mathcal{A}) \xrightarrow{\varphi} \mathbb{Z}$ given by the equality

$$\varphi([u]) = \text{Index}\,(PuP)\ ,\ u \in GL_1(\mathcal{A}) \tag{1}$$

where P is the projector $P = \frac{1+F}{2}$, $F = \text{Sign}\,(D)$.

This map is computed by the pairing of $K_1(\mathcal{A})$ with the following cyclic cocycle

$$\tau(a^0, \ldots, a^n) = \text{Trace}\,(a^0 [F, a^1] \ldots [F, a^n]) \qquad \forall\, a^j \in \mathcal{A} \tag{2}$$

where $F = \text{Sign}\ D$ and we assume that the dimension p of our space is finite, which means that $(D+i)^{-1}$ is of order $1/p$, also $n \geq p$ is an odd integer. There are similar formulas involving the grading γ in the even case, and it is quite satisfactory ([33] [34]) that both cyclic cohomology and the chern Character formula adapt to the infinite dimensional case in which the only hypothesis is that $\exp(-D^2)$ is a trace class operator.

It is difficult to compute the cocycle τ in general because the formula (2) involves the ordinary trace instead of the local trace $⨍$ and it is crucial to obtain a local form of the above cocycle.
This problem is solved by a general formula [35] which we now describe.
Let us make the following regularity hypothesis on $(\mathcal{A}, \mathcal{H}, D)$

$$a \text{ and } [D,a] \in \cap \operatorname{Dom} \delta^k, \ \forall a \in \mathcal{A} \tag{3}$$

where δ is the derivation $\delta(T) = [|D|, T]$ for any operator T.
We let $\mathcal{B}$ denote the algebra generated by $\delta^k(a)$, $\delta^k([D,a])$. The usual notion of *dimension* of a space is replaced by the *dimension spectrum* which is a subset of $\mathbb{C}$. The precise definition of the dimension spectrum is the subset $\Sigma \subset \mathbb{C}$ of singularities of the analytic functions

$$\zeta_b(z) = \operatorname{Trace}(b|D|^{-z}) \qquad \operatorname{Re} z > p\ , \ b \in \mathcal{B}\,. \tag{4}$$

The dimension spectrum of a manifold M is the set $\{0, 1, \dots, n\}$, $n = \dim M$; it is simple. Multiplicities appear for singular manifolds. Cantor sets provide examples of complex points $z \notin \mathbb{R}$ in the dimension spectrum.
We assume that Σ is discrete and simple, i.e. that ζ_b can be extended to $\mathbb{C}/\Sigma$ with simple poles in Σ.
We refer to [35] for the case of a spectrum with multiplicities. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple satisfying the hypothesis (3) and (4). The local index theorem is the following, [35]:

Theorem 7.

1. *The equality*
$$⨍ P = \operatorname{Res}_{z=0} \operatorname{Trace}(P|D|^{-z})$$
defines a trace on the algebra generated by $\mathcal{A}$, $[D, \mathcal{A}]$ and $|D|^z$, where $z \in \mathbb{C}$.

2. *There is only a finite number of non–zero terms in the following formula which defines the odd components $(\varphi_n)_{n=1,3,\dots}$ of a cocycle in the bicomplex (b, B) of $\mathcal{A}$,*
$$\varphi_n(a^0, \dots, a^n) = \sum_k c_{n,k} ⨍ a^0 [D, a^1]^{(k_1)} \dots [D, a^n]^{(k_n)} \, |D|^{-n-2|k|} \qquad \forall\, a^j \in \mathcal{A}$$
where the following notations are used: $T^{(k)} = \nabla^k(T)$ and $\nabla(T) = D^2 T - T D^2$, k is a multi-index, $|k| = k_1 + \dots + k_n$,
$$c_{n,k} = (-1)^{|k|} \sqrt{2i} (k_1! \dots k_n!)^{-1} \left((k_1+1) \dots (k_1+k_2+\dots+k_n+n)\right)^{-1} \Gamma\left(|k| + \frac{n}{2}\right).$$

3. *The pairing of the cyclic cohomology class $(\varphi_n) \in HC^*(\mathcal{A})$ with $K_1(\mathcal{A})$ gives the Fredholm index of D with coefficients in $K_1(\mathcal{A})$.*

For the normalization of the pairing between HC^* and $K(\mathcal{A})$ see [36]. In the even case, i.e. when $\mathcal{H}$ is $\mathbb{Z}/2$ graded by γ,

$$\gamma = \gamma^*, \quad \gamma^2 = 1, \quad \gamma a = a \gamma \quad \forall\, a \in \mathcal{A}, \ \gamma D = -D\gamma,$$

there is an analogous formula for a cocycle (φ_n), n even, which gives the Fredholm index of D with coefficients in K_0. However, φ_0 is not expressed in terms of the residue $⨍$ because it is not local for a finite dimensional $\mathcal{H}$.

XIV Diffeomorphism invariant Geometry

The power of the above operator theoretic local trace formula lies in its generality and in the existence of really new geometric examples to which it applies.
In this section we shall explain how the transverse structure of foliations is described by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with simple dimension spectrum. This allows moreover to give a precise meaning to diffeomorphism invariant geometry on a manifold M, by the construction of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where the algebra $\mathcal{A}$ is the crossed product of the algebra of smooth functions on the finite dimensional bundle P of metrics on M by the natural action of the diffeomorphism group of M. While ordinary geometric constructions are "covariant" with respect to diffeomorphisms, our construction ([37]) is "invariant" inasmuch as the algebra now incorporates the full group of diffeomorphisms and the metrics involved are canonical.
The operator D is an hypoelliptic operator ([38]) which is directly associated to the reduction of the structure group of the manifold P to a group of triangular matrices whose diagonal blocks are orthogonal. By construction the fiber of $P \xrightarrow{\pi} M$ is the quotient $F^+/SO(n)$ of the $GL^+(n)$–principal bundle F^+ of oriented frames on M by the action of the orthogonal group $SO(n) \subset GL^+(n)$. The space P admits a canonical foliation: the vertical foliation $V \subset TP$, $V = \operatorname{Ker} \pi_*$ and on the fibers V and on $N = (TP)/V$ the following Euclidean structures. A choice of $GL^+(n)$–invariant Riemannian metric on $GL^+(n)/SO(n)$ determines a metric on V. The metric on N is defined tautologically: for every $p \in P$ one has a metric on $T_{\pi(p)}(M)$ which is isomorphic to N_p by π_*.
We first consider the hypoelliptic signature operator Q on F^+. It is not a scalar operator but it acts in the tensor product

$$\mathcal{H}_0 = L^2(F^+, v) \otimes E \tag{1}$$

where E is a finite dimensional representation of $SO(n)$ specifically given by

$$E = \wedge P_n \otimes \wedge \mathbb{R}^n \ , \ P_n = S^2 \, \mathbb{R}^n \, . \tag{2}$$

The operator Q is the graded sum,

$$Q = (d_V^* \, d_V - d_V \, d_V^*) \oplus (d_H + d_H^*) \tag{3}$$

where the horizontal (resp. vertical) differentiation d_H (resp. d_V) is a matrix in the horizontal and vertical vector fields $\mathbf{X}_i$ and $\mathbf{Y}_\ell^k$ as well as their adjoints (which also involve scalars). When n is equal to 1 or 2 modulo 4 one has to replace F^+ by its product by S^1 so that the dimension of the vertical fiber is even (it is then $1 + \frac{n(n+1)}{2}$) and the vertical signature operator makes sense. The longitudinal part is not elliptic but only transversally elliptic with respect to the action of $SO(n)$. Thus to get an hypoelliptic operator one restricts Q to the Hilbert space,

$$\mathcal{H} = (L^2(F^+, v) \otimes E)^{SO(n)} \tag{4}$$

and one takes the following algebra $\mathcal{A}$,

$$\mathcal{A} = C_c^\infty(P) \rtimes \mathrm{Diff}^+ \ , \ P = F^+/SO(n) \, . \tag{5}$$

Let us note that the operator Q is in fact the image under the right regular representation of the affine group G_{affine} of a (matrix valued) hypoelliptic symmetric element in the envelopping algebra $\mathcal{U}(G_{affine})$. By an easy adaptation of a theorem of Nelson and Stinespring, it then follows that Q is essentially selfadjoint (with core any dense G_{affine}-invariant subspace of the space of C^∞ vectors of the right regular representation of G_{affine}).

Theorem 8. [37] *Let $\mathcal{A}$ be the crossed product $C_c^\infty(P) \rtimes \mathrm{Diff}^+$ acting in $\mathcal{H}$ as above.*

1. *The equality $D|D| = Q$ defines a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ which satisfies the hypotheses of theorem 7; its dimension spectrum is simple and given by $\Sigma = \{0, 1, ..., 2n + \frac{n(n+1)}{2}\}$.*
2. *The cocycle φ_j given by the local index formula (theorem 7) is the image by the characteristic map of a universal Gelfand-Fuchs cohomology class.*

The equality $D|D| = Q$ defining D while Q is a differential operator of second order, is characteristic of "quartic" geometries.

The computation of the local index formula for diffeomorphism invariant geometry [37] was quite complicated even in the case of codimension 1 foliations: there were innumerable terms to be computed; this could be done by hand, by 3 weeks of eight hours per day tedious computations, but it was of course hopeless to proceed by direct computations in the general case. Henri and I finally found how to get the answer for the general case after discovering that the computation generated a Hopf algebra $\mathcal{H}(n)$ which only depends on n= codimension of the foliation, and which allows to organize the computation provided cyclic cohomology is suitably adapted to Hopf algebras as in the next section.

The Hopf algebra $\mathcal{H}(n)$ only depends upon the integer n and is neither commutative nor cocommutative. We proved in [37] that it is isomorphic to the bicrossed product Hopf algebra ([70], [69], [71]) associated to the following pair of subgroups of $G = \mathrm{Diff}(\mathbb{R}^n)$. We let $G_1 \subset G$ be the subgroup of affine diffeomorphisms,

$$k(x) = Ax + b \qquad \forall\, x \in \mathbb{R}^n \tag{6}$$

and we let $G_2 \subset G$ be the subgroup,

$$\varphi \in G\,, \;\; \varphi(0) = 0\,, \;\; \varphi'(0) = 1\,. \tag{7}$$

Given $\varphi \in G$ it has a unique decomposition $\varphi = k\,\psi$ where $k \in G_1$, $\psi \in G_2$ which allows to perform the bicrossed product construction.

XV Characteristic classes for actions of Hopf algebras

Hopf algebras arise very naturally from their actions on noncommutative algebras [39]. Given an algebra A, an action of the Hopf algebra $\mathcal{H}$ on A is given by a linear map,

$$\mathcal{H} \otimes A \to A, \quad h \otimes a \to h(a) \tag{1}$$

satisfying $h_1(h_2 a) = (h_1 h_2)(a)$, $\forall h_i \in \mathcal{H}$, $a \in A$ and

$$h(ab) = \sum h_{(1)}(a) h_{(2)}(b) \qquad \forall a, b \in A, h \in \mathcal{H}. \tag{2}$$

where the coproduct of h is,

$$\Delta(h) = \sum h_{(1)} \otimes h_{(2)} \tag{3}$$

In concrete examples, the algebra A appears first, together with linear maps $A \to A$ satisfying a relation of the form (2) which dictates the Hopf algebra structure. This is exactly what occured in the above example (see [37] for the description of $\mathcal{H}(n)$ and its relation with $\mathrm{Diff}(\mathbb{R}^n)$).
The theory of characteristic classes for actions of $\mathcal{H}$ extends the construction [40] of cyclic cocycles from a Lie algebra of derivations of a C^* algebra A, together with an *invariant trace* τ on A.
This theory was developped in [37] in order to solve the above computational problem for diffeomorphism invariant geometry but it was shown in [41] that the correct framework for the cyclic cohomology of Hopf algebras is that of modular pairs in involution. It is quite satisfactory that exactly the same structure emerged from the analysis of locally compact quantum groups. The resulting cyclic cohomology appears to be the natural candidate for the analogue of Lie algebra cohomology in the context of Hopf algebras. We fix a group-like element σ and a character δ of $\mathcal{H}$ with $\delta(\sigma) = 1$. They will play the role of the module of locally compact groups.
We then introduce the twisted antipode,

$$\widetilde{S}(y) = \sum \delta(y_{(1)}) S(y_{(2)}) \ , \ y \in \mathcal{H}, \ \Delta y = \sum y_{(1)} \otimes y_{(2)}. \tag{4}$$

We shall say that the modular pair (σ, δ) is in involution if the (σ, δ)-twisted antipode is an involution,

$$(\sigma^{-1} \widetilde{S})^2 = I. \tag{5}$$

We associate a cyclic complex (in fact a Λ-module, where Λ is the cyclic category), to any Hopf algebra together with a modular pair in involution. More precisely the following graded vector space $\mathcal{H}^\natural_{(\delta,\sigma)} = \{\mathcal{H}^{\otimes n}\}_{n \geq 1}$ equipped with the operators given by the following formulas (6)–(8) defines a module over the cyclic category Λ. First, by transposing the standard simplicial operators underlying the Hochschild homology complex of an algebra, one associates to $\mathcal{H}$, viewed only as a coalgebra, the natural cosimplicial module $\{\mathcal{H}^{\otimes n}\}_{n \geq 1}$, with face operators $\delta_i : \mathcal{H}^{\otimes n-1} \to \mathcal{H}^{\otimes n}$,

$$\begin{aligned} &\delta_0(h^1 \otimes \ldots \otimes h^{n-1}) = 1 \otimes h^1 \otimes \ldots \otimes h^{n-1} \\ &\delta_j(h^1 \otimes \ldots \otimes h^{n-1}) = h^1 \otimes \ldots \otimes \Delta h^j \otimes \ldots \otimes h^n, \ \forall 1 \leq j \leq n-1, \\ &\delta_n(h^1 \otimes \ldots \otimes h^{n-1}) = h^1 \otimes \ldots \otimes h^{n-1} \otimes \sigma \end{aligned} \tag{6}$$

and degeneracy operators $\sigma_i : \mathcal{H}^{\otimes n+1} \to \mathcal{H}^{\otimes n}$,

$$\sigma_i(h^1 \otimes \ldots \otimes h^{n+1}) = h^1 \otimes \ldots \otimes \varepsilon(h^{i+1}) \otimes \ldots \otimes h^{n+1} \ , \ 0 \leq i \leq n. \tag{7}$$

The remaining two essential features of a Hopf algebra –*product* and *antipode* – are brought into play, to define the *cyclic operators* $\tau_n : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$,

$$\tau_n(h^1 \otimes \ldots \otimes h^n) = (\Delta^{n-1} \widetilde{S}(h^1)) \cdot h^2 \otimes \ldots \otimes h^n \otimes \sigma. \tag{8}$$

The theory of characteristic classes applies to actions of the Hopf algebra on an algebra endowed with a δ-invariant σ-trace. A linear form τ on A is a σ-trace under the action of $\mathcal{H}$ iff one has,

$$\tau(ab) = \tau(b\sigma(a)) \qquad \forall a, b \in A. \tag{9}$$

A σ-trace τ on A is δ-invariant under the action of $\mathcal{H}$ iff

$$\tau(h(a)b) = \tau(a\widetilde{S}(h)(b)) \qquad \forall a, b \in A,\ h \in \mathcal{H}. \tag{10}$$

Note that equation (9) is an excellent guide in order to construct Hopf algebra actions, since by the modular theory any positive linear functional τ on an algebra A gives rise to an (unbounded) automorphism σ of its weak closure fulfilling equation (9).

The theory of characteristic classes for actions of Hopf algebras is governed by the following general result:

Theorem 9. ([41]) *Let $\mathcal{H}$ be a Hopf algebra endowed with a modular pair in involution Then $\mathcal{H}^\natural_{\delta,\sigma} = \{\mathcal{H}^{\otimes n}\}_{n\geq 1}$ equipped with the operators given by (6)–(8) defines a module over the cyclic category Λ. Let $\mathcal{H}$ act on an algebra A endowed with a δ-invariant σ-trace τ , then the following defines a canonical map from $HC^*_{\delta,\sigma}(\mathcal{H})$ to $HC^*(A)$,*

$$\gamma(h^1 \otimes \ldots \otimes h^n) \in C^n(A),\ \gamma(h^1 \otimes \ldots \otimes h^n)(x^0, \ldots, x^n) =$$

$$\tau(x^0 h^1(x^1) \ldots h^n(x^n)).$$

We refer to [41] for the discussion of the remarkable agreement of this theory with the standard theory of quantum groups and their locally compact versions.

XVI Hopf algebras, Renormalization and the Riemann-Hilbert problem

We describe in this section our joint work with Dirk Kreimer. Perturbative renormalization is by far the most successful technique for computing physical quantities in quantum field theory. It is well known for instance that it accurately predicts the first ten decimal places of the anomalous magnetic moment of the electron.

The physical motivation behind the renormalization technique is quite clear and goes back to the concept of effective mass in nineteen century hydrodynamics. To appreciate it, one should dive under water with a ping-pong ball and start applying Newton's law,

$$F = m\, a \tag{1}$$

to compute the initial acceleration of the ball B when we let it loose (at zero speed relative to the water). If one naively applies 1, one finds (see the QFT course by Sidney Coleman) an unrealistic initial acceleration of about 20g! In fact as explained in loc. cit. due to the interaction of B with the surrounding field of water, the inertial mass m involved in 1 is not the bare mass m_0 of B but is modified to

$$m = m_0 + \tfrac{1}{2}\, M \tag{2}$$

where M is the mass of the water occupied by B.

It follows for instance that the initial acceleration a of B is given, using the Archimedean law, by

$$-(M - m_0)g = \left(m_0 + \tfrac{1}{2}M\right)a \tag{3}$$

and is always of magnitude less than $2g$.

The additional inertial mass $\delta m = m - m_0$ is due to the interaction of B with the surounding field of water and if this interaction could not be turned off (which is the case if we deal with an electron instead of a ping-pong ball) there would be no way to measure the bare mass m_0.

The analogy between hydrodynamics and electromagnetism led (through the work of Thomson, Lorentz, Kramers... [80]) to the crucial distinction between the bare parameters, such as m_0, which enter the field theoretic equations, and the observed parameters, such as the inertial mass m.

A quantum field theory in $D = 4$ dimensions, is given by a classical action functional,

$$S(A) = \int \mathcal{L}(A)\, d^4x \tag{4}$$

where A is a classical field and the Lagrangian is of the form,

$$\mathcal{L}(A) = (\partial A)^2/2 - \frac{m^2}{2}\,A^2 + \mathcal{L}_{\rm int}(A) \tag{5}$$

where $\mathcal{L}_{\rm int}(A)$ is usually a polynomial in A and possibly its derivatives.

One way to describe the quantum fields $\phi(x)$, is by means of the time ordered Green's functions,

$$G_N(x_1, \ldots, x_N) = \langle 0 | T\,\phi(x_1)\ldots\phi(x_N)| 0 \rangle \tag{6}$$

where the time ordering symbol T means that the $\phi(x_j)$'s are written in order of increasing time from right to left.

The probability amplitude of a classical field configuration A is given by,

$$e^{i\frac{S(A)}{\hbar}} \tag{7}$$

and if one could ignore the renormalization problem, the Green's functions would be computed as,

$$G_N(x_1, \ldots, x_N) = \mathcal{N} \int e^{i\frac{S(A)}{\hbar}}\; A(x_1)\ldots A(x_N)\,[dA] \tag{8}$$

where $\mathcal{N}$ is a normalization factor required to ensure the normalization of the vacuum state,

$$\langle 0 \mid 0 \rangle = 1\,. \tag{9}$$

If one could ignore renormalization, the functional integral 8 would be easy to compute in perturbation theory, i.e. by treating the term $\mathcal{L}_{\rm int}$ in 5 as a perturbation of

$$\mathcal{L}_0(A) = (\partial A)^2/2 - \frac{m^2}{2}\,A^2\,. \tag{10}$$

With obvious notations the action functional splits as

$$S(A) = S_0(A) + S_{\rm int}(A) \tag{11}$$

where the free action S_0 generates a Gaussian measure $\exp(i\, S_0(A))\, [dA] = d\mu$.

The series expansion of the Green's functions is then given in terms of Gaussian integrals of polynomials as,

$$G_N(x_1, \ldots, x_N) = \left(\sum_{n=0}^{\infty} i^n/n! \int A(x_1) \ldots A(x_N)\, (S_{\text{int}}(A))^n \, d\mu \right) \left(\sum_{n=0}^{\infty} i^n/n! \int S_{\text{int}}(A)^n \, d\mu \right)^{-1}$$

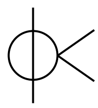
The various terms of this expansion are computed using integration by parts under the Gaussian measure μ. This generates a large number of terms $U(\Gamma)$, each being labelled by a Feynman graph Γ, and having a numerical value $U(\Gamma)$ obtained as a multiple integral in a finite number of space-time variables. As a rule the unrenormalized values $U(\Gamma)$ are given by nonsensical divergent integrals.

The conceptually really nasty divergences are called ultraviolet and are associated to the presence of arbitrarily large frequencies or equivalently to the unboundedness of momentum space on which integration has to be carried out. Equivalently, when one attempts to integrate in coordinate space, one confronts divergences along diagonals, reflecting the fact that products of field operators are defined only on the configuration space of distinct spacetime points.

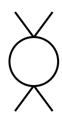
The physics resolution of this problem is obtained by first introducing a cut-off in momentum space (or any suitable regularization procedure) and then by cleverly making use of the unobservability of the bare parameters, such as the bare mass m_0. By adjusting, term by term of the perturbative expansion, the dependence of the bare parameters on the cut-off parameter, it is possible for a large class of theories, called renormalizable, to eliminate the unwanted ultraviolet divergences.

The main calculational complication of this subtraction procedure occurs for diagrams which possess non-trivial subdivergences, i.e. subdiagrams which are already divergent. In that situation the procedure becomes very involved since it is no longer a simple subtraction, and this for two obvious reasons: i) the divergences are no longer given by local terms, and ii) the previous corrections (those for the subdivergences) have to be taken into account.

To have an example for the combinatorial burden imposed by these difficulties consider the problem below of the renormalization of a two-loop four-point function in massless scalar ϕ^4 theory in four dimensions, given by the following Feynman graph:

$\Gamma^{(2)} =$ 

It contains a divergent subgraph:

$\Gamma^{(1)} =$ 

We work in the Euclidean framework and introduce a cut-off λ which we assume to be always much bigger than the square of any external momentum p_i. Analytic expressions for these Feynman graphs are obtained by utilizing a map Γ_λ which assigns integrals to them according to the Feynman rules and employs the cut-off λ to the momentum integrations. Then $\Gamma_\lambda[\Gamma^{[1,2]}]$ are given by

$$\Gamma_\lambda[\Gamma^{[1]}](p_i) = \int d^4k \frac{\Theta(\lambda^2 - k^2)}{k^2} \frac{1}{(k + p_1 + p_2)^2},$$

and

$$\Gamma_\lambda[\Gamma^{[2]}](p_i) = \int d^4l \frac{\Theta(\lambda^2 - l^2)}{l^2(l + p_1 + p_2)^2} \Gamma_\lambda[\Gamma^{[1]}(p_1, l, p_2, l)].$$

It is easy to see that $\Gamma_\lambda[\Gamma^{[1]}]$ decomposes into the form $b \log \lambda$ (where b is a real number) plus terms which remain finite for $\lambda \to \infty$, and hence will produce a divergence which is a non-local function of external momenta

$$\sim \log \lambda \int d^4l \frac{\Theta(\lambda^2 - l^2)}{l^2(l + p_1 + p_2)^2} \sim \log \lambda \; \log(p_1 + p_2)^2.$$

Fortunately, the counterterm $\mathcal{L}_{\Gamma^{[1]}} \sim \log \lambda$ generated to subtract the divergence in $\Gamma_\lambda[\Gamma^{[1]}]$ will precisely cancel this non-local divergence in $\Gamma^{[2]}$.
That this type of cancellation occurs at any order of perturbation theory, i. e. that the two diseases above actually cure each other in general is a very non-trivial fact that took decades to prove [79].
The detailed combinatorics is governed by the $\bar{R}$ operation of Bogoliubov and Parasiuk (for a 1PI graph Γ)

$$\bar{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \subsetneq \Gamma} C(\gamma)\, U(\Gamma/\gamma) \tag{12}$$

which prepares a given graph with unrenormalized value $U(\Gamma)$ by adding the counterterms $C(\gamma)$. The latter are constructed by induction using

$$C\,(\Gamma) = -T\left(U(\Gamma) + \sum_{\gamma \subsetneq \Gamma} C(\gamma)\, U(\Gamma/\gamma) \right) \tag{13}$$

where, using dimensional regularization T is just the extraction of the pole part in $D = 4 - \epsilon$. The renormalized graph is then given by

$$R\,(\Gamma) = U(\Gamma) + C(\Gamma) + \sum_{\gamma \subsetneq \Gamma} C(\gamma)\, U(\Gamma/\gamma)\,. \tag{14}$$

For a mathematician the intricacies of the detailed combinatorics and the lack of any obvious mathematical structure underlying it make it totally inaccessible, in spite of the existence of a satisfactory formal approach to the problem [81].
This situation was drastically changed by the discovery by Dirk Kreimer ([42]) who understood that the formula for the $\bar{R}$ operation in fact dictates a Hopf algebra coproduct on the free commutative algebra $\mathcal{H}_K$ generated by the 1PI graphs Γ,

$$\Delta\,\Gamma = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma\ , \tag{15}$$

(In fact he first formulated it in terms of rooted trees, but the graph formulation is easier to explain).
This Hopf algebra is commutative as an algebra and we showed in [44], [46] that it is the dual Hopf algebra of the envelopping algebra of a Lie algebra $\underline{G}$ whose basis is labelled by the one particle irreducible Feynman graphs. The Lie bracket of two such graphs is computed from insertions of one graph in the other and vice versa. The corresponding Lie group G is the group of characters of $\mathcal{H}$.

The next breakthrough came from our joint discovery [46] that identical formulas to equations (12, 13, 14) occur in the solution of the Riemann Hilbert problem for an arbitrary pronilpotent Lie group G!
This really unveils the true nature of this seemingly complicated combinatorics and shows that it is a special case of a general extraction of finite values based on the Riemann-Hilbert problem.
The Riemann-Hilbert problem comes from Hilbert's 21^{st} problem which he formulated as follows:

> "Prove that there always exists a Fuchsian linear differential equation with given singularities and given monodromy."

In this form it admits a positive answer due to Plemelj and Birkhoff (cf. [47] for a careful exposition). When formulated in terms of linear systems of the form,

$$y'(z) = A(z)\, y(z) \ , \ A(z) = \sum_{\alpha \in S} \frac{A_\alpha}{z - \alpha} \, , \tag{16}$$

(where S is the given finite set of singularities, $\infty \notin S$, the A_α are complex matrices such that

$$\sum A_\alpha = 0 \tag{17}$$

to avoid singularities at ∞), the answer is not always positive [48], but the solution exists when the monodromy matrices M_α are sufficiently close to 1. It can then be explicitly written as a series of polylogarithms [47].
Another formulation of the Riemann-Hilbert problem, intimately tied up to the classification of holomorphic vector bundles on the Riemann sphere $P_1(\mathbb{C})$, is in terms of the Birkhoff decomposition

$$\gamma\,(z) = \gamma_-(z)^{-1}\, \gamma_+(z) \qquad z \in C \tag{18}$$

where we let $C \subset P_1(\mathbb{C})$ be a smooth simple curve, C_- the component of the complement of C containing $\infty \notin C$ and C_+ the other component. Both γ and $\gamma_\pm$ are loops with values in $\mathrm{GL}_n(\mathbb{C})$,

$$\gamma\,(z) \in G = \mathrm{GL}_n(\mathbb{C}) \qquad \forall\, z \in \mathbb{C} \tag{19}$$

and $\gamma_\pm$ are boundary values of holomorphic maps (still denoted by the same symbol)

$$\gamma_\pm : C_\pm \to \mathrm{GL}_n(\mathbb{C})\,. \tag{20}$$

The normalization condition $\gamma_-(\infty) = 1$ ensures that, if it exists, the decomposition (18) is unique (under suitable regularity conditions).

The existence of the Birkhoff decomposition (18) is equivalent to the vanishing,

$$c_1\,(L_j) = 0 \tag{21}$$

of the Chern numbers $n_j = c_1\,(L_j)$ of the holomorphic line bundles of the Birkhoff-Grothendieck decomposition,

$$E = \oplus\, L_j \tag{22}$$

where E is the holomorphic vector bundle on $P_1(\mathbb{C})$ associated to γ, i.e. with total space:

$$(C_+ \times \mathbb{C}^n) \cup_\gamma (C_- \times \mathbb{C}^n)\,. \tag{23}$$

The above discussion for $G = \mathrm{GL}_n(\mathbb{C})$ extends to arbitrary complex Lie groups.

When G is a simply connected nilpotent complex Lie group the existence (and uniqueness) of the Birkhoff decomposition (18) is valid for any γ. When the loop $\gamma : C \to G$ extends to a holomorphic loop: $C_+ \to G$, the Birkhoff decomposition is given by $\gamma_+ = \gamma$, $\gamma_- = 1$. In general, for $z_0 \in C_+$ the evaluation,

$$\gamma \to \gamma_+(z_0) \in G \tag{24}$$

is a natural principle to extract a finite value from the singular expression $\gamma(z_0)$. This extraction of finite values is a multiplicative removal of the pole part for a meromorphic loop γ when we let C be an infinitesimal circle centered at z_0.

We are now ready to apply this procedure in Quantum Field Theory. First, using dimensional regularization, the bare (unrenormalized) theory gives rise to a meromorphic loop,

$$\gamma(z) \in G\,, \qquad z \in \mathbb{C} \tag{25}$$

Our main result [45, 46] is that the renormalized theory is just the evaluation at the integer dimension $z_0 = D$ of space-time of the holomorphic part γ_+ of the Birkhoff decomposition of γ.

In fact, the original loop $d \to \gamma(d)$ not only depends upon the parameters of the theory but also on the additional "unit of mass" μ required by dimensional analysis. We showed in [49] that the mathematical concepts developped in our earlier papers provide very powerful tools to lift the usual concepts of the β-function and renormalization group from the space of coupling constants of the theory to the complex Lie group G.

We first observed that even though the loop $\gamma(d)$ does depend on the additional parameter μ,

$$\mu \to \gamma_\mu(d)\,, \tag{26}$$

the negative part γ_{μ^-} in the Birkhoff decomposition,

$$\gamma_\mu(d) = \gamma_{\mu^-}(d)^{-1}\,\gamma_{\mu^+}(d) \tag{27}$$

is actually independent of μ,

$$\frac{\partial}{\partial\mu}\,\gamma_{\mu^-}(d) = 0\,. \tag{28}$$

This is a restatement of a well known fact and follows immediately from dimensional analysis. Moreover, by construction, the Lie group G turns out to be graded, with grading,

$$\theta_t \in \mathrm{Aut}\,G\,, \quad t \in \mathbb{R}\,, \tag{29}$$

inherited from the grading of the Hopf algebra $\mathcal{H}$ of Feynman graphs given by the loop number,

$$L(\Gamma) = \text{loop number of } \Gamma \tag{30}$$

for any 1PI graph Γ.
The straightforward equality,

$$\gamma_{e^t \mu}(d) = \theta_{t\varepsilon}(\gamma_\mu(d)) \qquad \forall\, t \in \mathbb{R}\,,\ \varepsilon = D - d \tag{31}$$

shows that the loops γ_μ associated to the unrenormalized theory satisfy the striking property that the negative part of their Birkhoff decomposition is unaltered by the operation,

$$\gamma(\varepsilon) \to \theta_{t\varepsilon}(\gamma(\varepsilon))\,, \tag{32}$$

In other words, if we replace $\gamma(\varepsilon)$ by $\theta_{t\varepsilon}(\gamma(\varepsilon))$ we dont change the negative part of its Birkhoff decomposition. We settled now for the variable,

$$\varepsilon = D - d \in \mathbb{C}\backslash\{0\}\,. \tag{33}$$

We give in [49] a complete characterization of the loops $\gamma(\varepsilon) \in G$ fulfilling the above striking invariance. This characterization only involves the negative part $\gamma_-(\varepsilon)$ of their Birkhoff decomposition which by hypothesis fulfills,

$$\gamma_-(\varepsilon)\,\theta_{t\varepsilon}(\gamma_-(\varepsilon)^{-1}) \text{ is convergent for } \varepsilon \to 0\,. \tag{34}$$

It is easy to see that the limit of (34) for $\varepsilon \to 0$ defines a one parameter subgroup,

$$F_t \in G\,,\ t \in \mathbb{R} \tag{35}$$

and that the generator $\beta = \left(\frac{\partial}{\partial t} F_t\right)_{t=0}$ of this one parameter group is related to the *residue* of γ

$$\operatorname*{Res}_{\varepsilon=0} \gamma = -\left(\frac{\partial}{\partial u}\gamma_-\left(\frac{1}{u}\right)\right)_{u=0} \tag{36}$$

by the simple equation,

$$\beta = Y \operatorname{Res}\gamma\,, \tag{37}$$

where $Y = \left(\frac{\partial}{\partial t}\theta_t\right)_{t=0}$ is the grading.
This is straightforward but our result is the following formula (39) which gives $\gamma_-(\varepsilon)$ in closed form as a function of β. We shall for convenience introduce an additional generator in the Lie algebra of G (i.e. primitive elements of $\mathcal{H}^*$) such that,

$$[Z_0, X] = Y(X) \qquad \forall\, X \in \text{Lie}\, G\,. \tag{38}$$

The scattering formula for $\gamma_-(\varepsilon)$ is then,

$$\gamma_-(\varepsilon) = \lim_{t\to\infty} e^{-t\left(\frac{\beta}{\varepsilon}+Z_0\right)}\, e^{tZ_0}\,. \tag{39}$$

Both factors in the right hand side belong to the semi direct product,

$$\widetilde{G} = G \rtimes_\theta \mathbb{R} \tag{40}$$

of the group G by the grading, but of course the ratio (39) belongs to the group G.

This shows ([49]) that the higher pole structure of the divergences is uniquely determined by the residue and gives a strong form of the t'Hooft relations, which will come as an immediate corollary.
The main new result of [49], specializing to the massless case and taking φ_6^3 as an illustrative example to fix ideas and notations, is that the formula for the bare coupling constant,

$$g_0 = g\, Z_1\, Z_3^{-3/2} \tag{41}$$

where both $g\, Z_1 = g + \delta g$ and the field strength renormalization constant Z_3 are thought of as power series (in g) of elements of the Hopf algebra $\mathcal{H}$, does define a Hopf algebra homomorphism,

$$\mathcal{H}_{CM} \xrightarrow{g_0} \mathcal{H}_K\,, \tag{42}$$

from the Hopf algebra $\mathcal{H}_{CM}$ of coordinates on the group of formal diffeomorphisms of $\mathbb{C}$ such that,

$$\varphi(0) = 0\,,\ \varphi'(0) = \mathrm{id} \tag{43}$$

to the Hopf algebra $\mathcal{H}_K$ of the massless theory. We had already constructed in [46] a Hopf algebra homomorphism from $\mathcal{H}_{\rm CM}$ to the Hopf algebra of rooted trees, but the physical significance of this construction was unclear.
The homomorphism (42) is quite different in that for instance the transposed group homomorphism,

$$G \xrightarrow{\rho} \mathrm{Diff}(\mathbb{C}) \tag{44}$$

lands in the subgroup of *odd* diffeomorphisms,

$$\varphi(-z) = -\varphi(z) \qquad \forall\, z\,. \tag{45}$$

Moreover its physical significance is transparent. In particular the image by ρ of $\beta = Y\,\mathrm{Res}\,\gamma$ is the usual β-function of the coupling constant g.
We discovered the homomorphism (42) by lengthy concrete computations which were an excellent test for the explicit ways of handling the coproduct, coassociativity, symmetry factors... that underly the theory.
As a corollary of the construction of ρ we get an *action* by (formal) diffeomorphisms of the group G on the space X of (dimensionless) coupling constants of the theory. We can then in particular formulate the Birkhoff decomposition *directly* in the group,

$$\mathrm{Diff}\,(X) \tag{46}$$

of formal diffeomorphisms of the space of coupling constants.

Theorem 10. ([49]) *Let the unrenormalized effective coupling constant $g_{\rm eff}(\varepsilon)$ be viewed as a formal power series in g and let $g_{\rm eff}(\varepsilon) = g_{{\rm eff}_+}(\varepsilon)\,(g_{{\rm eff}_-}(\varepsilon))^{-1}$ be its (opposite) Birkhoff decomposition in the group of formal diffeomorphisms. Then the loop $g_{{\rm eff}_-}(\varepsilon)$ is the bare coupling constant and $g_{{\rm eff}_+}(0)$ is the renormalized effective coupling.*

This result is now, in its statement, no longer depending upon our group G or the Hopf algebra $\mathcal{H}$. But of course the proof makes heavy use of the above ingredients. It is a challenge to physicists to find a direct proof of this result.
Finally the Birkhoff decomposition of a loop,

$$\delta(\varepsilon) \in \mathrm{Diff}\,(X)\,, \tag{47}$$

admits a beautiful geometric interpretation. If we let X be a complex manifold and pass from formal diffeomorphisms to actual ones, the data (47) is the initial data to perform, by the clutching operation, the construction of a complex bundle,

$$P = (S^+ \times X) \cup_\delta (S^- \times X) \tag{48}$$

over the sphere $S = P_1(\mathbb{C}) = S^+ \cup S^-$, and with fiber X,

$$X \longrightarrow P \xrightarrow{\pi} S \,. \tag{49}$$

The meaning of the Birkhoff decomposition,

$$\delta(\varepsilon) = \delta_-(\varepsilon)^{-1}\, \delta_+(\varepsilon) \tag{50}$$

is then exactly captured by an isomorphism of the bundle P with the trivial bundle,

$$S \times X \,. \tag{51}$$

XVII Number theory

I shall conclude these notes by giving a brief glimpse at the connection between noncommutative geometry and number theory. There are two points of contact of the two subjects, the first gives a spectral interpretation of zeros of zeta and L-functions in terms of a construction involving adeles, more specifically the noncommutative space of adele classes. The second has to do with the missing Galois theory at Archimedean places. For the specialists of quantum chaos looking for a spectral realization of the non-trivial zeros of the Riemann zeta function from the quantization of classical mechanical systems, the adeles might look rather exotic at first sight and we first need to explain briefly (for non specialists) why Ideles and Adeles are natural and important in number theory.

Let us start with the reciprocity law (Gauss 1801)

$$\left(\frac{\ell}{p}\right) = (-1)^{\varepsilon(p)\varepsilon(\ell)} \left(\frac{p}{\ell}\right) \,, \qquad \varepsilon(p) = \frac{p-1}{2} \pmod 2 \tag{1}$$

where ℓ and p are odd primes and $\left(\frac{\ell}{p}\right)$ is the Legendre symbol whose value is $+1$ if the equation

$$x^2 = \ell \qquad (\mathrm{mod}\, p) \tag{2}$$

admits a solution, and is -1 if it does not.

For instance, with $\ell = 5$ we see that whether the equation $x^2 = 5 (\mathrm{mod}\, p)$ admits a solution only depends upon $p (\mathrm{mod}\, 5)$, i.e. only on the last digit of p. Thus the answer is the same for $p = 7$ and $p = 1997$ or for $p = 19$ and $p = 1999$. It follows that the primes p thus fall into *classes*. The language of Adeles and Ideles extends this simple notion of *classes* of primes to those of *ideal classes* and then of *Idele classes*.

To the proof of Dirichlet of the existence of infinitely many primes in an arithmetic progression corresponds the construction of an L-function associated to a character χ modulo m,

$$L(s,\chi) = \prod \frac{1}{1-\chi(p)\,p^{-s}}\,. \tag{3}$$

More generally a Hecke L-function is associated to a character of the ideal class group modulo m and in fact also to a Grössencharakter which is a character of the Idele class group of a number field k.

The quickest way to introduce the Idele class group of a number field k is to understand (Cf. Iwasawa *Ann. of Math.* **57** (1953)) that such a field sits uniquely as a discrete cocompact subfield of a unique locally compact (semi-simple and non discrete) ring A

$$k \subset A\,, \qquad k \ \text{cocompact} \tag{4}$$

called the ring of *Adeles* of k. One then has,

$$\text{Idele class group of } k = \mathrm{GL}_1(k)\backslash \mathrm{GL}_1(A)\,, \tag{5}$$

and a Grössencharakter is a character of this locally compact group. Iwasawa and Tate showed how to use analysis on adeles to prove the basic properties of the Hecke L-functions which were then extended to L-functions associated to automorphic forms which appear in the action of $\mathrm{GL}_n(A)$ on the Hilbert space

$$L^2(\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(A))\,. \tag{6}$$

To understand the other language involved in the basic dictionary which underlies the Langlands program let us go back to the equation (2) say with $\ell = 5$ and simply adjoin $\sqrt{5}$ to the field $\mathbb{Q}$ of rational numbers which gives an algebraic extension $K = \mathbb{Q}(\sqrt{5})$ of $k = \mathbb{Q}$. The Galois group $\mathrm{Gal}(\mathbb{Q}(\sqrt{5}) : \mathbb{Q}) = \mathrm{Gal}(K : k)$ is of course $\mathbb{Z}/2$ in this case and admits an obvious non trivial one dimensional representation π. In general, the *Artin* L-function associated to a representation

$$\mathrm{Gal}(K : k) \to \mathrm{GL}(n, \mathbb{C}) \tag{7}$$

of the Galois group of a finite Galois extension K of k, is

$$L(s,\pi) = \prod_p L_p(s,\pi) \tag{8}$$

where p runs through the prime ideals of k and the local L factor is given at unramified p by,

$$L_p(s,\pi) = \det(1-\pi(\sigma)\,N(p)^{-s}) \tag{9}$$

where σ is the Frobenius automorphism of p.

When K/k is an abelian extension and π a one dimensional representation it follows from class field theory that π defines a character modulo the conductor of K/k and that the Artin L-function equals the Hecke L-function. This Artin reciprocity law is a far reaching extension of the Gauss reciprocity law (1).

The Langlands program extends Hecke's theory of Euler products associated to automorphic forms on GL(2) to arbitrary reductive groups G and gives a correspondance, extending Artin's reciprocity law to the non-Abelian case, between automorphic representations of G and representations,

$$\mathrm{Gal}(K : k) \to {}^L G \tag{10}$$

in the Langlands dual ${}^LG$ of G.

A basic tool of the theory is the trace formula [76] which extends to the adelic context the Selberg trace formula. The trace formula is the equality obtained by computing in two different ways the trace of operators of the form,

$$\mathrm{Trace}\,(C_\Lambda\,\pi(f)) \tag{11}$$

where (for $G = \mathrm{GL}_n$), π is the natural representation of $\mathrm{GL}_n(A)$ in the Hilbert space

$$L^2_\chi(\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(A)) \tag{12}$$

where χ is a Grössencharakter, and C_Λ is a cutoff. The Grössencharakter χ allows one to restrict to vectors with a fixed behaviour relative to GL_1.

The spectral side of the trace formula is obtained from the harmonic analysis of the representation π. The geometric side expresses the trace as a sum of orbital integrals.

The restriction imposed in (12) by the Grössencharakter χ shows that the case $n = 1$ becomes trivial and concentrates essentially on the SL_n aspect for $n \geq 2$. So far the zeros of L-functions do not appear in this language.

Our contribution to this subject is to show that both the zeros of L-functions and the Riemann-Weil explicit formulas appear directly in a refinement of the trace formula obtained as follows. Instead of restricting the Hilbert space,

$$L^2(\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(A)) \tag{13}$$

by the choice of Grössencharakter χ as above, one introduces on the full Hilbert space (13) a finer cutoff operator Q_Λ taking care of the "GL_1" behaviour of vectors.

To understand in which way the corresponding trace formula refines the Arthur trace formula, it is simplest to restrict to the case of GL_1. In order to simplify even further we shall replace the number field k by a function field of positive characteristic. This allows for a straightforward definition of the cutoff operators Q_Λ as the orthogonal projection on the subspace,

$$Q_\Lambda \subset L^2(\mathrm{GL}_1(k)\backslash \mathrm{GL}_1(A)) \tag{14}$$

spanned by the vectors $\xi \in \mathcal{S}(A)$ (averaged on $\mathrm{GL}_1(k)$) which vanish as well as their Fourier transform for $|x| > \Lambda$. Note that we use Fourier transform on the *additive* group of adeles so that the space $\mathrm{GL}_1(k)\backslash A$ of Adele classes is implicit in this construction. To define this Fourier transformation we needed to choose a basic character $\alpha = \prod \alpha_v$ of the additive group A for which the lattice k is selfdual.

The spectral computation of the trace of $Q_\Lambda\,\pi(f)$ involves all the nontrivial zeros of Hecke L-functions and is given by the following formula ([73]),

$$\mathrm{Trace}(Q_\Lambda\,\pi(f)) = 2\left(\sum_{\mathrm{GL}_1} f(k)\right)\log'\Lambda$$

$$+\widehat{f}(0)+\widehat{f}(1) - \sum_{\substack{L\left(\chi,\frac{1}{2}+\rho\right)=0\\ \rho\in B/N^\perp}} N\left(\chi,\frac{1}{2}+\rho\right)\int_{i\mathbb{R}} \widehat{f}(\chi,z)\,d\mu_\rho(z) + o(1) \tag{15}$$

where B is the open strip $B = \{\rho \in \mathbb{C}\,; |\mathrm{Re}\,\rho| < \frac{1}{2}\}$, $N\left(\chi, \frac{1}{2}+\rho\right)$ is the multiplicity of $\frac{1}{2}+\rho$ as a zero of the L function $L(\chi, s)$, χ varying through Grössencharakters (modulo principal ones), N being the module,

$$N = \mathrm{Mod}(k)\,. \tag{16}$$

Also $2\log'\Lambda = \int_{|\lambda|\in[\Lambda^{-1},\Lambda]} d^*\lambda$, and the measure $d\mu_\rho(z)$ is the harmonic measure of $\rho \in \mathbb{C}$ with respect to the line $i\mathbb{R}$. In particular if the zero $\frac{1}{2}+\rho$ is on the critical line $d\mu_\rho(z)$ is just the Dirac mass at $z = \rho$. Finally the Fourier transform of f is given by,

$$\widehat{f}(\chi, z) = \int_{\mathrm{GL}_1(A)} f(u^{-1})\,\chi(u)\,|u|^z\,d^*u\,. \tag{17}$$

The geometric side of the trace formula has so far only be fully justified in the simplified situation where only finitely many places are used. It is then given by the following formula ([73])

$$\mathrm{Trace}(Q_\Lambda\,\pi(f)) = 2\left(\sum_{\mathrm{GL}_1} f(k)\right)\log'\Lambda + \sum_{v,k}\int'_{k_v^*}\frac{f(ku)}{|1-u|}\,d^*u + o(1)\,; \tag{18}$$

where each k_v^* is embedded in $(\mathrm{GL}_1(k)\backslash\mathrm{GL}_1(A))$ by the map $u \to (1, 1, ..., u, ..., 1)$ and the principal value $\int'$ is uniquely determined by the pairing with the unique distribution on k_v which agrees with $\frac{du}{|1-u|}$ for $u \neq 1$ and whose Fourier transform relative to α_v vanishes at 1.
By proving that it entails the positivity of the Weil distribution, we showed in [73] that the validity of the geometric side, i.e., the global trace formula, is equivalent to the Riemann Hypothesis for all L-functions with Grössencharakter.

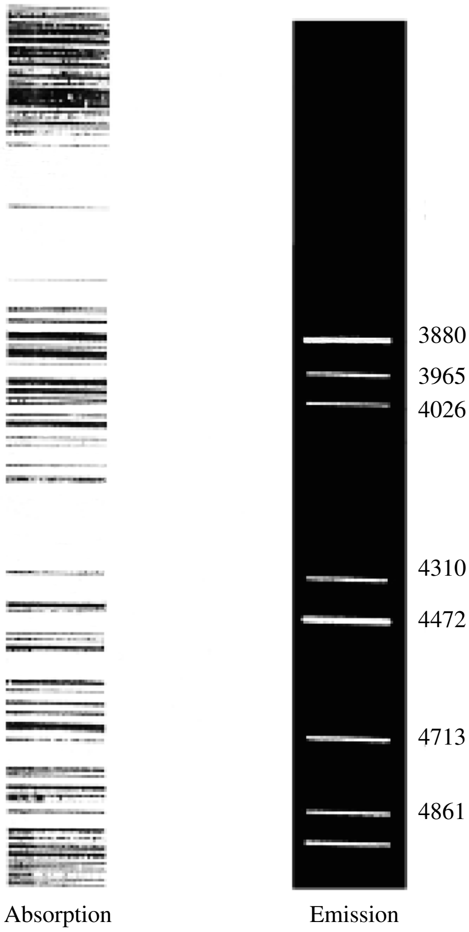
Theorem 11. *The following two conditions are equivalent:*

a) When $\Lambda \to \infty$, one has, for all $f \in \mathcal{S}(\mathrm{GL}_1(k)\backslash\mathrm{GL}_1(A))$ with compact support,

$$\mathrm{Trace}(Q_\Lambda\,\pi(f)) = 2\left(\sum_{\mathrm{GL}_1} f(k)\right)\log'\Lambda + \sum_{v,k}\int'_{k_v^*}\frac{f(ku)}{|1-u|}\,d^*u + o(1)\,; \tag{19}$$

b) All L functions with Grössencharakter on k satisfy the Riemann Hypothesis.

We have thus obtained a spectral interpretation of the zeros of zeta and L-functions as an absorption spectrum, i.e., as missing spectral lines. All zeros do play a role in the spectral side of the trace formula, but while the critical zeros do appear perse, the noncritical ones appear as resonances and enter in the trace formula through their harmonic potential with respect to the critical line. The spectral side is entirely canonical, and its validity is justified in the global case [73]. It is quite important to understand why a crucial negative sign in the analysis of the statistical fluctuations of the zeros of zeta indicated from the start that the spectral interpretation should be as an absorption spectrum, or equivalently should be of a cohomological nature.



The number of zeros of zeta whose imaginary part is less than $E > 0$,

$$N(E) = \# \text{ of zeros } \rho \ , \ 0 < \operatorname{Im} \rho < E \tag{20}$$

has an asymptotic expression ([26]) given by

$$N(E) = \frac{E}{2\pi} \left(\log \left(\frac{E}{2\pi} \right) - 1 \right) + \frac{7}{8} + o(1) + N_{\rm osc}(E) \tag{21}$$

where the oscillatory part of this step function is

$$N_{\rm osc}(E) = \frac{1}{\pi} \operatorname{Im} \log \zeta \left(\frac{1}{2} + iE \right) \tag{22}$$

which is of the order of $\operatorname{Log}(E)$ (We assume that E is not the imaginary part of a zero and take for the logarithm the branch which is 0 at $+\infty$). The Euler product formula for the zeta function yields (cf. [72]) a heuristic asymptotic formula for $N_{\rm osc}(E)$,

$$N_{\rm osc}(E) \simeq \frac{-1}{\pi} \sum_p \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{p^{m/2}} \sin (m \, E \, \log p) \, . \tag{23}$$

One can compare this formula with what appears in the direct atempt [72] to construct a spectral realization of zeros of zeta from quantization of a classical dynamical system.

In this theory the quantization of the classical dynamical system given by the phase space X and hamiltonian h gives rise to a Hilbert space $\mathcal{H}$ and a selfadjoint operator H whose spectrum is the essential physical observable of the system. For complicated systems the only useful information about this spectrum is that, while the average part of the counting function,

$$N(E) = \#\text{ eigenvalues of } H \text{ in } [0,E] \tag{24}$$

is computed by a semiclassical approximation mainly as a volume in phase space, the oscillatory part,

$$N_{\rm osc}(E) = N(E) - \langle N(E) \rangle \tag{25}$$

is the same as for a random matrix, governed by the statistic dictated by the symmetries of the system.

One can then ([72]) write down an asymptotic semiclassical approximation to the oscillatory function $N_{\rm osc}(E)$

$$N_{\rm osc}(E) = \frac{1}{\pi}\,\mathrm{Im}\int_0^\infty \mathrm{Trace}(H-(E+i\eta))^{-1}\,id\eta \tag{26}$$

using the stationary phase approximation of the corresponding functional integral. For a system whose configuration space is 2-dimensional, this gives ([72] (15)),

$$N_{\rm osc}(E) \simeq \frac{1}{\pi}\sum_{\gamma_p}\sum_{m=1}^{\infty}\frac{1}{m}\,\frac{1}{2\mathrm{sh}\left(\frac{m\lambda_p}{2}\right)}\,\sin(S_{\rm pm}(E)) \tag{27}$$

where the γ_p are the primitive periodic orbits, the label m corresponds to the number of traversals of this orbit, while the corresponding instability exponents are $\pm\lambda_p$. The phase $S_{\rm pm}(E)$ is up to a constant equal to $m\,E\,T_\gamma^\#$ where $T_\gamma^\#$ is the period of the primitive orbit γ_p.

Comparing the formulas one sees a fundamental mismatch (cf.[72]) which is the overall *minus sign* in front of formula (23) as opposed to the plus sign of (27). This problem is resolved in our spectral interpretation by the minus sign present in the spectral side of the trace formula (15). The point is that the spectral analysis of the action of the Idele class group on the Adele class space shows ([73]) white light with dark absorption lines labelled by the zeros of zeta and L-functions. This also provides the correct explanation for the asymptotic form of the formula for the average number of zeros

$$\langle N(E)\rangle \sim (E/2\pi)(\log(E/2\pi)-1) + 7/8 + o(1) \tag{28}$$

from a semiclassical computation for the number of quantum mechanical states in one degree of freedom which fulfill the conditions

$$|q| \le \Lambda,\, |p| \le \Lambda,\, |H| \le E\,, \tag{29}$$

where $H = 2\pi qp$ is the Hamiltonian which generates the group involved in the action of the Idele class group namely the scaling transformations (see ([73]) for precise normalization). We are thus computing the area of,

$$D = \{(p,q);\, pq \ge 0, |q| \le \Lambda, |p| \le \Lambda, |pq| \le E/2\pi\}. \tag{30}$$

(since we deal with zeta alone we restrict ourselves to even functions so that we exclude the region $pq \leq 0$ of the semiclassical (p,q) plane). The computation yields

$$1/2 \int_D dpdq = \frac{2E}{2\pi} \log \Lambda - \frac{E}{2\pi} \left(\log \frac{E}{2\pi} - 1 \right) . \tag{31}$$

In this formula we see in the right hand side the overall term $\langle N(E) \rangle$ which appears with a *minus* sign. This shows that the number of quantum mechanical states is equal to $\frac{4E}{2\pi} \log \Lambda$ minus the first approximation to the number of zeros of zeta whose imaginary part is less than E in absolute value (one just multiplies by 2 the equality (31)). Now $\frac{1}{2\pi}(2E)(2\log\Lambda)$ is the number of quantum states in the Hilbert space $L^2(\mathbb{R}_+^*, d^*x)$ which are localized in $\mathbb{R}_+^*$ between Λ^{-1} and Λ and are localized in the dual group $\mathbb{R}$ (for the pairing $\langle \lambda, t \rangle = \lambda^{it}$) between $-E$ and E. Thus we see clearly that the first approximation to $N(E)$ appears as the lack of surjectivity of the map which associates to quantum states ξ with support in D the function on $\mathbb{R}_+^*$,

$$E(\xi)(x) = |x|^{1/2} \sum_{n \in \mathbb{Z}} \xi(nx) \tag{32}$$

where we assume the additional conditions $\xi(0) = \int \xi(x)dx = 0$.

A finer analysis, which is just what the trace formula is doing, would yield the additional terms $7/8 + o(1) + N_{osc}(E)$. The above discussion yields an explicit construction of a large matrix whose spectrum approaches the zeros of zeta as $\Lambda \to \infty$.

While the above discussion clearly gives the sought for spectral interpretation of zeros of zeta it is unclear that one can expect to justify the (geometric side of) trace formula without a deeper understanding of the symmetries of the situation, which might well involve quantum groups.

As we mentionned earlier, the second point of contact between noncommutative geometry and number theory has to do with the missing Galois theory at Archimedean places.

Let k be a *global* field, when the characteristic of k is $p > 1$ so that k is a function field over $\mathbb{F}_q$, one has

$$k \subset k_{\rm un} \subset k_{\rm ab} \subset k_{\rm sep} \subset \overline{k} \,,$$

where $\overline{k}$ is an algebraic closure of k, $k_{\rm sep}$ the separable algebraic closure, $k_{\rm ab}$ the maximal abelian extension and $k_{\rm un}$ is obtained by adjoining to k all roots of unity of order prime to p.

One defines the Weil group W_k as the subgroup of ${\rm Gal}(k_{\rm ab} : k)$ of those automorphisms which induce on $k_{\rm un}$ an integral power of the Frobenius automorphism σ,

$$\sigma(\mu) = \mu^q \qquad \forall \, \mu \text{ root of 1 of order prime to } p \, .$$

The main theorem of global class field theory asserts the existence of a canonical isomorphism,

$$W_k \simeq C_k = GL_1(A)/GL_1(k) \,,$$

of locally compact groups.

When k is of characteristic 0, i.e. is a number field, one has a canonical isomorphism,

$$\mathrm{Gal}(k_{\rm ab} : k) \simeq C_k / D_k \,,$$

where D_k is the connected component of identity in the Idele class group C_k, but because of the Archimedean places of k there is no interpretation of C_k analogous to the Galois group interpretation for function fields. According to A. Weil [77], "La recherche d'une interprétation pour C_k si k est un corps de nombres, analogue en quelque manière à l'interprétation par un groupe de Galois quand k est un corps de fonctions, me semble constituer l'un des problèmes fondamentaux de la théorie des nombres à l'heure actuelle ; il se peut qu'une telle interprétation renferme la clef de l'hypothèse de Riemann ...".

Galois groups are by construction projective limits of the finite groups attached to finite extensions. To get connected groups one clearly needs to relax this finiteness condition which is the same as the finite dimensionality of the central simple algebras of the Brauer theory. Since Archimedean places of k are responsible for the non triviality of D_k it is natural to ask the following preliminary question,

"Is there a non trivial Brauer theory of central simple algebras over $\mathbb{C}$."

We showed in [3] that the *approximately finite dimensional* simple central algebras over $\mathbb{C}$ (called factors) provide a satisfactory answer to this question. They are classified by their module,

$$\mathrm{Mod}(M) \underset{\sim}{\subset} \mathbb{R}_+^*,$$

which is a virtual closed subgroup of $\mathbb{R}_+^*$.
One can in fact go much further and understand that the renormalization group, once properly formulated mathematically as we did in section XVI, really appears as a perfect ambiguity group between solutions to a (physics) problem. It hence plays a role very similar to that of the Galois group of an algebraic equation and is an ideal candidate for the missing Galois group at the Archimedean place. One can explore this idea further by making use of the relation between the (conjectural) Hopf algebra of Euler-Zagier numbers ([82], [83]) and the Kreimer Hopf algebra.

XVIII Appendix, the cyclic category

At the conceptual level, cyclic cohomology is a way to embed the nonadditive category of algebras and algebra homomorphisms in an additive category of modules. The latter is the additive category of Λ-modules where Λ is the cyclic category. Cyclic cohomology is then obtained as an *Ext* functor ([14]).
The cyclic category is a small category which can be defined by generators and relations. It has the same objects as the small category Δ of totally ordered finite sets and increasing maps which plays a key role in simplicial topology. Let us recall that Δ has one object $[n]$ for each integer n, and is generated by faces δ_i, $[n-1] \to [n]$ (the injection that misses i), and degeneracies σ_j, $[n+1] \to [n]$ (the surjection which identifies j with $j+1$), with the relations,

$$\delta_j \delta_i = \delta_i \delta_{j-1} \text{ for } i < j, \; \sigma_j \sigma_i = \sigma_i \sigma_{j+1} \qquad i \leq j \tag{1}$$

$$\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & i < j \\ 1_n & \text{if } i = j \text{ or } i = j+1 \\ \delta_{i-1} \sigma_j & i > j+1. \end{cases} \tag{2}$$

To obtain the cyclic category Λ one adds for each n a new morphism $\tau_n, [n] \to [n]$ such that,

$$\begin{aligned}
&\tau_n \delta_i = \delta_{i-1} \tau_{n-1} \quad 1 \le i \le n, \quad \tau_n \delta_0 = \delta_n \\
&\tau_n \sigma_i = \sigma_{i-1} \tau_{n+1} \quad 1 \le i \le n, \quad \tau_n \sigma_0 = \sigma_n \tau_{n+1}^2 \\
&\tau_n^{n+1} = 1_n .
\end{aligned} \tag{3}$$

The original definition of Λ (cf. [14]) used homotopy classes of non decreasing maps from S^1 to S^1 of degree 1, mapping $\mathbb{Z}/n$ to $\mathbb{Z}/m$ and is trivially equivalent to the above. Given an algebra A one obtains a module over the small category Λ by assigning to each integer $n \ge 0$ the vector space C^n of $n+1$-linear forms $\varphi(x^0, \ldots, x^n)$ on A, while the basic operations are given by

$$\begin{aligned}
(\delta_i \varphi)(x^0, \ldots, x^n) &= \varphi(x^0, \ldots, x^i x^{i+1}, \ldots, x^n), \quad i = 0, 1, \ldots, n-1 \\
(\delta_n \varphi)(x^0, \ldots, x^n) &= \varphi(x^n x^0, x^1, \ldots, x^{n-1}) \\
(\sigma_j \varphi)(x^0, \ldots, x^n) &= \varphi(x^0, \ldots, x^j, 1, x^{j+1}, \ldots, x^n), \quad j = 0, 1, \ldots, n \\
(\tau_n \varphi)(x^0, \ldots, x^n) &= \varphi(x^n, x^0, \ldots, x^{n-1}).
\end{aligned} \tag{4}$$

These operations satisfy the relations (1) (2) and (3). This shows that any algebra A gives rise canonically to a Λ-module and allows [14, 21] to interpret the cyclic cohomology groups $HC^n(A)$ as Ext^n functors. All of the general properties of cyclic cohomology such as the long exact sequence relating it to Hochschild cohomology are shared by Ext of general Λ-modules and can be attributed to the equality of the classifying space $B\Lambda$ of the small category Λ with the classifying space BS^1 of the compact one-dimensional Lie group S^1. One has

$$B\Lambda = BS^1 = P_\infty(\mathbb{C}) \tag{5}$$

XIX References

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