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NONCOMMUTATIVE FINITE-DIMENSIONAL MANIFOLDS

I. SPHERICAL MANIFOLDS AND RELATED EXAMPLES

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Abstract

We exhibit large classes of examples of noncommutative finite-dimensional manifolds which are (non-formal) deformations of classical manifolds. The main result of this paper is a complete description of noncommutative three-dimensional spherical manifolds, a noncommutative version of the sphere S^3 defined by basic K-theoretic equations. We find a 3-parameter family of deformations $S^3_{\mathbf{u}}$ of the standard 3-sphere S^3 and a corresponding 3-parameter deformation of the 4-dimensional Euclidean space $\mathbb{R}^4$. For generic values of the deformation parameters we show that the obtained algebras of polynomials on the deformed $\mathbb{R}^4_{\mathbf{u}}$ only depend on two parameters and are isomorphic to the algebras introduced by Sklyanin in connection with the Yang-Baxter equation. It follows that different $S^3_{\mathbf{u}}$ can span the same $\mathbb{R}^4_{\mathbf{u}}$. This equivalence generates a foliation of the parameter space Σ. This foliation admits singular leaves reduced to a point. These critical points are either isolated or fall in two 1-parameter families $C_\pm \subset \Sigma$. Up to the simple operation of taking the fixed algebra by an involution, these two families are identical and we concentrate here on C_+. For $u \in C_+$ the above isomorphism with the Sklyanin algebra breaks down and the corresponding algebras are special cases of θ-deformations, a notion which we generalize in any dimension and various contexts, and study in some details. Here, and this point is crucial, the dimension is not an artifact, i.e. the dimension of the classical model, but is the Hochschild dimension of the corresponding algebra which remains constant during the deformation. Besides the standard noncommutative tori, examples of θ-deformations include the recently defined noncommutative 4-sphere S^4_θ as well as m-dimensional generalizations, noncommutative versions of spaces $\mathbb{R}^m$ and quantum groups which are deformations of various classical groups. We develop general tools such as the twisting of the Clifford algebras in order to exhibit the spherical property of the hermitian projections corresponding to the noncommutative $2n$-dimensional spherical manifolds S^{2n}_θ. A key result is the differential self-duality properties of these projections which generalize the self-duality of the round instanton.

1 Introduction

Our aim in this paper is to describe large classes of tractable concrete examples of *noncommutative manifolds*. Our original motivation is the problem of classification of *spherical* noncommutative manifolds which arose from the basic

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discussion of Poincaré duality in K-homology [16], [18].

The algebra $\mathcal{A}$ of functions on a spherical noncommutative manifold S of dimension n is generated by the matrix components of a cycle x of the K theory of $\mathcal{A}$, whose dimension is the same as $n = \dim(S)$.
More specifically, for n even, $n = 2m$, the algebra $\mathcal{A}$ is generated by the matrix elements e^i_j of a self-adjoint idempotent

$$e = [e^i_j] \in M_q(\mathcal{A}), \ \ e = e^2 = e^*, \tag{1.1}$$

and one assumes that all the components $\mathrm{ch}_k(e)$ of the Chern character of e in cyclic homology satisfy,

$$\mathrm{ch}_k(e) = 0 \ \forall k = 0, 1, \ldots, m-1 \tag{1.2}$$

while $\mathrm{ch}_m(e)$ defines a non zero Hochschild cycle playing the role of the volume form of S.
For n odd the algebra $\mathcal{A}$ is similarly generated by the matrix components U^i_j of a unitary

$$U = [U^i_j] \in M_q(\mathcal{A}), \ \ UU^* = U^*U = 1 \tag{1.3}$$

and, with $n = 2m+1$, the vanishing condition (1.2) becomes

$$\mathrm{ch}_{k+\frac{1}{2}}(U) = 0 \ \ \forall k = 0, 1, \ldots, m-1. \tag{1.4}$$

The components ch_k of the Chern character in cyclic homology are the following explicit elements of the tensor product

$$\mathcal{A} \otimes (\tilde{\mathcal{A}})^{\otimes 2k} \tag{1.5}$$

where $\tilde{\mathcal{A}}$ is the quotient of $\mathcal{A}$ by the subspace $\mathbb{C}1$,

$$\mathrm{ch}_k(e) = \left(e^{i_0}_{i_1} - \frac{1}{2}\delta^{i_0}_{i_1}\right) \otimes e^{i_1}_{i_2} \otimes e^{i_2}_{i_3} \otimes \cdots \otimes e^{i_{2k}}_{i_0} \tag{1.6}$$

and

$$\mathrm{ch}_{k+\frac{1}{2}}(U) = U^{i_0}_{i_1} \otimes U^{*i_1}_{i_2} \otimes U^{i_2}_{i_3} \otimes \cdots \otimes U^{*i_{2k+1}}_{i_0} - U^{*i_0}_{i_1} \otimes \cdots \otimes U^{i_{2k+1}}_{i_0} \tag{1.7}$$

up to an irrelevant normalization constant.

It was shown in [16] that the Bott generator on the classical spheres S^n give solutions to the above equations (1.2), (1.4) and in [18] that non trivial noncommutative solutions exist for $n = 3$, $q = 2$ and $n = 4$, $q = 4$.

In fact, as will be explained in our next paper (Part II), consistency with the suspension functor requires a coupling between the dimension n of S and q. Namely q must be the same for $n = 2m$ and $n = 2m+1$ whereas it must be doubled when going from $n = 2m - 1$ to $n = 2m$. This implies that for dimensions $n = 2m$ and $n = 2m + 1$, one has $q = 2^m q_0$ for some $q_0 \in \mathbb{N}$. Furthermore the normalization $q_0 = 1$ is induced by the identification of S^2 with one-dimensional projective space $P_1(\mathbb{C})$ (which means $q = 2$ for $n = 2$). We shall take this convention (i.e. $q = 2^m$ for $n = 2m$ and $n = 2m + 1$) in the following.

The main result of the present paper is the complete description of the noncommutative solutions for $n = 3$ ($q = 2$). We find a three-parameter family of deformations of the standard three-sphere S^3 and a corresponding 3-parameter deformation of the 4-dimensional Euclidean space $\mathbb{R}^4$. For generic values of the deformation parameters we show that the obtained algebras of polynomials on the deformed $\mathbb{R}^4_{\mathbf{u}}$ only depend on two parameters and are isomorphic to the algebras introduced by Sklyanin in connection with the Yang-Baxter equation. It follows that different $S^3_{\mathbf{u}}$ can span the same $\mathbb{R}^4_{\mathbf{u}}$. This equivalence relation generates a foliation of the parameter space Σ. This foliation admits singular leaves reduced to a point. These critical points are either isolated or fall in two 1-parameter families $C_\pm \subset \Sigma$. Up to the simple operation of taking the fixed algebra by an involution, these two families are identical and we concentrate here on C_+. For $u \in C_+$ the above isomorphism with the Sklyanin algebra breaks down and the corresponding algebras are special cases of θ-deformations. It gives rise to a one-parameter deformation $\mathbb{C}^2_\theta$ of $\mathbb{C}^2$ (identified with $\mathbb{R}^4$) which is well suited for simple higher dimensional generalizations (i.e. $\mathbb{C}^2$ replaced by $\mathbb{C}^n \simeq \mathbb{R}^{2n}$). We shall describe and analyse them in details to understand this particular critical case, while the general case (of generic values of the parameters) will be treated in Part II.

First we shall show that, unlike most deformations used to produce noncommutative spaces from classical ones, the above deformations do not alter the Hochschild dimension. The latter is the natural generalization of the notion of dimension to the noncommutative case and is the smallest integer m such that the Hochschild homology of $\mathcal{A}$ with values in a bimodule $\mathcal{M}$ vanishes for $k > m$ ($H^k(\mathcal{A}, \mathcal{M}) = 0\ \forall k > m$). Second we shall describe the natural notion of differential forms on the above noncommutative spaces and obtain the natural quantum groups of symmetries as "θ-deformations" of the classical groups $GL(m, \mathbb{R})$, $SL(m, \mathbb{R})$ and $GL(n, \mathbb{C})$.

The algebraic versions of differential forms on the above quantum groups turn out to be graded involutive differential Hopf algebras, which implies that the corresponding differential calculi are bicovariant in the sense of [57]. It is worth noticing here that conversely as shown in [5], a bicovariant differential calculus on a quantum group always comes from a graded differential Hopf algebra as above.

Finally we shall come back to the metric aspect of the construction which was the original motivation for the definition of spherical manifolds from the polynomial operator equation fulfilled by the Dirac operator.

We shall check in detail that θ-deformations of Riemannian spin geometries fulfill all axioms of noncommutative geometry, thus completing the path, in the special case of θ-deformations, from the crudest level of the algebra $C_{\rm alg}(S)$ of polynomial functions on S to the full-fledged structure of noncommutative geometry [15].

In the course of the paper it will be shown that the self-duality property of the round instanton on S^4 extends directly to the self-adjoint idempotent identifying S^4_θ as a noncommutative 4-dimensional spherical manifold and that, more generally, the self-adjoint idempotents corresponding to the noncommutative $2n$-dimensional spherical manifolds S^{2n}_θ defined below satisfy a differential self-duality property which is a direct extension of the one satisfied by their classical counterparts as explained in [22].

In conclusion the above examples appear as an interesting point of contact between various approaches to noncommutative geometry. The original motivation came from the operator equation of degree n fulfilled by the Dirac operator of an n-dimensional spin manifold [15]. The simplest equation "quantizing" the corresponding Hochschild cycle c, namely $c = \mathrm{ch}(e)$ ([16]) led to the definition of spherical manifolds. What we show here is that in the simplest non trivial case ($n = 3$, $q = 2$) the answer is intimately related to the Sklyanin algebras which play a basic role in noncommutative algebraic geometry.

Many algebras occuring in this paper are finitely generated and finitely presented. These algebras are viewed as algebras of polynomials on the corresponding noncommutative space S and we denote them by $C_{\rm alg}(S)$. With these notations $C_{\rm alg}(S)$ has to be distinguished from $C^\infty(S)$, the algebra of smooth functions on S obtained as a suitable completion of $C_{\rm alg}(S)$. Basic algebraic properties such as Hochschild dimension are not necessarily preserved under the transition from $C_{\rm alg}(S)$ to $C^\infty(S)$. The topology of S is specified by the C^* completion of $C^\infty(S)$.
The plan of the paper is the following. After this introduction, in section 2, we give a complete description of noncommutative spherical manifolds for the lowest non trivial dimension : Namely for dimension $n = 3$ and for $q = 2$. These form a 3-parameter family $S^3_{\mathbf{u}}$ of deformations of the standard 3-sphere as explained above and correspondingly one has a homogeneous version which is a three-parameter family $\mathbb{R}^4_{\mathbf{u}}$ of deformations of the standard 4-dimensional Euclidean space $\mathbb{R}^4$. We then consider their suspensions and show that the suspension $S^4_{\mathbf{u}}$ of $S^3_{\mathbf{u}}$ is a four-dimensional noncommutative spherical manifold (with $q = 4 = 2^2$). In Section 3, we show that for generic values of the parameters, the algebra $C_{\rm alg}(\mathbb{R}^4_{\mathbf{u}})$ of polynomial functions on the noncommutative $\mathbb{R}^4_{\mathbf{u}}$ reduces to a Sklyanin algebra [50], [51].These Sklyanin algebras have been intensively studied [43], [52], from the point of view of noncommutative algebraic geometry but we postpone their analysis to Part II of this paper. We

concentrate instead on the determination of the scaling foliation of the parameter space Σ for 3-dimensional spherical manifolds $S^3_{\mathbf{u}}$. Different $S^3_{\mathbf{u}}$ can span isomorphic 4-dimensional $\mathbb{R}^4_{\mathbf{u}}$ and we shall determine here the corresponding foliation of Σ using the geometric data associated [43] [1][2] to such algebras. This will allow us to isolate the critical points in the parameter space Σ and we devote the end of the paper to the study of the corresponding algebras. The simplest way to analyse them is to view them as a special case of a general procedure of θ-deformation. In Section 4 we define a noncommutative deformation $\mathbb{R}^{2n}_\theta$ of $\mathbb{R}^{2n}$ for $n \geq 2$ which is coherent with the identification $\mathbb{C}^n = \mathbb{R}^{2n}$ as real spaces and is also consequently a noncommutative deformation $\mathbb{C}^n_\theta$ of $\mathbb{C}^n$. For $n = 2$, $\mathbb{R}^4_\theta$ reduces to the above one-parameter family of deformations of $\mathbb{R}^4$ which is included in the multiparameter deformation $\mathbb{R}^4_{\mathbf{u}}$ of Section 2. We introduce in this section a deformation of the generators of the Clifford algebra of $\mathbb{R}^{2n}$ which will be very useful for the computations of Sections 5 and 12. In Section 5 we define noncommutative versions $\mathbb{R}^{2n+1}_\theta$, S^{2n}_θ and S^{2n-1}_θ of $\mathbb{R}^{2n+1}$, S^{2n} and S^{2n-1} for $n \geq 2$. For $n = 2$, S^{2n}_θ reduces to the noncommutative 4-sphere S^4_θ of [18] whereas S^{2n-1}_θ reduces to the one-parameter family S^3_θ of deformation of S^3 associated to the non-generic values of $\mathbf{u}$. We generalize the results of [18] on S^4_θ to S^{2n}_θ for arbitrary $n \geq 2$ and we describe their counterpart for the odd-dimensional cases S^{2n-1}_θ showing thereby that these S^m_θ ($m \geq 3$) are noncommutative spherical manifolds. Furthermore, it will be shown later (in Section 12) that the defining hermitian projections of S^{2n}_θ possess differential self-duality properties which generalize the ones of their classical counterpart (i.e. for S^{2n}) as explained in [22]. In Section 6, we define algebraic versions of differential forms on the above noncommutative spaces. These definitions, which are essentially unique, provide dense subalgebras of the canonical algebras of smooth differential forms defined in Sections 11, 12, 13 for these particular cases. These differential calculi are diagonal [28] which implies that they are quotients of the corresponding universal diagonal differential calculi [24]. In Section 7 we construct quantum groups which are deformations (called θ-deformations) of the classical groups $GL(m,\mathbb{R})$, $SL(m,\mathbb{R})$ and $GL(n,\mathbb{C})$ for $m \geq 4$ and $n \geq 2$. The point of view for this construction is close to the one of [37] which is itself a generalization of a construction described in [38], [39]. It is pointed out that there is no such θ-deformation of $SL(n,\mathbb{C})$ although there is a θ-deformation of the subgroup of $GL(n,\mathbb{C})$ consisting of matrices with determinants of modulus one ($|\det_{\mathbb{C}}(M)|^2 = 1$). In Section 8 we define the corresponding deformations of the groups $O(m)$, $SO(m)$ and $U(n)$. As above there is no θ-deformation of $SU(n)$ which is the counterpart of the same statement for $SL(n,\mathbb{C})$. All the quantum groups G_θ considered in Section 7 and in Section 8 are matrix quantum groups [56] and in fact as coalgebra $C_{\mathrm{alg}}(G_\theta)$ is undeformed i.e. isomorphic to the classical coalgebra $C_{\mathrm{alg}}(G)$ of representative functions on G [23], (only the associative product is deformed). In Section 9, we analyse the structure of the algebraic version of differential forms on the above quantum groups. These graded-involutive differential algebras turn out to be *graded-involutive differential Hopf algebras* (with coproducts and counits extending the original ones) which, in view of [5], means that the corresponding differential calculi are bicovariant in the sense of [57]. It is worth noticing that the above θ-deformations of $\mathbb{R}^m$, of the differential calculus on $\mathbb{R}^m$ and of some classical groups have been already considered for instance in [3]. The quantum group setting analysis of [3] is clearly very interesting: There, $\mathbb{R}^m_\theta$ appears (with other notations) as a quantum space on which some quantum group acts (or more precisely as a quantum homogeneous space) and the differential calculus on $\mathbb{R}^m_\theta$ is the covariant one. Another powerful approach to the above quantum group aspects is to make use of the notion of Drinfeld twist [20] since it is clear that the θ-deformed quantum group of Sections 7 and 8 can be obtained by twisting (see e.g. in [49] for a particular case); thus many results of Sections 7 to 9 can be obtained by using for instance Proposition 2.3.8 of [35], its graded counterpart and the result of [36] for the differential calculus in this case. Here the emphasis is rather different. The noncommutative $\mathbb{R}^m_\theta$ appears as a solution of the K-theoretic equations (1.2) or (1.4) appropriate to the dimension m and the differential calculus which is essentially unique is used to produce the projective resolution of $C^\infty(\mathbb{R}^m_\theta)$ which ensures that the Hochschild dimension of $C^\infty(\mathbb{R}^m_\theta)$ is m (i.e. that R^m_θ is m-dimensional). It turns out that the differential calculus on $\mathbb{R}^m_\theta$ is covariant for some quantum group actions and that these quantum groups are again θ-deformations. However, our interest in θ-deformation is connected to the fact that it preserves the Hochschild dimension. Furthermore the analysis of Section 12 shows that in general the differential calculi over θ-deformations do not rely on the existence of quantum group actions, (see below). In Section 10, we define the splitting homomorphisms mapping the polynomial algebras C_{alg} of the various θ-deformations introduced previously into the polynomial algebras on the product of the corresponding classical spaces with the noncommutative n-torus T^n_θ. In Section 11 we use the splitting homomorphisms to produce smooth structures on the previously defined noncommutative spaces, that is the algebras of smooth functions and of smooth differential forms. In Section 12 we describe in general the construction which associates to each finite-dimensional manifold M endowed with a smooth action σ of the n-torus T^n a noncommutative deformation $C^\infty(M_\theta)$ of the algebra $C^\infty(M)$ of smooth functions on M (and of the algebra of smooth differential forms) which defines the noncommutative manifold M_θ and we explain why the Hochschild dimension of the deformed algebra remains constant and equal to the dimension of M. The construction of differential forms given in this section shows that the θ-deformation of differential forms does not rely on a quantum group action since generically there is no such an action on M_θ (beside the action of the n-torus). The deformation $C^\infty(M_\theta)$ of the algebra $C^\infty(M)$ is a special case of Rieffel's deformation quantization [47] and close to the form adopted in [48]. It is worth noticing here that at the formal level deformations of algebras for actions of $\mathbb{R}^n$ have been also analysed in [40]. It is however crucial to consider (non formal) actions of T^n; our results would be generically wrong for actions of $\mathbb{R}^n$.

In Section 13 we analyse the metric aspect of the construction showing that the deformation is isospectral in the sense of [18] and that our construction gives an alternate setting for results like Theorem 6 of [18]. We use the splitting homomorphism to show that when M is a compact riemannian spin manifold endowed with an isometric action of T^n the corresponding spectral triple ([18]) $(C^\infty(M_\theta), \mathcal{H}_\theta, D_\theta)$ satisfies the axioms of noncommutative geometry of [15]. We show moreover (theorem 9) that any T^n-invariant metric on S^m, ($m = 2n$, $2n-1$), whose volume form is rotation invariant yields a solution of the original polynomial equation for the Dirac operator on S^m_θ. Section 14 is our conclusion.

Throughout this paper n denotes an integer such that $n \geq 2$, $\theta \in M_n(\mathbb{R})$ is an antisymmetric real (n,n)-matrix with matrix elements denoted by $\theta_{\mu\nu}$ ($\mu, \nu = 1, 2, \dots, n$) and we set $\lambda^{\mu\nu} = e^{i\theta_{\mu\nu}} = \lambda_{\mu\nu}$. The reason for this double notation $\lambda^{\mu\nu}, \lambda_{\mu\nu}$ for the same object $e^{i\theta_{\mu\nu}}$ is to avoid ambiguities connected with the Einstein summation convention (of repeated up down indices) which is used throughout. The symbol $\otimes$ without other specification will always denote the tensor product over the field $\mathbb{C}$. A *self-adjoint idempotent* or a *hermitian projection* in a $*$-algebra is an element e satisfying $e^2 = e = e^*$. By a *graded-involutive algebra* we here mean a graded $\mathbb{C}$-algebra endowed with an antilinear involution $\omega \mapsto \bar\omega$ such that $\overline{\omega\omega'} = (-1)^{pp'}\bar\omega'\bar\omega$ for ω of degree p and ω' of degree p'. A *graded-involutive differential algebra* will be a graded-involutive algebra endowed with a real differential d such that $d(\bar\omega) = \overline{d(\omega)}$ for any ω. Given a graded vector space $V = \oplus_n V^n$, we denote by $(-I)^{\mathrm{gr}}$ the endomorphism of V which is the identity mapping on $\oplus_k V^{2k}$ and minus the identity mapping on $\oplus_k V^{2k+1}$. If Ω' and Ω'' are graded algebras one can endow $\Omega' \otimes \Omega''$ with the usual product $(x' \otimes x'')(y' \otimes y'') = x'y' \otimes x''y''$ or with the graded twisted one $(x' \otimes x'')(y' \otimes y'') = (-1)^{|x''||y'|}x'y' \otimes x''y''$ where $|x''|$ is the degree of x'' and $|y'|$ is the degree of y'; in the latter case we denote by $\Omega' \otimes_{\mathrm{gr}} \Omega''$ the corresponding graded algebra. If furthermore Ω' and Ω'' are graded differential algebras $\Omega' \otimes_{\mathrm{gr}} \Omega''$ will denote the corresponding graded algebra endowed with the differential $d = d' \otimes I + (-I)^{\mathrm{gr}} \otimes d''$. A bimodule over an algebra A is said to be *diagonal* if it is a subbimodule of A^I for some set I. Concerning locally convex algebras, topological modules, bimodules and resolutions we use the conventions of [10]. All our locally convex algebras and locally convex modules will be nuclear and complete. Finally we shall need some notations concerning matrix algebras $M_n(A) = M_n(\mathbb{C}) \otimes A$ with entries in an algebra A. For $M \in M_n(A)$, we denote by $\mathrm{tr}(M)$ the element $\sum_{\alpha=1}^n M^\alpha_\alpha$ of A. If M and N are in $M_n(A)$, we denote by $M \circledcirc N$ the element of $M_n(A \otimes A)$ defined by $(M \circledcirc N)^\alpha_\beta = M^\alpha_\gamma \otimes M^\gamma_\beta$.

2 Noncommutative 3-spheres and 4-planes

Our aim in this section is to give a complete description of noncommutative spherical three-manifolds. More specifically we give here a complete description of the class of complex unital $*$-algebras $\mathcal{A}^{(1)}$ satisfying the following conditions (I_1) and (II):

(I_1) $\mathcal{A}^{(1)}$ is generated as unital $*$-algebra by the matrix elements of a unitary

$U \in M_2(\mathcal{A}^{(1)}) = M_2(\mathbb{C}) \otimes \mathcal{A}^{(1)}$,

(II) U satisfies $\mathrm{ch}_{\frac{1}{2}}(U) = U^i_j \otimes U^{*j}_i - U^{*i}_j \otimes U^j_i = 0$

(i.e. with the notations explained above $\mathrm{tr}(U \circledcirc U^* - U^* \circledcirc U) = 0$).

It is convenient to consider the corresponding homogeneous problem, i.e. the class of unital $*$-algebras $\mathcal{A}$ such that

(I) $\mathcal{A}$ is generated by the matrix elements of a $U \in M_2(\mathcal{A}) = M_2(\mathbb{C}) \otimes \mathcal{A}$ satisfying $U^*U = UU^* \in \mathbb{1}_2 \otimes \mathcal{A}$ where $\mathbb{1}_2$ is the unit of $M_2(\mathbb{C})$ and,

(II) U satisfies $\tilde{\mathrm{ch}}_{\frac{1}{2}}(U) = U^i_j \otimes U^{*j}_i - U^{*i}_j \otimes U^j_i = 0$

i.e. $\mathrm{tr}(U \circledcirc U^* - U^* \circledcirc U) = 0$.
Notice that if $\mathcal{A}^{(1)}$ satisfies Conditions (I_1) and (II) or if $\mathcal{A}$ satisfies Conditions (I) and (II) with U as above, nothing changes if one makes the replacement

$$U \mapsto U' = uV_1UV_2 \tag{2.1}$$

with $u = e^{i\varphi} \in U(1)$ and $V_1, V_2 \in SU(2)$. This corresponds to a linear change in generators, $(\mathcal{A}^{(1)}, U')$ satisfies (I_1) and (II) whenever $(\mathcal{A}^{(1)}, U)$ satisfies (I_1) and (II) and $(\mathcal{A}, U')$ satisfies (I) and (II) whenever $(\mathcal{A}, U)$ satisfies (I) and (II).

Let $\mathcal{A}$ be a unital $*$-algebra and $U \in M_2(\mathcal{A})$. We use the standard Pauli matrices σ_k to write U as

$$U = \mathbb{1}_2 z^0 + i\sigma_k z^k \tag{2.2}$$

where z^μ are elements of $\mathcal{A}$ for $\mu = 0, 1, 2, 3$. In terms of the z^μ, the transformations (2.1) reads

$$z^\mu \mapsto uS^\mu_\nu z^\nu \tag{2.3}$$

with $u \in U(1)$ as above and where S^μ_ν are the matrix elements of the real rotation $S \in SO(4)$ corresponding to $(V_1, V_2) \in SU(2) \times SU(2) = \mathrm{Spin}(4)$. The pair $(\mathcal{A}, U)$ fulfills (I) if and only if $\mathcal{A}$ is generated by the z^μ as unital $*$-algebra and the z^μ satisfy

$$z^k z^{0*} - z^0 z^{k*} + \epsilon_{k\ell m} z^\ell z^{m*} = 0 \tag{2.4}$$

$$z^{0*} z^k - z^{k*} z^0 + \epsilon_{k\ell m} z^{\ell *} z^m = 0 \tag{2.5}$$

$$\sum_{\mu=0}^{3} (z^\mu z^{\mu *} - z^{\mu *} z^\mu) = 0 \tag{2.6}$$

for $k = 1, 2, 3$, where $\epsilon_{k\ell m}$ is completely antisymmetric in $k, \ell, m \in \{1, 2, 3\}$ with $\epsilon_{123} = 1$. Condition (I_1) is satisfied if and only if one has in addition $\sum_\mu z^{\mu *} z^\mu = \mathbb{1}$. The following lemma shows that there is no problem to pass from (I) to (I_1) just imposing the relation $\sum_\mu z^{\mu *} z^\mu - \mathbb{1} = 0$.

LEMMA 1 *Let $\mathcal{A}$, U satisfy* (I) *as above. Then* $\sum_{\mu=0}^{3} z^{\mu *} z^{\mu}$ *is in the center of* $\mathcal{A}$.

This result is easily verified using relations (2.4), (2.5), (2.6) above.

Let us now investigate condition (II). In terms of the representation (2.2), for U, condition (II) reads

$$\sum_{\mu=0}^{3} (z^{\mu *} \otimes z^{\mu} - z^{\mu} \otimes z^{\mu *}) = 0 \tag{2.7}$$

for the $z^{\mu} \in \mathcal{A}$. One has the following result.

LEMMA 2 *Condition* (II)*, i.e. equation (2.7), is satisfied if and only if there is a symmetric unitary matrix* $\Lambda \in M_4(\mathbb{C})$ *such that* $z^{\mu *} = \Lambda^{\mu}_{\nu} z^{\nu}$ *for* $\mu \in \{0, \cdots, 3\}$.

The existence of $\Lambda \in M_4(\mathbb{C})$ as above clearly implies Equation (2.7). Conversely assume that (2.7) is satisfied. If the (z^{μ}) are linearly independent, the existence and uniqueness of a matrix Λ such that $z^{\mu *} = \Lambda^{\mu}_{\nu} z^{\nu}$ is immediate, and the symmetry and unitarity of Λ follow from its uniqueness. Thus the only difficulty is to take care in general of the linear dependence between the (z^{μ}). We let $I \subset \{0, \dots, 3\}$ be a maximal subset of $\{0, \dots, 3\}$ such that the $(z^i)_{i \in I}$ are linearly independent. Let I' be its complements ; one has $z^{i'} = \bar{L}^{i'}_{i} z^i$ for some $L^{i'}_{i} \in \mathbb{C}$. On the other hand one has $z^{i *} = C^i_j z^j + E^i_A y^A$ where the y^A are linearily independent elements of $\mathcal{A}$ which are independent of the z^i and C^i_j, E^i_A are complex numbers. This implies in particular that $z^{i' *} = L^{i'}_{i} C^i_j z^j + L^{i'}_{i} E^i_A y^A$. By expanding Equation (2.7) in terms of the linearily independent elements $z^i \otimes z^j$, $z^i \otimes y^A$, $y^A \otimes z^i$ of $\mathcal{A} \otimes \mathcal{A}$ one obtains

$$(\mathbb{1} + L^* L) C = ((\mathbb{1} + L^* L) C)^t \tag{2.8}$$

for the complex matrices $L = (L^{i'}_{j})$ and $C = (C^i_j)$ (C is a square matrix whereas L is generally rectangular) and

$$(\mathbb{1} + L^* L) E_A = 0$$

for the E^i_A which implies $E^i_A = 0$ (since $\mathbb{1} + L^* L > 0$). Thus one has $z^{i *} = C^i_j z^j$ which implies $\bar{C} C = \mathbb{1}$ for the matrix C, $z^{i'} = \bar{L}^{i'}_{i} z^i$, $z^{i' *} = L^{i'}_{i} C^i_j z^j$ together with Equation (2.8). This implies $z^{\mu *} = \Lambda^{\mu}_{\nu} z^{\nu}$ together with $\Lambda^{\mu}_{\nu} = \Lambda^{\nu}_{\mu}$ for $\Lambda \in M_4(\mathbb{C})$ given by

$$\begin{aligned} \Lambda^i_j &= C^i_j - \textstyle\sum_{n'} C^m_i L^{n'}_{m} \bar{L}^{n'}_{j} \\ \Lambda^{i'}_{j} &= L^{i'}_{m} C^m_j = \Lambda^j_{i'} \\ \Lambda^{i'}_{j'} &= 0 \end{aligned}$$

With an obvious relabelling of the z^μ, one can write Λ in block from

$$\Lambda = \begin{pmatrix} C - C^t L^t \bar{L} & C^t L^t \\ \\ LC & 0 \end{pmatrix}$$

The equality $\Lambda z = z^*$ and the symmetry of Λ show that $\Lambda^* z^* = z$ so that $\Lambda^*\Lambda z = z$. Let $\Lambda = U|\Lambda|$ be the polar decomposition of Λ. Since Λ is symmetric, the matrix U is also symmetric (symmetry means $\Lambda^* = J\Lambda J^{-1}$ where J is the antilinear involution defining the complex structure, one has $\Lambda = |\Lambda^*|U$ so that $\Lambda^* = U^*|\Lambda^*|$ and the uniqueness of the polar decomposition gives $U^* = JUJ^{-1}$). Moreover the equality $\Lambda^*\Lambda z = z$ shows that

(1) $\Lambda z = Uz$, $Pz = 0$ where $P = (1 - U^*U)$

One has $(1 - UU^*) = JPJ^{-1}$ and with e_j an orthonormal basis of $P\mathbb{C}^4$, $f_j = Je_j$ the corresponding orthornormal basis of $JP\mathbb{C}^4$ one checks that the following matrix is symmetric,

(2) $S = \sum |f_j\rangle\langle e_j|$

Let now $\tilde{\Lambda} = U + S$. By (1) one has $\tilde{\Lambda} z = z^*$ since $Sz = 0$ and $Uz = \Lambda z = z^*$. Since $\tilde{\Lambda}$ is symmetric and unitary we get the conclusion.

Under the transformation (2.3), Λ transforms as

$$\Lambda \mapsto u^2\ {}^t S \Lambda S$$

so one can diagonalize the symmetric unitary Λ by a real rotation S and fix its first eigenvalue to be 1 by chosing the appropriate $u \in U(1)$ which shows that one can take Λ in diagonal form

$$\Lambda = \begin{pmatrix} 1 & & & \\ & e^{-2i\varphi_1} & & \\ & & e^{-2i\varphi_2} & \\ & & & e^{-2i\varphi_k} \end{pmatrix} \tag{2.9}$$

i. e. one can assume that $z^0 = x^0$ and $z^k = e^{i\varphi_k} x^k$ with $e^{i\varphi_k} \in U(1) \subset \mathbb{C}$ for $k \in \{1,2,3\}$ and $x^{\mu *} = x^\mu$ $(\in \mathcal{A})$ for $\mu \in \{0, \cdots, 3\}$.

Setting $z^0 = x^0 = x^{0*}$ and $z^k = e^{i\varphi_k} x^k$, $x^k = x^{k*}$ relations (2.4) and (2.5) read

$$\cos(\varphi_k)[x^0, x^k]_- = i \sin(\varphi_\ell - \varphi_m)[x^\ell, x^m]_+ \tag{2.10}$$
$$\cos(\varphi_\ell - \varphi_m)[x^\ell, x^m]_- = -i \sin(\varphi_k)[x^0, x^k]_+ \tag{2.11}$$

for $k = 1,2,3$ where (k, ℓ, m) is the cyclic permutation of $(1,2,3)$ starting with k and where $[x, y]_\pm = xy \pm yx$. Let $\mathbf{u}$ be the element $(e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3})$ of T^3, we denote by $\mathcal{A}_{\mathbf{u}}$ the complex unital $*$-algebra generated by four hermitian elements x^μ, $\mu \in \{0, \cdots, 3\}$, with relations (2.10), (2.11) above. It follows from the above discussion that all $\mathcal{A}$ satisfying (I) and (II) are quotient of $\mathcal{A}_{\mathbf{u}}$ for some $\mathbf{u}$. However it is straightforward that the pair $(\mathcal{A}_{\mathbf{u}}, U_{\mathbf{u}})$ with $U_{\mathbf{u}} = \mathbb{1}_2 x^0 + i\sum_{k=1}^3 e^{i\varphi_k}\sigma_k x^k$ satisfies (I) and (II) so the $\mathcal{A}_{\mathbf{u}}$ are the maximal solutions of (I), (II) and any other solution is a quotient of some $\mathcal{A}_{\mathbf{u}}$. In particular each maximal solution of (I_1), (II) is the quotient $\mathcal{A}^{(1)}_{\mathbf{u}}$ of $\mathcal{A}_{\mathbf{u}}$ by the ideal generated by $\sum_\mu (x^\mu)^2 - \mathbb{1}$ for some $\mathbf{u}$. This quotient does not contain other relations since $\sum_\mu (x^\mu)^2$ is in the center of $\mathcal{A}_{\mathbf{u}}$ (Lemma 1). In summary one has the following theorem.

THEOREM 1 (*i*) *For any $\mathbf{u} \in T^3$ the complex unital $*$-algebra $\mathcal{A}_{\mathbf{u}}$ satisfies conditions (I) et (II). Moreover, if $\mathcal{A}$ is a complex unital $*$-algebra satisfying conditions (I) and (II) then $\mathcal{A}$ is a quotient of $\mathcal{A}_{\mathbf{u}}$ (i.e. a homomorphic image of $\mathcal{A}_{\mathbf{u}}$) for some $\mathbf{u} \in T^3$.*

(*ii*) *For any $\mathbf{u} \in T^3$, the complex unital $*$-algebra $\mathcal{A}^{(1)}_{\mathbf{u}}$ satisfies conditions (I_1) and (II). Moreover, if $\mathcal{A}^{(1)}$ is a complex unital $*$-algebra satisfying conditions (I_1) and (II) then $\mathcal{A}^{(1)}$ is a quotient of $\mathcal{A}^{(1)}_{\mathbf{u}}$ for some $\mathbf{u} \in T^3$.*

By construction the algebras $\mathcal{A}^{(1)}_{\mathbf{u}}$ are all quotients of the universal Grassmannian $\mathcal{A}$ generated by (I_1) i. e. by the matrix components $x_1, \dots, x_4$ of a two by two unitary matrix.
One can show that the intersection $\mathcal{J}$ of the kernels of the representations ρ of $\mathcal{A}$ such that $\mathrm{ch}_{\frac{1}{2}}(\rho(U)) = 0$ is a non-trivial two sided ideal of $\mathcal{A}$. More precisely let $\mu = [x_1, \dots, x_4]$, be the multiple commutator $\Sigma\ \varepsilon(\sigma)\, x_{\sigma(1)} \dots x_{\sigma(4)}$ (where the sum is over all permutations and $\varepsilon(\sigma)$ is the signature of the permutation) then $[\mu, \mu^*] \neq 0$ in $\mathcal{A}$ and $[\mu, \mu^*]$ belongs to $\mathcal{J}$ (see the Appendix for the detailed proof). Thus the odd Grassmannian $\mathcal{B}$ which was introduced in [18] is a nontrivial quotient of $\mathcal{A}$.

There is another way to write relations (2.10) and (2.11) which will be useful for the description of the suspension below, it is given in the following lemma.

LEMMA 3 *Let $\gamma_\mu = \gamma^*_\mu \in M_4(\mathbb{C})$ be the generators of the Clifford algebra of $\mathbb{R}^4$, that is $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}\mathbb{1}$, and let $\tilde{\gamma}_\mu$ be defined by $\tilde{\gamma}_0 = \gamma_0$, $\tilde{\gamma}_k = e^{i\frac{1}{2}\varphi_k\gamma}\gamma_k e^{-i\frac{1}{2}\varphi_k\gamma}$ for $k \in \{1,2,3\}$ with $\gamma = \gamma_0\gamma_1\gamma_2\gamma_3$ $(= \gamma_5)$. Then the relations (2.10) and (2.11) defining $\mathcal{A}_{\mathbf{u}}$ are equivalent to the relation*

$$(\tilde{\gamma}_\mu x^\mu)^2 = \mathbb{1} \otimes \sum_\mu (x^\mu)^2$$

in $M_4(\mathcal{A}_{\mathbf{u}}) = M_4(\mathbb{C}) \otimes \mathcal{A}_{\mathbf{u}}$.

This is easy to check using $\gamma\gamma_\mu = -\gamma_\mu\gamma$ and $\gamma^2 = \mathbb{1}$. On the right-hand side of the above equality appears the central element $\sum_\mu (x^\mu)^2$ of $\mathcal{A}_{\mathbf{u}}$; the algebra $\mathcal{A}_{\mathbf{u}}$ has another central element described in the following lemma.

LEMMA 4 *The element* $\sum_{k=1}^{3}\cos(\varphi_k-\varphi_\ell-\varphi_m)\cos(\varphi_k)\sin(\varphi_k)(x^k)^2$ *is in the center of* $\mathcal{A}_{\mathbf{u}}$*, where in the summation* (k,ℓ,m) *is the cyclic permutation of* $(1,2,3)$ *starting with* k *for* $k\in\{1,2,3\}$.

This can be checked directly using (2.10), (2.11). So one has two quadratic elements in the x^μ which belong to the center $Z(\mathcal{A}_{\mathbf{u}})$ of $\mathcal{A}_{\mathbf{u}}$. In fact, for generic $\mathbf{u}$, the center is generated by these two quadratic elements.

By changing x_k in $-x_k$ one can replace φ_k by $\varphi_k+\pi$ and by a rotation of $SO(3)$ one can permute the φ_k without changing the algebra $\mathcal{A}_{\mathbf{u}}$ nor the algebra $\mathcal{A}^{(1)}_{\mathbf{u}}$. It follows that it is sufficient to take $\mathbf{u}$ in the 3-cell defined by

$$\{(e^{i\varphi_k})\in T^3|\pi>\varphi_1\geq\varphi_2\geq\varphi_3\geq 0\}\tag{2.12}$$

to cover all the $\mathcal{A}_{\mathbf{u}}$ and $\mathcal{A}^{(1)}_{\mathbf{u}}$.

It is apparent that $\mathcal{A}_{\mathbf{u}}$ is a deformation of the commutative $*$-algebra $C_{\mathrm{alg}}(\mathbb{R}^4)$ of complex polynomial functions on $\mathbb{R}^4$; it reduces to the latter for $\varphi_1=\varphi_2=\varphi_3=0$ that is for $\mathbf{u}=\mathbf{e}$ where $\mathbf{e}=(1,1,1)$ is the unit of T^3. We shall denote $\mathcal{A}_{\mathbf{u}}$ by $C_{\mathrm{alg}}(\mathbb{R}^4_{\mathbf{u}})$ defining thereby the noncommutative 4-plane $\mathbb{R}^4_{\mathbf{u}}$ as dual object. Similarly, the quotient $\mathcal{A}^{(1)}_{\mathbf{u}}$ of $\mathcal{A}_{\mathbf{u}}$ by the ideal generated by $\sum_\mu(x^\mu)^2-\mathbb{1}$ is a deformation of the $*$-algebra $C_{\mathrm{alg}}(S^3)$ of polynomial functions on S^3 that is of functions on S^3 which are restrictions to $S^3\subset\mathbb{R}^4$ of elements of $C_{\mathrm{alg}}(\mathbb{R}^4)$; we shall denote this quotient $\mathcal{A}^{(1)}_{\mathbf{u}}$ by $C_{\mathrm{alg}}(S^3_{\mathbf{u}})$ defining thereby the noncommutative 3-sphere $S^3_{\mathbf{u}}$ by duality.

Let $C_{\mathrm{alg}}(\mathbb{R}^5_{\mathbf{u}})$ be the unital $*$-algebra obtained by adding a central hermitian generator x^4 to $C_{\mathrm{alg}}(\mathbb{R}^4_{\mathbf{u}})=\mathcal{A}_{\mathbf{u}}$, i.e. $C_{\mathrm{alg}}(\mathbb{R}^5_{\mathbf{u}})$ is the unital $*$-algebra generated by hermitian elements x^μ, $\mu\in\{0,\dots,3\}$, and x^4 such that the x^μ satisfy (2.10), (2.11) and that one has $x^\mu x^4=x^4x^\mu$ for $\mu\in\{0,\dots,3\}$; the noncommutative 5-plane $\mathbb{R}^5_{\mathbf{u}}$ being defined by duality. Let $C_{\mathrm{alg}}(S^4_{\mathbf{u}})$ be the unital $*$-algebra quotient of $C_{\mathrm{alg}}(\mathbb{R}^5_{\mathbf{u}})$ by two-sided ideal generated by the hermitian central element $\sum_{\mu=0}^{3}(x^\mu)^2+(x^4)^2-\mathbb{1}$. The noncommutative 4-sphere $S^4_{\mathbf{u}}$ defined as dual object is in the obvious sense the *suspension* of $S^3_{\mathbf{u}}$. This is a 3-parameter deformation of the sphere S^4 which reduces to S^4_θ for $\varphi_1=\varphi_2=-\frac{1}{2}\theta$ and $\varphi_3=0$, (see below). We denote by u^μ,u the canonical images of $x^\mu,x^4\in C_{\mathrm{alg}}(\mathbb{R}^5_{\mathbf{u}})$ in $C_{\mathrm{alg}}(S^4_{\mathbf{u}})$ and by v^μ the canonical images of $x^\mu\in C_{\mathrm{alg}}(\mathbb{R}^4_{\mathbf{u}})$ in $C_{\mathrm{alg}}(S^3_{\mathbf{u}})$, i.e. one has $\sum(u^\mu)^2+u^2=\mathbb{1}$ and $\sum(v^\mu)^2=\mathbb{1}$, etc.. It will be convenient for further purpose to summarize some important points discussed above by the following theorem.

THEOREM 2 (i) *One obtains a hermitian projection* $e\in M_4(C_{\mathrm{alg}}(S^4_{\mathbf{u}}))$ *by setting* $e=\frac{1}{2}(\mathbb{1}+\tilde\gamma_\mu u^\mu+\gamma u)$*. Furthermore one has* $\mathrm{ch}_0(e)=0$ *and* $\mathrm{ch}_1(e)=0$.
(ii) *One obtains a unitary* $U\in M_2(C_{\mathrm{alg}}(S^3_{\mathbf{u}}))$ *by setting* $U=\mathbb{1}v^0+i\tilde\sigma_k v^k$ *where* $\tilde\sigma_k=\sigma_k e^{i\varphi_k}$*. Furthermore one has* $\mathrm{ch}_{\frac{1}{2}}(U)=0$.

Statement (ii) is just a reformulation of what has be done previously. Concerning Statement (i), the fact that e is a hermitian projection with $\mathrm{ch}_0(e)=0$ follows directly from the definition and Lemma 3 whereas $\mathrm{ch}_1(e)=0$ is a consequence of $\mathrm{ch}_{\frac{1}{2}}(U)=0$ in (ii).

We shall now compute $\tilde{\mathrm{ch}}_{\frac{3}{2}}(U)$ and check that, except for exceptional values of $\mathbf{u}$ for which $\tilde{\mathrm{ch}}_{\frac{3}{2}}(U)=0$, it is a non trivial Hochschild cycle on $\mathcal{A}_{\mathbf{u}}$.

One has by construction

$$\tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}})=\mathrm{tr}(U_{\mathbf{u}}\circledcirc U^*_{\mathbf{u}}\circledcirc U_{\mathbf{u}}\circledcirc U^*_{\mathbf{u}}-U^*_{\mathbf{u}}\circledcirc U_{\mathbf{u}}\circledcirc U^*_{\mathbf{u}}\circledcirc U_{\mathbf{u}})$$

which is an element of $\mathcal{A}_{\mathbf{u}}\otimes\mathcal{A}_{\mathbf{u}}\otimes\mathcal{A}_{\mathbf{u}}\otimes\mathcal{A}_{\mathbf{u}}$ and can be considered as a $\mathcal{A}_{\mathbf{u}}$-valued Hochschild 3-chain. One obtains using (2.10), (2.11)

$$\begin{aligned}\tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}}) \quad = \quad & -\sum_{3\geq\alpha,\beta,\gamma,\delta\geq 0}\epsilon_{\alpha\beta\gamma\delta}\cos(\varphi_\alpha-\varphi_\beta+\varphi_\gamma-\varphi_\delta)x^\alpha\otimes x^\beta\otimes x^\gamma\otimes x^\delta\\ & + i\sum_{3\geq\mu,\nu\geq 0}\sin(2(\varphi_\mu-\varphi_\nu))x^\mu\otimes x^\nu\otimes x^\mu\otimes x^\nu \qquad (2.13)\end{aligned}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is completely antisymmetric with $\epsilon_{0123}=1$ and where we have set $\varphi_0=0$. Using (2.13), (2.10), (2.11) one checks that $\tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}})$ is in fact a Hochschild cycle, i.e. $b(\tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}}))=0$. Actually, this follows on general grounds from the fact that $\tilde{\mathrm{ch}}_{\frac{1}{2}}(U_{\mathbf{u}})=0$ and that $U^*_{\mathbf{u}}U_{\mathbf{u}}=U_{\mathbf{u}}U^*_{\mathbf{u}}$ is an element of the center $\mathbb{1}_2\otimes Z(\mathcal{A}_{\mathbf{u}})$ of $M_2(\mathcal{A}_{\mathbf{u}})$ in view of Lemma 1. In fact the $\mathcal{A}_{\mathbf{u}}$-valued Hochschild 3-cycle $\tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}})$ is trivial (i.e. is a boundary) if and only if it vanishes (which means that all coefficients vanish in formula (2.13)). Indeed $\mathcal{A}_{\mathbf{u}}$ is a $\mathbb{N}$-graded algebra with $\mathcal{A}^0_{\mathbf{u}}=\mathbb{C}\mathbb{1}$ and $\mathcal{A}^1_{\mathbf{u}}=$ linear span of the $\{x^\mu|\mu\in\{0,\cdots,3\}\}$ and the Hochschild boundary preserves the degree. It follows that $\tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}})$ can only be the boundary of linear combinations of terms which are in $\otimes^5\mathcal{A}_{\mathbf{u}}$ of total degree 4 and contain therefore at least one tensor factor equal to $\mathbb{1}$. Among these terms, the $\mathbb{1}\otimes x^\alpha\otimes x^\beta\otimes x^\gamma\otimes x^\delta$ are the only ones which contain in their boundaries tensor products of four x^μ. One has for these terms

$$b(\mathbb{1}\otimes x^\alpha\otimes x^\beta\otimes x^\gamma\otimes x^\delta)=x^\alpha\otimes x^\beta\otimes x^\gamma\otimes x^\delta+x^\delta\otimes x^\alpha\otimes x^\beta\otimes x^\gamma$$

$$-\mathbb{1}\otimes(x^\alpha x^\beta\otimes x^\gamma\otimes x^\delta-x^\alpha\otimes x^\beta x^\gamma\otimes x^\delta+x^\alpha\otimes x^\beta\otimes x^\gamma x^\delta)$$

however the $x^\alpha\otimes x^\beta\otimes x^\gamma\otimes x^\delta+x^\delta\otimes x^\alpha\otimes x^\beta\otimes x^\gamma$ cannot produce by linear combination a term with the kind of generalized antisymmetry of $\tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}})$ excepted of course if $\tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}})=0$. Thus $\tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}})$ is non trivial if not zero.
The $\mathcal{A}^{(1)}_{\mathbf{u}}$-valued Hochschild 3-cycle $\mathrm{ch}_{\frac{3}{2}}(U)$ on $\mathcal{A}^{(1)}_{\mathbf{u}}$ is the image of $\tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}})$ by the projection of $\mathcal{A}_{\mathbf{u}}$ onto $\mathcal{A}^{(1)}_{\mathbf{u}}$. In particular $\mathrm{ch}_{\frac{3}{2}}(U)$ vanishes if $\tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}})$ vanishes which occurs on Σ^3 for $\varphi_1=\varphi_2=\varphi_3=\frac{\pi}{2}$ and for $\varphi_1=\frac{\pi}{2}$, $\varphi_2=\varphi_3=0$.

For these two values of $\mathbf{u}$, the algebras $\mathcal{A}_{\mathbf{u}}$ are isomorphic, one passes from $\varphi_1 = \varphi_2 = \varphi_3 = \frac{\pi}{2}$ to $\varphi_1 = \frac{\pi}{2}$, $\varphi_2 = \varphi_3 = 0$ by the exchange of x^0 and x^1; this is of course the same for $\mathcal{A}^{(1)}_{\mathbf{u}}$. One can furthermore check that the Hochschild dimension of $\mathcal{A}^{(1)}_{\mathbf{u}}$ for these values of $\mathbf{u}$ is one.
To obtain the Hochschild 4-cycle on $\mathcal{A}_{\mathbf{u}}$ corresponding to the volume form on the noncommutative 4-plane $\mathbb{R}^4_{\mathbf{u}}$, we shall just apply to $\tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}})$ the natural extension of the de Rham coboundary in the noncommutative case, namely the operator $B : \mathcal{A}_{\mathbf{u}} \otimes \tilde{\mathcal{A}}_{\mathbf{u}}^{\otimes^3} \to \mathcal{A}_{\mathbf{u}} \otimes \tilde{\mathcal{A}}_{\mathbf{u}}^{\otimes^4}$ ([10] [33]). Since $\tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}})$ is not only a Hochschild cycle but also fulfills the cyclicity condition, it follows that, up to an irrelevant normalization B reduces there to the tensor product by $\mathbb{1}$, thus

$$B\tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}}) = \mathbb{1} \otimes \tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}})$$

and the Hochschild 4-cycle $B\tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}})$ which plays the role of the volume form of $\mathbb{R}^4_{\mathbf{u}}$ is thus given by

$$\begin{aligned} v \quad = \quad & - \sum_{3\geq\alpha,\beta,\gamma,\delta\geq 0} \epsilon_{\alpha\beta\gamma\delta} \cos(\varphi_\alpha - \varphi_\beta + \varphi_\gamma - \varphi_\delta) \mathbb{1} \otimes x^\alpha \otimes x^\beta \otimes x^\gamma \otimes x^\delta \\ & + i \sum_{3\geq\mu,\nu\geq 0} \sin(2(\varphi_\mu - \varphi_\nu)) \mathbb{1} \otimes x^\mu \otimes x^\nu \otimes x^\mu \otimes x^\nu \end{aligned} \tag{2.14}$$

It turns out that this 4-cycle is non trivial whenever it does not vanish as can be verified by evaluation at the origin which is the classical point of $\mathbb{R}^4_{\mathbf{u}}$. The nontriviality of $\mathrm{ch}_{\frac{3}{2}}(U)$ follows since $B\tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}})$ is its suspension.

3 The Scaling Foliation and relation to Sklyanin algebras

We let as above $\Sigma = T^3$ be the parameter space for 3-dimensional spherical manifolds $S^3_{\mathbf{u}}$. Different $S^3_{\mathbf{u}}$ can span isomorphic 4-dimensional $\mathbb{R}^4_{\mathbf{u}}$ and we shall analyse here the corresponding foliation of Σ.

More precisely, let us say that $S^3_{\mathbf{u}}$ is "scale-equivalent" to $S^3_{\mathbf{v}}$ and write $u \sim v$ when the quadratic algebras corresponding to $\mathbb{R}^4_{\mathbf{u}}$ and $\mathbb{R}^4_{\mathbf{v}}$ are isomorphic. This generates a foliation of Σ which is completely described by the orbits of the flow of the following vector field:

$$Z \;=\; \sum_{k=1}^{3} \sin(2\varphi_k)\, \sin(\varphi_\ell + \varphi_m - \varphi_k)\, \frac{\partial}{\partial \varphi_k} \tag{3.1}$$

as shown by,

THEOREM 3 *Let* $\mathbf{u} \in \Sigma$. *There exists a neighborhood* V *of* $\mathbf{u}$ *such that* $\mathbf{v} \in V$ *is scale-equivalent to* $\mathbf{u}$ *if and only if it belongs to the orbit of* $\mathbf{u}$ *under the flow of* Z.

Let us first show that if $\mathbf{v}$ belongs to the orbit of $\mathbf{u}$ under Z then the corresponding quadratic algebras are isomorphic. To the action of the group of permutations $\mathcal{S}_4$ of the 4 generators of the quadratic algebra there corresponds an action of $\mathcal{S}_4$ on the parameter space Σ. This action is the obvious one on the subgroup $\mathcal{S}_3$ of permutations fixing 0 and the action of the permutation $(1,0,3,2)$ of $(0,1,2,3)$ is given by the following transformation,

$$w(\varphi_1,\varphi_2,\varphi_3) = (-\varphi_1,\varphi_3-\varphi_1,\varphi_2-\varphi_1) \tag{3.2}$$

The transformation w and its conjugates under the action of $\mathcal{S}_3$ by permutations of the φ_j generate an abelian group K of order 4 which is a normal subgroup of the group $W=\mathcal{S}_4$ generated by w and $\mathcal{S}_3$. By construction $g(\mathbf{u})$ is scale-equivalent to $\mathbf{u}$ for any $g\in W$. At a more conceptual level the group W is the Weyl group of the symmetric space used in lemma 2, of symmetric unitary (unimodular) 4 by 4 matrices. Moreover the flow of Z is invariant under the action of W. This is obvious for $g\in\mathcal{S}_3$ and can be checked directly for w.
Let C be the set of critical points for Z, i.e. $C=\{\mathbf{u},Z_{\mathbf{u}}=0\}$. For $\mathbf{u}\in C$ the orbit of $\mathbf{u}$ is reduced to $\mathbf{u}$ and the required equivalence is trivial. To handle the case $\mathbf{u}\notin C$ we let $D\subset\Sigma$ be the zero set of the function,

$$\delta(\mathbf{u}) = \prod_{k=1}^{3}\sin(\varphi_k)\,\cos(\varphi_l-\varphi_m) \tag{3.3}$$

The inclusion $\cap\, gD\subset C$ where g varies in K shows that we can assume that $\mathbf{u}\notin D$. We can then find 4 non-zero scalars s^μ, $\mu\in\{0,\cdots,3\}$ such that,

$$\begin{aligned}
s^0s^1\cos(\varphi_2-\varphi_3)+s^2s^3\sin(\varphi_1) &= 0\\
s^0s^2\cos(\varphi_3-\varphi_1)+s^3s^1\sin(\varphi_2) &= 0\\
s^0s^3\cos(\varphi_1-\varphi_2)+s^1s^2\sin(\varphi_3) &= 0
\end{aligned} \tag{3.4}$$

The solution is unique (up to an overall normalization and choices of sign) and can be written in the form,

$$\begin{aligned}
s^0 &= \left(\textstyle\prod_j \sin\varphi_j\right)^{1/2}\\
s^k &= \left(\sin\varphi_k \textstyle\prod_{\ell\neq k}\cos(\varphi_k-\varphi_\ell)\right)^{1/2}
\end{aligned}$$

where the square roots are chosen so that $\prod s^\mu = -\delta(\mathbf{u})$. Then, provided that $\cos(\varphi_j)\neq 0\ \forall j$, the relations (2.10), (2.11) can be written

$$[S_0,S_k]_- = iJ_{\ell m}[S_\ell,S_m]_+ \tag{3.5}$$
$$[S_\ell,S_m]_- = i[S_0,S_k]_+ \tag{3.6}$$

where $J_{\ell m} = -\tan(\varphi_\ell - \varphi_m)\tan(\varphi_k)$ for any cyclic permutation (k, ℓ, m) of $(1, 2, 3)$ and where

$$S_\mu = s^\mu x^\mu \tag{3.7}$$

So defined the three real numbers $J_{k\ell}$ satisfy the relation

$$J_{12} + J_{23} + J_{31} + J_{12}J_{23}J_{31} = 0 \tag{3.8}$$

as easily verified. The relations (3.5), (3.6) together with (3.8) for the constants $J_{k\ell}$ characterize the algebra introduced by Sklyanin in connection with the Yang-Baxter equation [50], [51].

In the case when the s^μ are real, the transformation (3.7) preserves the involution which on the Sklyanin algebra $S(J_{k\ell})$ is given by

$$S_\mu^* = S_\mu \qquad \mu = 0, 1, 2, 3.$$

In general, however, one cannot choose the s^μ's to be real and the involutive algebra $\mathcal{A}_\mathbf{u}$ gives a different real form of the Sklyanin algebra.

The invariance of the $J_{k\ell}$ under the flow Z, $Z(J_{k\ell}) = 0$, thus gives the required scale-equivalence on the orbit of $\mathbf{u}$ provided $\cos(\varphi_j) \neq 0\ \forall j$. The condition $\varphi_j = \pi/2$ is invariant under the flow Z and this special case is handled in the same way (note that if moreover $\varphi_l = \varphi_m$ one of the relations becomes trivial, the corresponding algebra is not a Sklyanin algebra but is constant on the orbit of Z).
We have thus shown that two points on the same orbit of Z are scale-equivalent. Let us now prove the converse in the form stated in theorem 3. In order to distinguish the quadratic algebras $\mathcal{A}_\mathbf{u}$ we shall use an invariant called the associated geometric data.
The Sklyanin algebras $S(J_{k\ell})$ have been extensively studied from the point of view of noncommutative algebraic geometry. An important role is played by the associated geometric data

$$\{E, \sigma, \mathcal{L}\}$$

consisting of an elliptic curve $E \subset P_3(\mathbb{C})$, an automorphism σ of E and an invertible $\mathcal{O}_E$-module $\mathcal{L}$ (cf. [1], [2], [43], [52]). This geometric data is invariantly defined for any graded algebra and in the above case of $S(J_{k\ell})$, it degenerates when one of the parameters $J_{k\ell}$ vanishes (or in the case $J_{k\ell} = 1, J_{\ell r} = -1$, cf. [52] for a careful discussion).
It is straightforward to extend the computations of [52] to the present situation in order to cover all cases. Up to the action of the group W the critical set C is the union of the point $P = (\pi/2, \pi/2, \pi/2)$ with the two circles,

$$C_+ = \{\mathbf{u}\,; \varphi_1 = \varphi_2, \varphi_3 = 0\}\,,\ \ C_- = \{\mathbf{u}\,; \varphi_1 = \frac{\pi}{2} + \varphi_3, \varphi_2 = \frac{\pi}{2}\}$$

For $\mathbf{u} = P$, the geometric data is very degenerate, $E = P_3(\mathbb{C})$, while σ is a symmetry of determinant -1. In fact there are two other W-orbits, those of $P' = (\pi/2, \pi/2, 0)$ and of $O = (0,0,0)$ for which $E = P_3(\mathbb{C})$. For P', the correspondence σ is a symmetry of determinant 1, while for O it is the identity. For $\mathbf{u} \in C_+$, $\mathbf{u} \neq O$, $\mathbf{u} \notin W(P')$ the geometric data degenerates to the union of 6 projective lines $P_1(\mathbb{C})$, with σ given by multiplication by 1 for two of them, by $e^{2i\varphi_1}$ for two others and $e^{-2i\varphi_1}$ for the last two. The case $\mathbf{u} \in C_-$ is similar, but not identical. E is the union of six lines but σ is given by multiplication by -1 for two of them, it exchanges two of the remaining lines with σ^2 given by multiplication by $e^{4i\varphi_1}$ and exchanges the last two with σ^2 given by multiplication by $e^{-4i\varphi_1}$.

For $\mathbf{u} \notin C$, we can assume as above that $\mathbf{u} \notin D$. Then, provided that $\cos(\varphi_j) \neq 0 \ \forall j$ we can reduce as above to Sklyanin algebras. In that case ([52]) the geometric data $E \subset P_3(\mathbb{C})$ is the union of 4 points with a non-singular elliptic curve, except (up to signed permutations) for the following degenerate case:

$$F_1 = \{\mathbf{u};\, J_{23} = -a, J_{31} = a, J_{12} = 0\}$$

In that case, E is the union of 2 points, one line and 2 circles, the correspondence σ fixes the 2 points and the line pointwise. It restricts to both circles $\Gamma_j \sim P_1(\mathbb{C})$ and is given in terms of a rational parameter as the multiplication by $(i + a^{1/2})/(i - a^{1/2})$ where each circle corresponds to a different choice of the square root $a^{1/2}$.
In the case

$$F_2 = \{\mathbf{u};\, \varphi_1 = \frac{\pi}{2}\ ,\ \varphi_2 \neq \varphi_3,\, \varphi_2 \neq \frac{\pi}{2},\, \varphi_3 \neq \frac{\pi}{2}\}$$

where $\mathbf{u} \notin D$ but $\cos(\varphi_j) = 0$ for some j say j=1, the above change of variables breaks down, but the direct computation shows that as for $\mathbf{u} \in F_1$, E is the union of 2 points, one line and 2 circles. However the correspondence σ is different from that case. It fixes the 2 points and is multiplication by -1 on the line. It exchanges the two circles $\Gamma_j \sim P_1(\mathbb{C})$ and its square σ^2 is given in terms of a rational parameter as the multiplication by the square of $(i + b^{1/2})/(i - b^{1/2})$, $b = -J_{31}$, where each circle corresponds to a different choice of the square root $b^{1/2}$.

On the circle,

$$L = \{\mathbf{u};\, \varphi_1 = \frac{\pi}{2}\ ,\ \varphi_2 = \varphi_3\}$$

the first of the six relations (2.11) becomes trivial and the quadratic algebra is independent of the value of $\varphi_2 = \varphi_3$ except for the isolated values 0 and $\pi/2$,

which correspond to the orbit $W(P)$ of the point $P = (\pi/2, \pi/2, \pi/2)$ discussed above and for which $\tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}})$ vanishes as explained in the last section. For points of the circle L not on this orbit, E is the union of six lines. The correspondence σ is 1 on one line, -1 on another line, and permutes cyclically the remaining 4 lines, inducing twice an isomorphism and twice the coarse correspondence.
Finally on the circle,

$$L' = \{\mathbf{u};\, \varphi_1 = \frac{\pi}{2} \ ,\ \varphi_2 = \frac{\pi}{2}\}$$

except for the special cases treated above E is the union of a point with $P_2(\mathbb{C})$ and the correspondence σ is a symmetry of determinant -1.

Let us now end the proof of theorem 3. We work modulo W. For $\mathbf{u} \in C$ the geometric data allows to distinguish it from any $\mathbf{v}$ in a neighborhood (one checks this for $\mathbf{u} = P$ and $\mathbf{u} \in C_{+,-}$). For $\mathbf{u} \notin C$ the flow line through $\mathbf{u}$ is non-trivial. For $\mathbf{u} \in L$ or L' the nearby points having the same geometric data are necessarily on L or L' and the scaling flow is locally transitive on both, so the answer follows. Each of the faces F_j is globally invariant under the flow Z. For $\mathbf{u} \in F_j$ the nearby points having the same geometric data are necessarily on F_j and the correspondence σ gives the required information to conclude that scale-equivalent nearby points are on the same flow line. Finally for points not W-equivalent to those treated so far, the geometric data is a non-degenerate elliptic curve E whose j-invariant is given by

$$j = 256(\lambda^2 - \lambda + 1)^3/(\lambda^2(1-\lambda)^2)$$

$$\lambda = \sin(2\varphi_1)\sin(2(\varphi_2 - \varphi_3))/\sin(2\varphi_2)\sin(2(\varphi_1 - \varphi_3))$$

and a translation σ which together allow for the local determination of the parameters $J_{k\ell}$ and hence of the flow line of $\mathbf{u}$.

COROLLARY 1 *The critical points of the scaling foliation are given by the union of the W-orbits of P, of C_+ and of C_-.*

We shall now analyse the noncommutative 3-spheres associated to the critical points in C_+. The easiest way to understand them is as special cases of the general procedure of θ-deformation (applied here to the usual 3-sphere and also to $\mathbb{R}^4$) which lends itself to easy higher dimensional generalization. (The case of C_- can be reduced to C_+ thanks to an easy "involutive twist" which will be described in general in part II).

4 The θ-deformed $2n$-plane $\mathbb{R}^{2n}_\theta$ and its Clifford algebra

In the previous sections, we have obtained a multiparameter noncommutative deformation $C_{\mathrm{alg}}(\mathbb{R}^4_{\mathbf{u}})$ of the graded algebra $C_{\mathrm{alg}}(\mathbb{R}^4)$ of polynomial functions on $\mathbb{R}^4$ which induces a corresponding deformation $C_{\mathrm{alg}}(S^3_{\mathbf{u}})$ of the algebra of polynomial functions on S^3 in such a way that all dimensions are preserved as will be shown in Part II. Moreover this is the generic deformation under the above conditions. We also extracted from this multiparameter deformation of $C_{\mathrm{alg}}(\mathbb{R}^4)$ a one-parameter deformation $C_{\mathrm{alg}}(\mathbb{R}^4_\theta)$ of $C_{\mathrm{alg}}(\mathbb{R}^4)$ which is also a one-parameter deformation $C_{\mathrm{alg}}(\mathbb{C}^2_\theta)$ of $C_{\mathrm{alg}}(\mathbb{C}^2)$ whence $\mathbb{C}^2$ is identified with $\mathbb{R}^4$ through (for instance) $z^1 = x^0 + ix^3$, $z^2 = x^1 + ix^2$. The parameter θ corresponds to the curve $\theta \mapsto \mathbf{u}(\theta)$ defined by $\mathbf{u}_1 = \mathbf{u}_2 = e^{-\frac{i}{2}\theta}$ and $\mathbf{u}_3 = 1$, i.e. to $\varphi_1 = \varphi_2 = -\frac{1}{2}\theta$ and $\varphi_3 = 0$ in terms of the previous parameters. Indeed for these values of $\mathbf{u}$, the relations (2.10), (2.11) for x^0, x^1, x^2, x^3 read in terms of $z^1 = x^0 + ix^3, \bar z^1 = x^0 - ix^3$, $z^2 = x^1 + ix^2$, $\bar z^2 = x^1 - ix^2$, (one has $z^{1*} = \bar z^1$ and $z^{2*} = \bar z^2$) $z^1z^2 = \lambda z^2 z^1$, $\bar z^1 \bar z^2 = \lambda \bar z^2 \bar z^1$, $z^1 \bar z^1 = \bar z^1 z^1$, $z^2 \bar z^2 = \bar z^2 z^2$, $\bar z^1 z^2 = \lambda^{-1} z^2 \bar z^1$, $z^1 \bar z^2 = \lambda^{-1} \bar z^2 z^1$ where we have set $\lambda = e^{i\theta}$. This one-parameter deformation is well suited for simple higher-dimensional generalizations (i.e. $\mathbb{C}^2$ is replaced by $\mathbb{C}^n$ and $\mathbb{R}^4$ by $\mathbb{R}^{2n}$, $n \geq 2$). In the following we shall describe and analyze them in details. For this we shall generalize θ as explained at the end of the introduction as an antisymmetric matrix $\theta \in M_n(\mathbb{R})$, the previous one being identified as $\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \in M_2(\mathbb{R})$, and we shall use the notations explained at the end of the introduction.
Let $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ be the complex unital associative algebra generated by $2n$ elements $z^\mu, \bar z^\mu$ $(\mu,\nu = 1,\dots,n)$ with relations

$$z^\mu z^\nu = \lambda^{\mu\nu} z^\nu z^\mu,\ \bar z^\mu \bar z^\nu = \lambda^{\mu\nu} \bar z^\nu \bar z^\mu,\ \bar z^\mu z^\nu = \lambda^{\nu\mu} z^\nu \bar z^\mu \tag{4.1}$$

for $\mu, \nu = 1, \dots, n$ ($\lambda^{\mu\nu} = e^{i\theta_{\mu\nu}}$, $\theta_{\mu\nu} = -\theta_{\nu\mu} \in \mathbb{R}$). Notice that one has $\lambda^{\nu\mu} = 1/\lambda^{\mu\nu} = \overline{\lambda^{\mu\nu}}$ and that $\lambda^{\mu\mu} = 1$. We endow $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ with the unique $\mathbb{C}$-algebra involution $x \mapsto x^*$ such that $z^{\mu *} = \bar z^\mu$. Clearly the $*$-algebra $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ is a deformation of the commutative $*$-algebra $C_{\mathrm{alg}}(\mathbb{R}^{2n})$ of complex polynomial functions on $\mathbb{R}^{2n}$, (it reduces to the latter for $\theta = 0$). The algebra $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ will be refered to as the *algebra of complex polynomials on the noncommutative $2n$-plane $\mathbb{R}^{2n}_\theta$*.

In fact the relations (4.1) define a deformation $\mathbb{C}^n_\theta$ of $\mathbb{C}^n$ and we can identify $\mathbb{C}^n_\theta$ and $\mathbb{R}^{2n}_\theta$ by writing $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta) = C_{\mathrm{alg}}(\mathbb{C}^n_\theta)$. Correspondingly, the unital subalgebra $H_{\mathrm{alg}}(\mathbb{C}^n_\theta)$ generated by the z^μ is a deformation of the algebra of holomorphic polynomial functions on $\mathbb{C}^n$.

There is a unique group-homomorphism $s \mapsto \sigma_s$ of the abelian group T^n into the group $\mathrm{Aut}(C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta))$ of unital $*$-automorphisms of $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ which is such that $\sigma_s(z^\nu) = e^{2\pi i s_\nu} z^\nu$, $(\sigma_s(\bar z^\nu) = e^{-2\pi i s_\nu} \bar z^\nu)$. This definition is independent of θ, in particular $s \mapsto \sigma_s$ is well defined as a group-homomorphism of T^n into $\mathrm{Aut}(C_{\mathrm{alg}}(\mathbb{R}^{2n}))$ where it is induced by a smooth action of T^n on the manifold $\mathbb{R}^{2n}$. It follows from the relations (4.1) that the $z^\mu z^{\mu *} = z^{\mu *} z^\mu$ $(1 \leq \mu \leq n)$ are in the center of $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$. Furthermore these hermitian elements generate the center as unital subalgebra of $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ whenever θ is *generic*, i.e. for $\theta_{\mu\nu}$ irrational $\forall \mu, \nu$ with $1 \leq \mu < \nu \leq n$. On the other hand these elements $\bar{z}^\mu z^\mu$ generate the subalgebra $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)^\sigma$ of elements of $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ which are invariant by the action σ of T^n. Thus $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)^\sigma$ is contained in the center of $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$. This is not an accident, moreover the subalgebra of invariant elements of $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ is not deformed (i.e. does not depend on θ) and is canonically isomorphic to $C_{\mathrm{alg}}(\mathbb{R}^{2n})^\sigma$.

Let $\mathrm{Cliff}(\mathbb{R}^{2n}_\theta)$ be the unital associative $\mathbb{C}$-algebra generated by $2n$ elements Γ^μ, $\Gamma^{\nu *}$ $(\mu, \nu = 1, \dots, n)$ with relations

$$\Gamma^\mu \Gamma^\nu + \lambda^{\nu\mu} \Gamma^\nu \Gamma^\mu = 0 \tag{4.2}$$

$$\Gamma^{\mu *} \Gamma^{\nu *} + \lambda^{\nu\mu} \Gamma^{\nu *} \Gamma^{\mu *} = 0 \tag{4.3}$$

$$\Gamma^{\mu *} \Gamma^\nu + \lambda^{\mu\nu} \Gamma^\nu \Gamma^{\mu *} = \delta^{\mu\nu} \mathbb{1} \tag{4.4}$$

where $\mathbb{1}$ denotes the unit of the algebra. For $\theta = 0$ one recovers the usual Clifford algebra of $\mathbb{R}^{2n}$; the familiar generators γ^a $(a = 1, 2, \dots, 2n)$ associated to the canonical basis of $\mathbb{R}^{2n}$ being then given by $\gamma^\mu = \Gamma^\mu + \Gamma^{\mu *}$ and $\gamma^{\mu+n} = -i(\Gamma^\mu - \Gamma^{\mu *})$. There is a unique involution $\Lambda \mapsto \Lambda^*$ such that $(\Gamma^\mu)^* = \Gamma^{\mu *}$ for which $\mathrm{Cliff}(\mathbb{R}^{2n}_\theta)$ is a unital complex $*$-algebra. One also endows $\mathrm{Cliff}(\mathbb{R}^{2n}_\theta)$ with a $\mathbb{Z}_2$-grading of algebra by giving odd degree to the $\Gamma^\mu, \Gamma^{\nu *}$. The relations (4.2), (4.3) and (4.4) imply that the hermitian element $[\Gamma^{\mu *}, \Gamma^\mu] = \Gamma^{\mu *}\Gamma^\mu - \Gamma^\mu \Gamma^{\mu *}$ anticommutes with Γ^μ and $\Gamma^{\mu *}$ whereas it commutes with Γ^ν and $\Gamma^{\nu *}$ for $\nu \neq \mu$ and that furthermore one has $([\Gamma^{\mu *}, \Gamma^\mu])^2 = \mathbb{1}$. It follows that $\gamma \in \mathrm{Cliff}(\mathbb{R}^{2n}_\theta)$ defined by

$$\gamma = [\Gamma^{1*}, \Gamma^1] \dots [\Gamma^{n*}, \Gamma^n] = \prod_{\mu=1}^{n} [\Gamma^{\mu *}, \Gamma^\mu] \tag{4.5}$$

is hermitian $(\gamma = \gamma^*)$ and satisfies

$$\gamma^2 = 1, \quad \gamma \Gamma^\mu + \Gamma^\mu \gamma = 0, \ \gamma \Gamma^{\mu *} + \Gamma^{\mu *} \gamma = 0 \tag{4.6}$$

in fact $\Lambda \mapsto \gamma \Lambda \gamma$ is the $\mathbb{Z}_2$-grading. The very reason why we have imposed the relations (4.2), (4.3) and (4.4) is the following easy lemma.

LEMMA 5 *In the algebra* $\mathrm{Cliff}(\mathbb{R}^{2n}_\theta) \otimes C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$, *the elements* $\Gamma^{\mu *} z^\mu$ *and* $\Gamma^\rho \bar{z}^\rho = \Gamma^\rho z^{\rho *}$, $\mu, \rho = 1, \dots, n$, *satisfy the following anticommutation relations* $\Gamma^{\mu *} z^\mu \Gamma^{\rho *} z^\rho + \Gamma^{\rho *} z^\rho \Gamma^{\mu *} z^\mu = 0$ $(\Gamma^\mu \bar{z}^\mu \Gamma^\rho \bar{z}^\rho + \Gamma^\rho \bar{z}^\rho \Gamma^\mu \bar{z}^\mu = 0)$ *and* $\Gamma^{\mu *} z^\mu \Gamma^\rho \bar{z}^\rho + \Gamma^\rho \bar{z}^\rho \Gamma^{\mu *} z^\mu = \delta^{\mu\rho} z^\mu \bar{z}^\mu$ *which do not depend on* θ.

This straightforward result is a key to reduce lots of computations to the classical case $\theta = 0$, (see below). The next result shows that $\mathrm{Cliff}(\mathbb{R}^{2n}_\theta)$ is isomorphic to the usual $\mathrm{Cliff}(\mathbb{R}^{2n})$ as $*$-algebra and as $\mathbb{Z}_2$-graded algebra.

PROPOSITION 1 *The following equality gives a faithful $*$-representation π of $\mathrm{Cliff}(\mathbb{R}^{2n}_\theta)$ in the Hilbert space $\otimes^n \mathbb{C}^2$,*

$$\begin{aligned}\pi(\Gamma^{\mu *}) &= \begin{pmatrix} -\lambda^{1\mu} & 0 \\ 0 & 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} -\lambda^{\mu-1\mu} & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2 \\ &= \pi(\Gamma^\mu)^*\end{aligned}$$

and π is the unique irreducible $*$-representation of $\mathrm{Cliff}(\mathbb{R}^{2n}_\theta)$ up to a unitary equivalence.

The proof is straightforward. Note that $\otimes^n \mathbb{C}^2$, viewed as the graded tensor product of $\mathbb{C}^2$ graded by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a $\mathbb{Z}_2$-graded $\mathrm{Cliff}(\mathbb{R}^{2n}_\theta)$-module. One has $\pi(\gamma) = \otimes^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In the following we will use the above representation to identify $\mathrm{Cliff}(\mathbb{R}^{2n}_\theta)$ with $M_{2^n}(\mathbb{C})$.

5 Spherical property of θ-deformed spheres

Let $C_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta)$, *the algebra of polynomial functions on the noncommutative $(2n+1)$-plane* $\mathbb{R}^{2n+1}_\theta$, be the unital complex $*$-algebra obtained by adding an hermitian generator x to $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ with relations $xz^\mu = z^\mu x$ ($\mu = 1, \dots, n$), i.e. $C_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta) \simeq C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta) \otimes \mathbb{C}[x] \simeq C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta) \otimes C_{\mathrm{alg}}(\mathbb{R})$. One knows that the $z^\mu \bar{z}^\mu = \bar{z}^\mu z^\mu$ and x are in the center so $\sum_{\mu=1}^n z^\mu \bar{z}^\mu + x^2$ is also in the center $C_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta)$. We let $C_{\mathrm{alg}}(S^{2n}_\theta)$ be the $*$-algebra quotient of $C_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta)$ by the ideal generated by $\sum_{\mu=1}^n z^\mu \bar{z}^\mu + x^2 - \mathbb{1}$. In the following, we shall denote by $u^\mu, \bar{u}^\nu = u^{\nu *}$, u the canonical images of $z^\mu, \bar{z}^\nu, x$ in $C_{\mathrm{alg}}(S^{2n}_\theta)$. On the unital complex $*$-algebra $C_{\mathrm{alg}}(S^{2n}_\theta)$ there is a greatest C^*-seminorm which is a norm; the C^*-algebra $C(S^{2n}_\theta)$ obtained by completion will be refered to as *the algebra of continuous functions on the noncommutative $2n$-sphere S^{2n}_θ*.

It is worth noticing that the noncommutative $2n$-sphere S^{2n}_θ can be viewed as "one-point compactification" of the noncommutative $2n$-plane $\mathbb{R}^{2n}_\theta$. To explain this, let us slightly enlarge the $*$-algebra $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ by adjoining a hermitian central generator $(1 + \sum_{\mu=1}^n \bar{z}^\mu z^\mu)^{-1} = (1 + |z|^2)^{-1}$ with relation $(1 + \sum_{\mu=1}^n \bar{z}^\mu z^\mu)(1+|z|^2)^{-1} = (1+|z|^2)^{-1}(1+\sum_{\mu=1}^n \bar{z}^\mu z^\mu) = 1$. As will become clear $(1+|z|^2)^{-1}$ is smooth so that in fact we are staying in the algebra $C^\infty(\mathbb{R}^{2n}_\theta)$ of smooth functions on $\mathbb{R}^{2n}_\theta$. By setting

$$\tilde{u}^\mu = 2z^\mu(1+|z|^2)^{-1},\ \tilde{u}^{\nu *} = 2\bar{z}^\mu(1+|z|^2)^{-1},\ \tilde{u} = (1 - \sum_{\mu=1}^n \bar{z}^\mu z^\mu)(1+|z|^2)^{-1},$$

one sees that the $\tilde{u}^\mu$, $\tilde{u}^{\nu *}, \tilde{u}$ satisfy the same relations as the u^μ, $u^{\nu *}, u$. The "only difference" is that the classical point $u^\mu = 0, \bar{u}^\mu = 0, u = -1$ of S^{2n}_θ does not belong to the spectrum of $\tilde{u}^\mu, \tilde{u}^{\nu *}, \tilde{u}$. In the same spirit, one can cover S^{2n}_θ by two "charts" with domain $\mathbb{R}^{2n}_\theta$ with transition on $\mathbb{R}^{2n}_\theta \backslash \{0\}$, ($z^\mu = 0, \bar{z}^\nu = 0$ being a classical point of $\mathbb{R}^{2n}_\theta$).

Let $C_{\text{alg}}(S^{2n-1}_\theta)$ be the quotient of the $*$-algebra $C_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ by the two-sided ideal generated by the element $\sum^n_{\mu=1} z^\mu \bar{z}^\mu - \mathbb{1}$ of the center of $C_{\text{alg}}(\mathbb{R}^{2n}_\theta)$. This defines by duality the noncommutative $(2n-1)$-sphere S^{2n-1}_θ. In the following, we shall denote by v^μ, $\bar{v}^\nu$ the canonical images of z^μ, $\bar{z}^\nu$ in $C_{\text{alg}}(S^{2n-1}_\theta)$. Again there is a greatest C^*-seminorm which is a norm on $C_{\text{alg}}(S^{2n-1}_\theta)$; the C^*-algebra obtained by completion will be refered to as *the algebra of continuous functions on the noncommutative* $(2n-1)$*-sphere* S^{2n-1}_θ. It is clear that, in an obvious sense, S^{2n}_θ is the suspension of S^{2n-1}_θ.

As for the case of $\mathbb{R}^{2n}_\theta$, one has an action σ of T^n on $\mathbb{R}^{2n+1}_\theta$, S^{2n}_θ and S^{2n-1}_θ which is induced by an action on the corresponding classical spaces. More precisely the group-homomorphism $s \mapsto \sigma_s$ of T^n into $\text{Aut}(C_{\text{alg}}(\mathbb{R}^{2n}_\theta))$ extends as a group-homomorphism $s \mapsto \sigma_s$ of T^n into $\text{Aut}(C_{\text{alg}}(\mathbb{R}^{2n+1}_\theta))$ and these group-homomorphisms induce group homomorphisms $s \mapsto \sigma_s$ of T^n into $\text{Aut}(C_{\text{alg}}(S^{2n-1}_\theta))$ and of T^n into $\text{Aut}(C_{\text{alg}}(S^{2n}_\theta))$. As for $\mathbb{R}^{2n}_\theta$, one checks that the subalgebras of σ-invariant elements are in the respective centers, are not deformed, and are isomorphic to the subalgebras of σ-invariant elements of $C_{\text{alg}}(\mathbb{R}^{2n+1})$, $C_{\text{alg}}(S^{2n})$ and $C_{\text{alg}}(S^{2n-1})$ respectively.

In order to formulate the last part of the next theorem, let us notice that, in view of (4.5) and (4.6), there is an injective representation of $\text{Cliff}(\mathbb{R}^{2n}_\theta)$ for which $\gamma = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ where $\mathbb{1}$ denotes the unit of $M_{2^{n-1}}(\mathbb{C})$. In such a representation one has in view of (4.6)

$$\Gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^{\mu *} & 0 \end{pmatrix}, \quad \Gamma^{\mu *} = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^{\mu *} & 0 \end{pmatrix}$$

where σ^μ and $\bar{\sigma}^\mu$ are in $M_{2^{n-1}}(\mathbb{C})$.

THEOREM 4 (*i*) *One obtains a hermitian projection* $e \in M_{2^n}(C_{\text{alg}}(S^{2n}_\theta))$ *by setting* $e = \frac{1}{2}(\mathbb{1} + \sum^n_{\mu=1}(\Gamma^{\mu *} u^\mu + \Gamma^\mu u^{\mu *}) + \gamma u)$. *Furthermore one has* $\text{ch}_m(e) = 0$ *for* $0 \leq m \leq n-1$.

(*ii*) *One obtains a unitary* $U \in M_{2^{n-1}}(C_{\text{alg}}(S^{2n-1}_\theta))$ *by setting* $U = \sum^n_{\mu=1}(\bar{\sigma}^\mu v^\mu + \sigma^\mu \bar{v}^\mu)$ *, where* σ^μ *and* $\bar{\sigma}^\mu$ *are as above. Furthermore one has* $\text{ch}_{m-\frac{1}{2}}(U) = 0$ *for* $1 \leq m \leq n-1$.

The relation $e = e^*$ is obvious. It follows from Lemma 5 that

$$\left(\sum^n_{\mu=1} (\Gamma^{\mu *} z^\mu + \Gamma^\mu z^{\mu *}) \right)^2 = \sum^n_{\mu=1} z^\mu \bar{z}^\mu,$$

which in terms of the σ^μ reads

$$(\bar\sigma^\mu z^\mu + \sigma^\mu \bar z^\mu)(\bar\sigma^\mu z^\mu + \sigma^\mu \bar z^\mu)^* = (\bar\sigma^\mu z^\mu + \sigma^\mu \bar z^\mu)^*(\bar\sigma^\mu z^\mu + \sigma^\mu \bar z^\mu) = \sum_{\mu=1}^n z^\mu \bar z^\mu.$$

On the other hand relations (4.6) imply then

$$\left(\sum_{\mu=1}^n (\Gamma^{\mu *} z^\mu + \Gamma^\mu z^{\mu *}) + \gamma x\right)^2 = \sum_{\mu=1}^n z^\mu \bar z^\mu + x^2$$

which reduces to $\mathbb{1} \in M_{2^n}(C_{\mathrm{alg}}(S^{2n}_\theta))$. This is equivalent to $e^2 = e$. Using again Lemma 5, $\mathrm{ch}_m(e) = 0$ for $m < n$ follows from the vanishing of the corresponding traces of products of the $\Gamma^\mu, \Gamma^{\mu *}, \gamma$ in the representation of Proposition 1. The unitarity of $U \in M_{2^{n-1}}(C_{\mathrm{alg}}(S^{2n-1}_\theta))$ is clear whereas one has

$$\mathrm{ch}_{m-\frac{1}{2}}(U) = \mathrm{tr}\left((U \circledcirc U^*)^{\circledcirc m} - (U^* \circledcirc U)^{\circledcirc m}\right) \tag{5.1}$$

which implies

$$\mathrm{ch}_{m-\frac{1}{2}}(U) = \mathrm{tr}\left(\frac{1+\gamma}{2}\Gamma^{\circledcirc 2m} - \frac{1-\gamma}{2}\Gamma^{\circledcirc 2m}\right) = \mathrm{tr}(\gamma\Gamma^{\circledcirc 2m}) \tag{5.2}$$

where $\Gamma = \sum_\mu (\Gamma^{\mu *} v^\mu + \Gamma^\mu \bar v^\mu) \in M_{2^n}(C_{\mathrm{alg}}(S^{2n-1}_\theta))$ and where in (5.2) tr and $\circledcirc$ are taken for M_{2^n} instead of $M_{2^{n-1}}$ as in (5.1), (see the definitions at the end of the introduction). It follows from (5.2) that one has $\mathrm{ch}_{m-\frac{1}{2}}(U) = 0$ for $1 \leq m \leq n-1$ for the same reasons as $\mathrm{ch}_m(e) = 0$ for $m \leq n-1$.

This theorem combined with the last theorem of Section 12 and the last theorem of Section 13 implies that S^m_θ is an m-dimensional noncommutative spherical manifold.

It follows from $\mathrm{ch}_m(e) = 0$ for $0 \leq m \leq n-1$ that $\mathrm{ch}_n(e)$ is a Hochschild cycle which corresponds to the volume form on S^{2n}_θ. In fact it is obvious that the whole analysis of Section III and IV of [18] generalizes from S^4_θ to S^{2n}_θ. This is in particular the case of Theorem 3 of [18] (with the appropriate changes e.g. $4 \mapsto 2n$ and $M_4(\mathbb{C}) \mapsto M_{2^n}(\mathbb{C})$). The odd case is obviously similar. This will be discussed in more details in Section 13.

The projection e is a noncommutative version of the projection-valued field P_+ on the sphere S^{2n} described in Section 2.7 of [22]; one has $P_+ = e|_{\theta=0}$. As was shown there, P_+ satisfies the following self-duality equation

$$*P_+(dP_+)^n = i^n P_+(dP_+)^n \tag{5.3}$$

where $*$ is the usual Hodge duality of forms on S^{2n}. Since $*$ is conformally invariant on forms of degree n, this equation is conformally invariant. The above equation generalizes to e (i.e. on S^{2n}_θ) once the appropriate differential

calculus and metric are defined, (see Theorem 6 of section 12 below). For n even, Equation (5.3) describes an intanton (the "round" one) for a conformally invariant generalization of the classical Yang-Mills action on S^{2n} (which reduces to the Yang-Mills action on S^4), [22]. The fact, which was pointed out and used in [25], that classical gauge theory can be formulated in terms of projection-valued fields is a direct consequence of the theorem of Narasimhan and Ramanan on the existence of universal connections [41], [42], (see also in [21] for a short economical proof of this theorem).
It is clear that by changing (u^μ, u) into $(-u^\mu, -u)$ one also obtains a hermitian projection $e_- \in M_{2^n}(C_{\text{alg}}(S^{2n}_\theta))$ satisfying $\text{ch}_m(e_-) = 0$ for $0 \leq m \leq n-1$. For $\theta = 0$, e_- coincides with the projection-valued field P_- on S^{2n} of [22] which satisfies $*P_-(dP_-)^n = -i^n P_-(dP_-)^n$. What replaces $e \mapsto e_-$ for the odd-dimensional case is $U \mapsto U^*$.

6 The graded differential algebras $\Omega_{\text{alg}}(\mathbb{R}^m_\theta)$ and $\Omega_{\text{alg}}(S^m_\theta)$

There are canonical differential calculi, $\Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ and $\Omega_{\text{alg}}(\mathbb{R}^{2n+1}_\theta)$, on the noncommutative planes $\mathbb{R}^{2n}_\theta$ and $\mathbb{R}^{2n+1}_\theta$, which are deformations of the differential algebras of polynomial differential forms on $\mathbb{R}^{2n}$ and $\mathbb{R}^{2n+1}$ and which are such that the $z^\mu \bar z^\mu = \bar z^\mu z^\mu$ are in the center of $\Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ and $\Omega_{\text{alg}}(\mathbb{R}^{2n+1}_\theta)$ as well as x in the case $\Omega_{\text{alg}}(\mathbb{R}^{2n+1}_\theta)$. Let us first give a detailed description of the graded differential algebra $\Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)$.

As a complex unital associative graded algebra $\Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta) = \oplus_{p\in\mathbb{N}}\Omega^p_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ is generated by $2n$ elements $z^\mu, \bar z^\nu$ of degree 0 with relations (4.1) and by $2n$ elements $dz^\mu, d\bar z^\nu$ of degree 1 with relations

$$dz^\mu dz^\nu + \lambda^{\mu\nu} dz^\nu dz^\mu = 0,\ d\bar z^\mu d\bar z^\nu + \lambda^{\mu\nu} d\bar z^\nu d\bar z^\mu = 0,\ d\bar z^\mu dz^\nu + \lambda^{\nu\mu} dz^\nu d\bar z^\mu = 0 \tag{6.1}$$

$$z^\mu dz^\nu = \lambda^{\mu\nu} dz^\nu z^\mu,\ \bar z^\mu d\bar z^\nu = \lambda^{\mu\nu} d\bar z^\nu \bar z^\mu,\ \bar z^\mu dz^\nu = \lambda^{\nu\mu} dz^\nu \bar z^\mu,\ z^\mu d\bar z^\nu = \lambda^{\nu\mu} d\bar z^\nu z^\mu \tag{6.2}$$

for any $\mu, \nu \in \{1, \dots, n\}$. There is a unique differential d of $\Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)$, (i.e. a unique antiderivation d satisfying $d^2 = 0$), which extends the mapping $z^\mu \mapsto dz^\mu$, $\bar z^\nu \mapsto d\bar z^\nu$. One extends $z^\mu \mapsto \bar z^\mu$, $dz^\nu \mapsto d\bar z^\nu = \overline{(dz^\nu)}$ as an antilinear involution $\omega \mapsto \bar\omega$ of $\Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ such that $\overline{\omega\omega'} = (-1)^{pp'}\bar\omega'\bar\omega$ for $\omega \in \Omega^p_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ and $\omega' \in \Omega^{p'}_{\text{alg}}(\mathbb{R}^{2n}_\theta)$. One has $d\bar\omega = \overline{d\omega}$, $\forall \omega \in \Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)$. Elements $\omega \in \Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ satisfying $\omega = \bar\omega$ will be refered to as *real elements*. Notice that the $\bar z^\mu z^\mu$, $\bar z^\mu dz^\mu$, $z^\mu d\bar z^\mu$, $d\bar z^\mu dz^\mu$ for $\mu \in \{1, \dots, n\}$ generate a graded differential subalgebra of the graded center of $\Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ which coincides with this graded center whenever θ is generic. Notice also that these elements are invariant by the canonical extension to $\Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ of the action σ of T^n on $C_{\text{alg}}(\mathbb{R}^{2n}_\theta) = \Omega^0_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ (see the end of this section).

There is another useful way to construct $\Omega_{\rm alg}(\mathbb{R}^{2n}_\theta)$ which we now describe. Consider the graded algebra $C_{\rm alg}(\mathbb{R}^{2n}_\theta)\otimes_{\mathbb{R}}\wedge\mathbb{R}^{2n} = C_{\rm alg}(\mathbb{R}^{2n}_\theta)\otimes\wedge_c\mathbb{R}^{2n}$ where $\wedge_c\mathbb{R}^{2n}$ is the complexified exterior algebra of $\mathbb{R}^{2n}$. The graded algebra $C_{\rm alg}(\mathbb{R}^{2n}_\theta)\otimes\wedge_c\mathbb{R}^{2n}$ is the unital complex graded algebra generated by $2n$ elements of degree zero, $z^\mu, \bar z^\nu$ $(\mu,\nu=1,\dots,n)$ satisfying relations (4.1) and by $2n$ elements of degree one, ξ^μ, $\bar\xi^\nu$ $(\mu,\nu=1,\dots,n)$ with relations

$$\xi^\mu\xi^\nu+\xi^\nu\xi^\mu=0, \bar\xi^\mu\bar\xi^\nu+\bar\xi^\nu\bar\xi^\mu=0, \bar\xi^\mu\xi^\nu+\xi^\nu\bar\xi^\mu=0 \tag{6.3}$$

$$z^\mu\xi^\nu=\xi^\nu z^\mu,\ \bar z^\mu\xi^\nu=\xi^\nu\bar z^\mu, z^\mu\bar\xi^\nu=\bar\xi^\nu z^\mu,\ \bar z^\mu\bar\xi^\nu=\bar\xi^\nu\bar z^\mu \tag{6.4}$$

for $\mu,\nu\in\{1,\dots,n\}$. The $2n$ elements $\xi^\mu,\bar\xi^\nu$ satisfying (6.3) generate the complexified exterior algebra $\wedge_c\mathbb{R}^{2n}$. An involution $\omega\mapsto\bar\omega$ of graded algebra on $C_{\rm alg}(\mathbb{R}^{2n}_\theta)\otimes\wedge_c\mathbb{R}^{2n}$ is obtained by setting $\overline{z^\mu}=\bar z^\mu, \overline{\bar z^\mu}=z^\mu$ as before and by setting $\overline{\xi^\mu}=\bar\xi^\mu$, $\overline{\bar\xi^\mu}=\xi^\mu$. There is a unique differential d on the graded differential algebra $C_{\rm alg}(\mathbb{R}^{2n}_\theta)\otimes\wedge_c\mathbb{R}^{2n}$ such that

$$d\xi^\mu=0 \quad , \quad d\bar\xi^\mu=0 \tag{6.5}$$

$$dz^\mu=z^\mu\xi^\mu \quad , \quad d\bar z^\mu=\bar z^\mu\bar\xi^\mu \tag{6.6}$$

for $\mu=1,\dots,n$. One then has $d\bar\omega=\overline{d\omega}$ for any $\omega\in C_{\rm alg}(\mathbb{R}^{2n}_\theta)\otimes\wedge_c\mathbb{R}^{2n}$. It is readily verified that the $dz^\mu, d\bar z^\nu$ defined by (6.6) satisfy relations (6.1) to (6.2). In other words $\Omega_{\rm alg}(\mathbb{R}^{2n}_\theta)$ is the differential subalgebra of $C_{\rm alg}(\mathbb{R}^{2n}_\theta)\otimes\wedge_c\mathbb{R}^{2n}$ generated by the $z^\mu,\bar z^\nu$ $(\mu,\nu=1,\dots,n)$. Furthermore the involution $\omega\mapsto\bar\omega$ of $C_{\rm alg}(\mathbb{R}^{2n}_\theta)\otimes\wedge_c\mathbb{R}^{2n}$ induces on $\Omega_{\rm alg}(\mathbb{R}^{2n}_\theta)$ the previously defined involution. As $C_{\rm alg}(\mathbb{R}^{2n}_\theta)$-bimodule, one has $\Omega^p_{\rm alg}(\mathbb{R}^{2n}_\theta)\subset C_{\rm alg}(\mathbb{R}^{2n}_\theta)\otimes\wedge^p_c\mathbb{R}^{2n}$ so that $\Omega^p_{\rm alg}(\mathbb{R}^{2n}_\theta)$ is a sub-bimodule of the diagonal bimodule $(C_{\rm alg}(\mathbb{R}^{2n}_\theta))^{C^p_{2n}}$, thus the $\Omega^p_{\rm alg}(\mathbb{R}^{2n}_\theta)$ are *diagonal bimodules* over $C_{\rm alg}(\mathbb{R}^{2n}_\theta)$ [28]. This implies in particular that $\Omega_{\rm alg}(\mathbb{R}^{2n}_\theta)$ is a quotient of the graded differential algebra $\Omega_{\rm Diag}(C_{\rm alg}(\mathbb{R}^{2n}_\theta))$ [24].

The differential algebra $C_{\rm alg}(\mathbb{R}^{2n}_\theta)\otimes\wedge_c\mathbb{R}^{2n}$ has the following interpretation. Let us "suppress" the classical points $z^\mu=0$ $(\mu=1,\dots,n)$ of $\mathbb{R}^{2n}_\theta$ by adjoining n real (hermitian) central generators of degree zero $|z^\mu|^{-2}$ to $\Omega_{\rm alg}(\mathbb{R}^{2n}_\theta)$ with relations

$$\bar z^\mu z^\mu|z^\mu|^{-2}=|z^\mu|^{-2}\bar z^\mu z^\mu=\mathbb{1}$$

for $\mu=1,\dots,n$. This becomes a graded differential algebra $\tilde\Omega_{\rm alg}(\mathbb{R}^{2n}_\theta)$ if one sets $d|z^\mu|^{-2}=-(|z^\mu|^{-2})^2 d(\bar z^\mu z^\mu)$ for $\mu=1,\dots,n$.
Then the algebra $C_{\rm alg}(\mathbb{R}^{2n}_\theta)\otimes\wedge_c\mathbb{R}^{2n}$ is the subalgebra generated by the $z^\mu,\bar z^\nu$ and the $\xi^\mu=|z^\mu|^{-2}\bar z^\mu dz^\mu$, $\bar\xi^\nu=|z^\nu|^{-2}z^\nu d\bar z^\nu$ and it is a graded differential subalgebra of $\tilde\Omega_{\rm alg}(\mathbb{R}^{2n}_\theta)$. The algebra $\tilde\Omega_{\rm alg}(\mathbb{R}^{2n}_\theta)$ is the θ-deformation of the algebra of complex polynomial differential forms on $(\mathbb{C}\backslash\{0\})^n\subset\mathbb{R}^{2n}$.
The complex unital associative graded algebra $\Omega_{\rm alg}(\mathbb{R}^{2n+1}_\theta)$ is defined as the graded tensor product $\Omega_{\rm alg}(\mathbb{R}^{2n}_\theta)\otimes_{\rm gr}\Omega_{\rm alg}(\mathbb{R})$. More concretely one adjoins to $\Omega_{\rm alg}(\mathbb{R}^{2n}_\theta)$ one generator x of degree zero and one generator dx of degree one with relations

$$xdx=dxx,\ x\omega=\omega x,\ dx\omega=(-1)^p\omega dx \tag{6.7}$$

for $\omega \in \Omega^p_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$. One extends the differential d of $\Omega_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ as the unique differential d of $\Omega_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta)$ mapping x on dx. The graded involution of $\Omega_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ is extended into a graded involution $\omega \mapsto \bar{\omega}$ of $\Omega_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta)$ by setting $\bar{x} = x$ and $\overline{dx} = dx$. One has again $d\bar{\omega} = \overline{d\omega}$ for $\omega \in \Omega_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta)$.

Again $\Omega_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta)$ is the differential subalgebra of $C_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta) \otimes \wedge_c \mathbb{R}^{2n+1}$ generated by the $z^\mu, \bar{z}^\nu, x$ where the $(2n+1)$-th basis element of $\mathbb{R}^{2n+1}$ is identified with dx i.e. $C_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta) \otimes \wedge_c \mathbb{R}^{2n+1} \simeq (C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta) \otimes \wedge_c \mathbb{R}^{2n}) \otimes \wedge(x, dx)$. Thus again the $\Omega^p_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta)$ are diagonal bimodules over $C_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta)$ which implies that $\Omega_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta)$ is a quotient of $\Omega_{\mathrm{Diag}}(C_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta))$. Notice that these identifications are compatible with the involutions of the corresponding graded differential algebras.

Let now $\Omega_{\mathrm{alg}}(S^{2n-1}_\theta)$ be the graded differential algebra quotient of $\Omega_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ by the differential two-sided ideal generated by $\sum^n_{\mu=1} z^\mu \bar{z}^\mu - \mathbb{1}$ and similarly $\Omega_{\mathrm{alg}}(S^{2n}_\theta)$ be the quotient of $\Omega_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta)$ by the differential two-sided ideal generated by $\sum^n_{\mu=1} z^\mu \bar{z}^\mu + x^2 - \mathbb{1}$. These are again graded-involutive algebras with real differentials. Furthermore, it will be shown using the splitting homomorphism that they are diagonal bimodules over $C_{\mathrm{alg}}(S^{2n-1}_\theta)$ and over $C_{\mathrm{alg}}(S^{2n}_\theta)$ repectively from which it follows that they are quotient of $\Omega_{\mathrm{Diag}}(C_{\mathrm{alg}}(S^{2n-1}_\theta))$ and of $\Omega_{\mathrm{Diag}}(C_{\mathrm{alg}}(S^{2n}_\theta))$ respectively.

Let $m = 2n$ or $2n+1$. The actions $s \mapsto \sigma_s$ of T^n on $C_{\mathrm{alg}}(\mathbb{R}^m_\theta)$ and $C_{\mathrm{alg}}(S^{m-1}_\theta)$ extend canonically to actions of T^n as automorphisms of graded-involutive differential algebras, $s \mapsto \sigma_s \in \mathrm{Aut}(\Omega_{\mathrm{alg}}(\mathbb{R}^m_\theta))$, and $s \mapsto \sigma_s \in \mathrm{Aut}(\Omega_{\mathrm{alg}}(S^{m-1}_\theta))$. The differential subalgebras $\Omega_{\mathrm{alg}}(\mathbb{R}^m_\theta)^\sigma$ and $\Omega_{\mathrm{alg}}(S^{m-1}_\theta)^\sigma$ of σ-invariant elements are in the graded centers of $\Omega_{\mathrm{alg}}(\mathbb{R}^m_\theta)$ and $\Omega_{\mathrm{alg}}(S^{m-1}_\theta)$ and they are undeformed, i.e. isomorphic to the corresponding subalgebras $\Omega_{\mathrm{alg}}(\mathbb{R}^m)^\sigma$ and $\Omega_{\mathrm{alg}}(S^{m-1})^\sigma$of $\Omega_{\mathrm{alg}}(\mathbb{R}^m)$ and $\Omega_{\mathrm{alg}}(S^{m-1})$.

7 The quantum groups $GL_\theta(m, \mathbb{R})$, $SL_\theta(m, \mathbb{R})$ and $GL_\theta(n, \mathbb{C})$

In this section we shall give a concrete explicit description of the various quantum groups of symmetries of the noncommutative spaces $\mathbb{R}^m_\theta$ and $\mathbb{C}^n_\theta$ for $m \geq 4$ and $n \geq 2$. There are other approaches to quantum groups of symmetries of S^4_θ and $\mathbb{R}^4_\theta$ and some generalizations [49], [54], [3]. In [49] the dual point of view is adopted and what is produced is the deformation of the universal enveloping algebra whereas in [54] the deformation is on the same side of the duality as developed here; both points of view are of course useful. However it must be stressed that, beside the fact that our approach is closely related to the differential calculus, the important point here is the observation that the quantum groups we introduce arise with their expected Hochschild dimensions which equals the dimensions of the corresponding classical groups. They are deformations (called *θ-deformations*) of the classical groups $GL(m,\mathbb{R})$, $SL(m,\mathbb{R})$, $GL(n,\mathbb{C})$ and as will be shown in Section 12, the Hochschild dimension is an invariant of these deformations. It is worth noticing here that there is no corresponding θ-deformation of $SL(n,\mathbb{C})$; the reason being that $dz^1\cdots dz^n$ is not central and not σ-invariant in $\Omega_{\mathrm{alg}}(\mathbb{C}^n_\theta)=\Omega_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$. However, there is a θ-deformation of the subgroup of $GL(n,\mathbb{C})$ consisting of matrices with determinants of modulus one because $dz^1\cdots dz^n d\bar z^1\cdots d\bar z^n$ is σ-invariant and (consequently) central.

Let $M_\theta(2n,\mathbb{R})$ be the unital associative $\mathbb{C}$-algebra generated by $4n^2$ element $a^\mu_\nu, b^\mu_\nu, \bar a^\mu_\nu, \bar b^\mu_\nu$ $(\mu,\nu=1,\dots,n)$ with relations such that the elements y^μ, $\bar y^\mu$, ζ^μ, $\bar\zeta^\mu$ of $M_\theta(2n,\mathbb{R})\otimes\Omega_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ defined by

$$y^\mu=a^\mu_\nu\otimes z^\nu+b^\mu_\nu\otimes\bar z^\nu,\ \ \bar y^\mu=\bar a^\mu_\nu\otimes\bar z^\nu+\bar b^\mu_\nu\otimes z^\nu,$$

$$\zeta^\mu=a^\mu_\nu\otimes dz^\nu+b^\mu_\nu\otimes d\bar z,\ \ \bar\zeta^\mu=\bar a^\mu_\nu\otimes d\bar z^\nu+\bar b^\mu_\nu\otimes dz^\nu$$

satisfy the relation

$$y^\mu y^\nu=\lambda^{\mu\nu}y^\nu y^\mu,\ \bar y^\mu\bar y^\nu=\lambda^{\mu\nu}\bar y^\nu\bar y^\mu,\ \bar y^\mu y^\nu=\lambda^{\nu\mu}y^\nu\bar y^\mu,$$

$$\zeta^\mu\zeta^\nu+\lambda^{\mu\nu}\zeta^\nu\zeta^\mu=0,\ \bar\zeta^\mu\bar\zeta^\nu+\lambda^{\mu\nu}\bar\zeta^\nu\bar\zeta^\mu=0,\ \bar\zeta^\mu\zeta^\nu+\lambda^{\nu\mu}\zeta^\nu\bar\zeta^\mu=0.$$

There is a unique $*$-algebra involution $a\mapsto a^*$ on $M_\theta(2n,\mathbb{R})$ such that $(a^\mu_\nu)^*=\bar a^\mu_\nu$, $(b^\mu_\nu)^*=\bar b^\mu_\nu$. The relations between the generators are easy to write explicitely, they read

$$a^\mu_\nu a^\tau_\rho=\lambda^{\mu\tau}\lambda_{\rho\nu}a^\tau_\rho a^\mu_\nu\quad,\quad a^\mu_\nu\bar a^\tau_\rho=\lambda^{\tau\mu}\lambda_{\nu\rho}\bar a^\tau_\rho a^\mu_\nu \tag{7.1}$$
$$a^\mu_\nu b^\tau_\rho=\lambda^{\mu\tau}\lambda_{\nu\rho}b^\tau_\rho a^\mu_\nu\quad,\quad a^\mu_\nu\bar b^\tau_\rho=\lambda^{\tau\mu}\lambda_{\rho\nu}\bar b^\tau_\rho a^\mu_\nu \tag{7.2}$$
$$b^\mu_\nu b^\tau_\rho=\lambda^{\mu\tau}\lambda_{\rho\nu}b^\tau_\rho b^\mu_\nu\quad,\quad b^\mu_\nu\bar b^\tau_\rho=\lambda^{\tau\mu}\lambda_{\nu\rho}\bar b^\tau_\rho b^\mu_\nu \tag{7.3}$$

plus the relations obtained by hermitian conjugation, where we have also used the notation $\lambda_{\nu\rho}$ for $\lambda^{\nu\rho}$ to indicate that there is no summation in the above formulas. This $*$-algebra becomes a $*$-bialgebra with coproduct Δ and counit ε if we endow it with the unique algebra-homomorphism

$$\Delta: M_\theta(2n,\mathbb{R})\to M_\theta(2n,\mathbb{R})\otimes M_\theta(2n,\mathbb{R})$$

and the unique character $\varepsilon: M_\theta(2n,\mathbb{R})\to\mathbb{C}$ such that

$$\Delta a^\mu_\nu = a^\mu_\lambda\otimes a^\lambda_\nu+b^\mu_\lambda\otimes\bar b^\lambda_\nu,\ \ \varepsilon(a^\mu_\nu)=\delta^\mu_\nu \tag{7.4}$$
$$\Delta\bar a^\mu_\nu = \bar a^\mu_\lambda\otimes\bar a^\lambda_\nu+\bar b^\mu_\lambda\otimes b^\lambda_\nu,\ \ \varepsilon(\bar a^\mu_\nu)=\delta^\mu_\nu \tag{7.5}$$
$$\Delta b^\mu_\nu = a^\mu_\lambda\otimes b^\lambda_\nu+b^\mu_\lambda\otimes\bar a^\lambda_\nu,\ \ \varepsilon(b^\mu_\nu)=0 \tag{7.6}$$
$$\Delta\bar b^\mu_\nu = \bar a^\mu_\lambda\otimes\bar b^\lambda_\nu+\bar b^\mu_\lambda\otimes a^\lambda_\nu,\ \ \varepsilon(\bar b^\mu_\nu)=0 \tag{7.7}$$

for any $\mu,\nu\in\{1,\dots,n\}$. It is easy to verify that there is a unique algebra-homomorphism $\delta:\Omega_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)\to M_\theta(2n,\mathbb{R})\otimes\Omega_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ such that $\delta z^\mu=y^\mu$,

$\delta\bar z^\mu = \bar y^\mu$, $\delta dz^\mu = \zeta^\mu$, $\delta d\bar z^\mu = \bar\zeta^\mu$ and that this is furthermore a graded-involutive algebra-homomorphism. In fact, this is another way to obtain $\Omega_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ starting from $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ and from the θ-twisted complexified exterior algebra $\wedge_c\mathbb{R}^{2n}_\theta$ generated by the $dz^\mu, d\bar z^\nu$ satisfying (6.1). One has

$$(\Delta\otimes I)\circ\delta = (I\otimes\delta)\circ\delta,\quad (\varepsilon\otimes I)\circ\delta = I \tag{7.8}$$

and $\delta\Omega^p_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)\subset M_\theta(2n,\mathbb{R})\otimes\Omega^p_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$, $\forall p\in\mathbb{N}$.
One has of course $\delta C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)\subset M_\theta(2n,\mathbb{R})\otimes C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$, (this is the previous result for $p=0$ since $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)=\Omega^0_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$), and $\delta\wedge_c\mathbb{R}^{2n}_\theta\subset M_\theta(2n,\mathbb{R})\otimes\wedge_c\mathbb{R}^{2n}_\theta$ with $\delta\wedge^p_c\mathbb{R}^{2n}_\theta\subset M_\theta(2n,\mathbb{R})\otimes\wedge^p_c\mathbb{R}^{2n}_\theta$ for any $p\in\mathbb{N}$. Since $\wedge^{2n}_c\mathbb{R}^{2n}_\theta$ is of dimension 1 and spanned by $d\bar z^1dz^1\dots d\bar z^ndz^n=\prod^n_{\mu=1}d\bar z^\mu dz^\mu$, it follows that one defines an element $\det_\theta\in M_\theta(2n,\mathbb{R})$ by setting

$$\delta\prod^n_{\mu=1}d\bar z^\mu dz^\mu = \det_\theta\otimes\prod^n_{\mu=1}d\bar z^\mu dz^\mu \tag{7.9}$$

which satisfies

$$\begin{aligned}\Delta\det_\theta &= \det_\theta\otimes\det_\theta &(7.10)\\ \varepsilon(\det_\theta) &= 1 &(7.11)\end{aligned}$$

and from the fact that $\prod^n_{\mu=1}d\bar z^\mu dz^\mu$ is central in $\Omega_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ and from the very definition of $M_\theta(2n,\mathbb{R})$ it also follows that $\det_\theta$ *belongs to the center of* $M_\theta(2n,\mathbb{R})$. The element $\det_\theta$ of $M_\theta(2n,\mathbb{R})$ is clearly hermitian, $(\det_\theta)^*=\det_\theta$.

<u>Remark.</u> It is worth noticing that Relations (7.1), (7.2), (7.3) and their hermitian conjugate are the quadratic relations associated with a R-matrix $\hat R$ satisfying the braid equation (Yang-Baxter) and which is of square equal to 1, (i.e. $\hat R$ represents an elementary transposition). In other words, the bialgebra $M_\theta(2n,\mathbb{R})$ is the bialgebra of the R-matrix $\hat R$.

Let $C_{\mathrm{alg}}(GL_\theta(2n,\mathbb{R}))$ be the $*$-bialgebra obtained by adding to $M_\theta(2n,\mathbb{R})$ a hermitian central element $\det^{-1}_\theta$ with relation $\det_\theta.\det^{-1}_\theta=\mathbb{1}=\det^{-1}_\theta.\det_\theta$ and by setting $\Delta\det^{-1}_\theta=\det^{-1}_\theta\otimes\det^{-1}_\theta$ and $\varepsilon(\det^{-1}_\theta)=1$. It is not hard (but cumbersome) to see that the introduction of $\det^{-1}_\theta$ allows to invert the $(2n,2n)$ matrix $L=\begin{pmatrix}A & B\\ \bar B & \bar A\end{pmatrix}$ in $M_{2n}(C_{\mathrm{alg}}(GL_\theta(2n,\mathbb{R}))$ and to obtain an antipode S on $C_{\mathrm{alg}}(GL_\theta(2n,\mathbb{R}))$ which of course satisfies $S(\det_\theta)=\det^{-1}_\theta$ and $S(\det^{-1}_\theta)=\det_\theta$. Thus $C_{\mathrm{alg}}(GL_\theta(2n,\mathbb{R}))$ *is a $*$-Hopf algebra* and the quantum group $GL_\theta(2n,\mathbb{R})$ is defined to be the dual object.
The quotient $C_{\mathrm{alg}}(SL_\theta(2n,\mathbb{R}))$ of $M_\theta(2n,\mathbb{R})$ by the relation $\det_\theta=\mathbb{1}$ is also the quotient of $C_{\mathrm{alg}}(GL_\theta(2n,\mathbb{R})$ by the two-sided ideal generated by $\det_\theta-\mathbb{1}$ and $\det^{-1}_\theta-\mathbb{1}$ which is a $*$-Hopf ideal. So $C_{\mathrm{alg}}(SL_\theta(2n,\mathbb{R}))$ is again a $*$-Hopf algebra which defines the quantum group $SL_\theta(2n,\mathbb{R})$ by duality.
Replacing $\Omega_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ by $\Omega_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta)$ one defines in a similar way the $*$-bialgebra

$M_\theta(2n+1,\mathbb{R})$, the $*$-Hopf algebras $C_{\rm alg}(GL_\theta(2n+1,\mathbb{R}))$, $C_{\rm alg}(SL_\theta(2n+1,\mathbb{R}))$ and therefore the quantum groups $GL_\theta(2n+1,\mathbb{R})$ and $SL_\theta(2n+1,\mathbb{R})$.

Finally, we let $C_{\rm alg}(GL_\theta(n,\mathbb{C}))$ be the quotient of $C_{\rm alg}(GL_\theta(2n,\mathbb{R}))$ by the ideal generated by the b^μ_ν and the $\bar{b}^\mu_\nu$ which is a $*$-Hopf ideal. The coaction of the corresponding Hopf algebra on $\Omega_{\rm alg}(\mathbb{C}^n_\theta)$ is straightforwardly obtained. This defines the quantum group $GL_\theta(n,\mathbb{C})$ and its action on $\mathbb{C}^n_\theta$. The ideal generated by the image of $\det_\theta - \mathbb{1}$ in $C_{\rm alg}(GL_\theta(n,\mathbb{C}))$ is a $*$-Hopf ideal and the corresponding quotient Hopf algebra defines by duality a quantum group which is a deformation (θ-deformation) of the subgroup of $GL(n,\mathbb{C})$ which consists of matrices with determinants of modulus one.

8 The quantum groups $O_\theta(m)$, $SO_\theta(m)$ and $U_\theta(n)$

Let $C_{\rm alg}(O_\theta(2n))$ be the quotient of $M_\theta(2n,\mathbb{R})$ by the two-sided ideal generated by

$$\sum^n_{\mu=1}(\bar{a}^\mu_\alpha a^\mu_\beta + b^\mu_\alpha \bar{b}^\mu_\beta) - \delta_{\alpha\beta}\mathbb{1},\ \sum^n_{\mu=1}(\bar{a}^\mu_\alpha b^\mu_\beta + b^\mu_\alpha \bar{a}^\mu_\beta),\ \sum^n_{\mu=1}(\bar{b}^\mu_\alpha a^\mu_\beta + a^\mu_\alpha \bar{b}^\mu_\beta)$$

for $\alpha,\beta = 1,\dots,n$. This ideal is $*$-invariant and is also a coideal. It follows that $C_{\rm alg}(O_\theta(2n))$ is again a $*$-bialgebra. Furthermore, one can show that $(\det_\theta)^2 - \mathbb{1}$ is in the above ideal (see below) so $C_{\rm alg}(O_\theta(2n))$ is a $*$-Hopf algebra which is a quotient of $C_{\rm alg}(GL_\theta(2n,\mathbb{R}))$. One verifies that the homomorphism $\delta : \Omega_{\rm alg}(\mathbb{R}^{2n}_\theta) \to M_\theta(2n,\mathbb{R}) \otimes \Omega_{\rm alg}(\mathbb{R}^{2n}_\theta)$ yields a homomorphism

$$\delta_R : \Omega_{\rm alg}(\mathbb{R}^{2n}_\theta) \to C_{\rm alg}(O_\theta(2n)) \otimes \Omega_{\rm alg}(\mathbb{R}^{2n}_\theta)$$

of graded-involutive algebras. This yields the quantum group $O_\theta(2n)$ which is a deformation of the group of rotations in dimension $2n$ and its action on $\mathbb{R}^{2n}_\theta$ (cf. [3]). Indeed one has

$$\delta_R(\sum^n_{\mu=1} \bar{z}^\mu z^\mu) = \mathbb{1} \otimes (\sum^n_{\mu=1} \bar{z}^\mu z^\mu)$$

by the very definition of $C_{\rm alg}(O_\theta(2n))$. One can notice here that $C_{\rm alg}(O_\theta(2n))$ is a quotient of the Hopf algebra of the quantum group of the non-degenerate bilinear form B on $\mathbb{C}^{2n}$ with matrix $\begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}$ defined in [26], the later bilinear form is equivalent to the metric of $\mathbb{R}^{2n}$, (the involution being defined accordingly). The coaction δ_R passes to the quotient to give the coaction

$$\delta_R : \Omega_{\rm alg}(S^{2n-1}_\theta) \to C_{\rm alg}(O_\theta(2n)) \otimes \Omega_{\rm alg}(S^{2n-1}_\theta)$$

which is also a homomorphism of graded-involutive algebras. By taking a further quotient by the relation $\det_\theta = \mathbb{1}$, one obtains the $*$-Hopf algebra $C_{\rm alg}(SO_\theta(2n))$

defining the quantum group $SO_\theta(2n)$. Let $\rho : M_\theta(2n,\mathbb{R}) \to C_{\mathrm{alg}}(O_\theta(2n))$ be the canonical projection. The algebra $C_{\mathrm{alg}}(O_\theta(2n))$ is the unital $*$-algebra generated by the $4n^2$ elements $\rho(a^\mu_\nu)$, $\rho(b^\mu_\nu)$, $\rho(\bar a^\mu_\nu)$, $\rho(\bar b^\mu_\nu)$ with relations induced by (7.1), (7.2), (7.3) and the relations

$$\sum_\mu (\rho(\bar a^\mu_\alpha)\rho(a^\mu_\beta)+\rho(b^\mu_\alpha)\rho(\bar b^\mu_\beta)) = \delta_{\alpha\beta}\mathbb{1},\ \sum_\mu (\rho(\bar a^\mu_\alpha)\rho(b^\mu_\beta)+\rho(b^\mu_\alpha)\rho(\bar a^\mu_\beta)) = 0$$

(for $\alpha,\beta = 1,\dots,n$), together with $\rho(\bar a^\mu_\nu)=\rho(a^\mu_\nu)^*$ and $\rho(\bar b^\mu_\nu)=\rho(b^\mu_\nu)^*$. It follows that, for any C^*-semi-norm ν on $C_{\mathrm{alg}}(O_\theta(2n))$ one has $\nu(a^\mu_\nu)=\nu(\bar a^\mu_\nu)\leq 1$ and $\nu(b^\mu_\nu)=\nu(\bar b^\mu_\nu)\leq 1$ so that there is a greatest C^*-semi-norm on $C_{\mathrm{alg}}(O_\theta(2n))$ which is a norm and the corresponding completion $C(O_\theta(2n))$ of $C_{\mathrm{alg}}(O_\theta(2n))$ is a C^*-algebra. This defines $O_\theta(2n)$ as a *compact matrix quantum group* [56]. The same applies to $SO_\theta(2n)$ which is therefore also a compact matrix quantum group.

One proceeds similarily (with obvious modifications) to obtain the quantum groups $O_\theta(2n+1)$ and $SO_\theta(2n+1)$ which are again compact matrix quantum groups. One has also the coaction

$$\delta_R : \Omega_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta) \to C_{\mathrm{alg}}(O_\theta(2n+1))\otimes\Omega_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta)$$

which passes to the quotient to yield the coaction

$$\delta_R : \Omega_{\mathrm{alg}}(S^{2n}_\theta) \to C_{\mathrm{alg}}(O_\theta(2n+1))\otimes\Omega_{\mathrm{alg}}(S^{2n}_\theta)$$

these coactions are homomorphisms of graded-involutive algebras. This gives the action of the quantum group $O_\theta(2n+1)$ on the noncommutative $2n$-sphere S^{2n}_θ. One obtains similarily the action of $SO_\theta(2n)$ on S^{2n-1}_θ and of $SO_\theta(2n+1)$ on S^{2n}_θ.

Finally one lets $C_{\mathrm{alg}}(U_\theta(n))$ be the quotient of $C_{\mathrm{alg}}(O_\theta(2n))$ by the ideal generated by the $\rho(b^\mu_\nu)$ and $\rho(\bar b^\mu_\nu)$ which is also a $*$-Hopf ideal. The coactions δ_R of $C_{\mathrm{alg}}(O_\theta(2n))$ on $\Omega_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)=\Omega_{\mathrm{alg}}(\mathbb{C}^n_\theta)$ and on $C_{\mathrm{alg}}(S^{2n-1}_\theta)$ pass to quotient to give corresponding coactions of $C_{\mathrm{alg}}(U_\theta(n))$.
Again there is no corresponding θ-deformation of $SU(n)$.

Let us denote by $z_\mu, \bar z_\nu = z^*_\nu$ the generators of $C_{\mathrm{alg}}(\mathbb{R}^{2n}_{-\theta}) = C_{\mathrm{alg}}(\mathbb{C}^n_{-\theta})$ satisfying $z_\mu z_\nu = \lambda_{\nu\mu} z_\nu z_\mu$ and $\bar z_\mu z_\nu = \lambda_{\mu\nu} z_\nu \bar z_\mu$. One verifies that one obtains a unique $*$-homomorphism φ of $M_\theta(2n,\mathbb{R})$ into $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)\otimes C_{\mathrm{alg}}(\mathbb{R}^{2n}_{-\theta})$ by setting $\varphi(a^\mu_\nu) = z^\mu\otimes z_\nu$ and $\varphi(b^\mu_\nu) = z^\mu\otimes z^*_\nu$. This homomorphism is injective and its image is invariant by the action $\sigma\otimes\sigma$ of $T^n\times T^n$ on $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)\otimes C_{\mathrm{alg}}(\mathbb{R}^{2n}_{-\theta})$. We shall again denote by $\sigma\otimes\sigma$ the corresponding action of $T^n\times T^n$ on $M_\theta(2n,\mathbb{R})$, i.e. the group-homomorphism of $T^n\times T^n$ into $\mathrm{Aut}(M_\theta(2n,\mathbb{R}))$, e.g. one writes $\sigma_s\otimes\sigma_t(a^\mu_\nu) = e^{2\pi i(s_\mu+t_\nu))}a^\mu_\nu$, $\sigma_s\otimes\sigma_t(b^\mu_\nu) = e^{2\pi i(s_\mu-t_\nu))}b^\mu_\nu$, etc. . This induces a group-homomorphism (also denoted by $\sigma\otimes\sigma$) of $T^n\times T^n$ into the group of automorphisms of unital $*$-algebras (not necessarily preserving the coalgebra structure) of the polynomial algebra C_{alg} on each of the quantum groups defined in this section and in Section 7. In each case, the subalgebra of $\sigma \otimes \sigma$-invariant elements is in the center and is undeformed, that is isomorphic to the corresponding subalgebra for $\theta = 0$.

9 The graded differential algebras $\Omega_{\mathrm{alg}}(G_\theta)$ as graded differential Hopf algebras

The relations (7.1) to (7.3) define the $*$-algebra $M_\theta(2n, \mathbb{R})$ as $C_{\mathrm{alg}}(\mathbb{R}^{2N}_\Theta)$ with $N = 2n^2$ and where $\Theta \in M_N(\mathbb{R})$ is the appropriate antisymmetric matrix (which depends on $\theta \in M_n(\mathbb{R})$). Let $\Omega_{\mathrm{alg}}(\mathbb{R}^{2N}_\Theta)$ be the corresponding graded-involutive differential algebra as in Section 6.

PROPOSITION 2 *The coproduct Δ of $M_\theta(2n, \mathbb{R})$ has a unique extension as homomorphism of graded differential algebras, again denoted by Δ, of $\Omega_{\mathrm{alg}}(\mathbb{R}^{2N}_\Theta)$ into $\Omega_{\mathrm{alg}}(\mathbb{R}^{2N}_\Theta) \otimes_{\mathrm{gr}} \Omega_{\mathrm{alg}}(\mathbb{R}^{2N}_\Theta)$. The counit ε of $M_\theta(2n, \mathbb{R})$ has a unique extension as algebra-homomorhism, again denoted by ε, of $\Omega_{\mathrm{alg}}(\mathbb{R}^{2N}_\Theta)$ into $\mathbb{C}$ with $\varepsilon \circ d = 0$. The coaction $\delta : C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta) \to M_\theta(2n, \mathbb{R}) \otimes C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ has a unique extension as homomorphism of graded differential algebras, again denoted by δ, of $\Omega_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ into $\Omega_{\mathrm{alg}}(\mathbb{R}^{2N}_\Theta) \otimes_{\mathrm{gr}} \Omega_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$. The extended Δ is coassociative and the extended ε is a counit for it and one has $(\Delta \otimes I) \circ \delta = (I \otimes \delta) \circ \delta$, $(\varepsilon \otimes I) \circ \delta = I$. These extended homomorphisms are real.*

In this proposition, $N = 2n^2$ and Θ are as explained above and one endows $\Omega_{\mathrm{alg}}(\mathbb{R}^{2N}_\Theta) \otimes_{\mathrm{gr}} \Omega(\mathbb{R}^{2N}_\Theta)$ of the involution $\omega' \otimes \omega' \mapsto \overline{\omega' \otimes \omega''} = \bar{\omega}' \otimes \bar{\omega}''$. So equipped $\Omega_{\mathrm{alg}}(\mathbb{R}^{2N}_\Theta) \otimes_{\mathrm{gr}} \Omega_{\mathrm{alg}}(\mathbb{R}^{2N}_\Theta)$ is a graded-involutive differential algebra and the reality of Δ means $\overline{\Delta(\omega)} = \Delta(\bar{\omega})$. The uniqueness in the proposition is obvious and the only thing to verify is the compatibility of the extension with the relations $da^\mu_\nu da^\tau_\rho + \lambda^{\mu\tau}\lambda_{\rho\nu} da^\tau_\rho da^\mu_\nu = 0, \dots, a^\mu_\nu da^\tau_\rho = \lambda^{\mu\tau}\lambda_{\rho\nu} da^\tau_\rho a^\mu_\nu, \dots$, etc. which is easy. One proceeds similarly for δ. In short, $\Omega_{\mathrm{alg}}(\mathbb{R}^{2N}_\Theta)$ is a *graded-involutive differential bialgebra* and $\Omega_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ is a *graded-involutive differential comodule* over $\Omega_{\mathrm{alg}}(\mathbb{R}^{2N}_\Theta)$. Notice that to say that Δ is a homomorphism of graded differential algebras means that Δ is a homomorphism of graded algebras and that one has *the graded co-Leibniz rule* $\Delta \circ d = (d \otimes I + (-I)^{\mathrm{gr}} \otimes d) \circ \Delta$.

By a *graded differential Hopf algebra* we mean a graded differential bialgebra which admits an antipode; the antipode S is then necessarily unique and satisfies $S \circ d = d \circ S$. The notion of graded-involutive differential Hopf algebra is clear. By adding $\det^{-1}_\theta$ to $M_\theta(2n, \mathbb{R}) = \Omega^0_{\mathrm{alg}}(\mathbb{R}^{2N}_\Theta)$ as in Section 7 to obtain the Hopf algebra $C_{\mathrm{alg}}(GL_\theta(2n, \mathbb{R}))$ and by setting

$$\begin{aligned} [\det^{-1}_\theta, \omega] &= 0, \ \forall \omega \in \Omega_{\mathrm{alg}}(\mathbb{R}^{2N}_\Theta) \\ d(\det^{-1}_\theta) &= -(\det^{-1}_\theta)^2 d(\det_\theta) \end{aligned}$$

one defines the graded-involutive differential algebra $\Omega_{\mathrm{alg}}(GL_\theta(2n, \mathbb{R}))$ (writing $\Omega^0_{\mathrm{alg}}(GL_\theta(2n, \mathbb{R})) = C_{\mathrm{alg}}(GL_\theta(2n, \mathbb{R}))$, etc.) which is naturally a graded-involutive differential bialgebra and it is easy to show that the antipode S of $C_{\mathrm{alg}}(GL_\theta(2n,\mathbb{R}))$ extends (uniquely) as an antipode, again denoted by S, of $\Omega_{\mathrm{alg}}(GL_\theta(2n,\mathbb{R}))$. One proceeds similarily to define $\Omega_{\mathrm{alg}}(GL_\theta(2n+1,\mathbb{R}))$. One thus gets the following result.

THEOREM 5 *Let m be either $2n$ or $2n+1$. Then the differential algebra $\Omega_{\mathrm{alg}}(GL_\theta(m,\mathbb{R}))$ is a graded-involutive differential Hopf algebra and $\Omega_{\mathrm{alg}}(\mathbb{R}^m_\theta)$ is canonically a graded-involutive differential comodule over $\Omega_{\mathrm{alg}}(GL_\theta(m,\mathbb{R}))$.*

Let G_θ be any of the quantum groups defined in Sections 7 and 8. Then $C_{\mathrm{alg}}(G_\theta)$ is a $*$-Hopf algebra which is a quotient of $C_{\mathrm{alg}}(GL_\theta(m,\mathbb{R}))$ by a real Hopf ideal $I(G_\theta)$ for $m=2n$ or $m=2n+1$. Let $[I(G_\theta)]$ be the closed graded two-sided ideal of $\Omega_{\mathrm{alg}}(GL_\theta(m,\mathbb{R}))$ generated by $I(G_\theta)$ and let $\Omega_{\mathrm{alg}}(G_\theta)$ be the quotient of $\Omega_{\mathrm{alg}}(GL_\theta(m,\mathbb{R}))$ by $[I(G_\theta)]$. The above result has the following corollary.

COROLLARY 2 *The differential algebra $\Omega_{\mathrm{alg}}(G_\theta)$ is a graded-involutive differential Hopf algebra and $\Omega_{\mathrm{alg}}(\mathbb{R}^m_\theta)$ is a graded-involutive differential comodule over $\Omega_{\mathrm{alg}}(G_\theta)$.*

Similarly the algebra $\Omega_{\mathrm{alg}}(S^m_\theta)$ is a graded-involutive differential comodule over $\Omega_{\mathrm{alg}}(SO_\theta(m+1))$ and a similar result holds for $GL_\theta(n,\mathbb{C})$, $m=2n$ and $\Omega_{\mathrm{alg}}(\mathbb{C}^n_\theta)$ $=\Omega_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$.

10 The splitting homomorphisms

We let $C_{\mathrm{alg}}(T^n_\theta)$ be the $*$-algebra of polynomials on the noncommutative n-torus T^n_θ i. e. the unital $*$-algebra generated by n unitary elements U^μ with relations

$$U^\mu U^\nu = \lambda^{\mu\nu} U^\nu U^\mu \tag{10.1}$$

for $\mu,\nu=1,\dots,n$. We denote by $s\mapsto \tau_s\in \mathrm{Aut}(C_{\mathrm{alg}}(T^n_\theta))$ the natural action of T^n on T^n_θ ([9]) such that $\tau_s(U^\mu)=e^{2\pi i s_\mu}U^\mu$ $\forall s\in T^n$ and $\mu\in\{1,\dots,n\}$.

We let as in Section 4, $s\mapsto \sigma_s\in \mathrm{Aut}(C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta))$ be the natural action of T^n on $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$. It is defined for any θ (real antisymmetric (n,n)-matrix) and in particular for $\theta=0$. This yields two actions σ and τ of T^n on $\mathbb{R}^{2n}\times T^n_\theta$ given by the group-homomorphisms $s\mapsto \sigma_s\otimes I$ and $s\mapsto I\otimes\tau_s$ of T^n into $\mathrm{Aut}(C_{\mathrm{alg}}(\mathbb{R}^{2n})\otimes C_{\mathrm{alg}}(T^n_\theta))$ with obvious notations. The noncommutative space $\mathbb{R}^{2n}\times T^n_\theta$ is here defined by $C_{\mathrm{alg}}(\mathbb{R}^{2n}\times T^n_\theta)=C_{\mathrm{alg}}(\mathbb{R}^{2n})\otimes C_{\mathrm{alg}}(T^n_\theta)$. We shall use the actions σ and *the diagonal action* $\sigma\times\tau^{-1}$ of T^n on $\mathbb{R}^{2n}\times T^n_\theta$, where $\sigma\times\tau^{-1}$ is defined by $s\mapsto\sigma_s\otimes\tau_{-s}=(\sigma\times\tau^{-1})_s$ (as group homomorphism of T^n into $\mathrm{Aut}(C_{\mathrm{alg}}(\mathbb{R}^{2n}\times T^n_\theta))$).

In the following statement, ${z^\mu}_{(0)}$ denotes the classical coordinates of $\mathbb{C}^n$ corresponding to z^μ for $\theta=0$.

THEOREM 6 *a) There is a unique homomorphism of unital $*$-algebra*

$$st : C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta) \to C_{\mathrm{alg}}(\mathbb{R}^{2n}) \otimes C_{\mathrm{alg}}(T^n_\theta)$$

such that $st(z^\mu) = {z^\mu}_{(0)} \otimes U^\mu$ for $\mu = 1, \dots, n$.
b) The homomorphism st induces an isomorphism of $C_{\mathrm{alg}}(\mathbb{R}^{2n}_\theta)$ onto the subalgebra $C_{\mathrm{alg}}(\mathbb{R}^{2n} \times T^n_\theta)^{\sigma\times\tau^{-1}}$ of $C_{\mathrm{alg}}(\mathbb{R}^{2n} \times T^n_\theta)$ of fixed points of the diagonal action of T^n.

One has $st(\bar z^\mu) = st(z^\mu)^*$ and, using (10.1), one checks that $st(z^\mu)$, $st(\bar z^\mu)$ fulfill the relations (4.1). On the other hand, it is obvious that the $st(z^\mu)$ are invariant by the diagonal action of T^n. Thus the only non-trivial parts of the statement, which are not difficult to show, are the injectivity of st and the fact that $C_{\mathrm{alg}}(\mathbb{R}^{2n} \times T^n_\theta)^{\sigma\times\tau^{-1}}$ is generated by the z^μ as unital $*$-algebra.
This extends trivially to

$$st : C_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta) \to C_{\mathrm{alg}}(\mathbb{R}^{2n+1}) \otimes C_{\mathrm{alg}}(T^n_\theta) = C_{\mathrm{alg}}(\mathbb{R}^{2n+1} \times T^n_\theta)$$

with $st(x) = x_{(0)} \otimes \mathbb{1}$ and $st(z^\mu) = {z^\mu}_{(0)} \otimes U^\mu$. This is again an isomorphism of $C_{\mathrm{alg}}(\mathbb{R}^{2n+1}_\theta)$ onto $C_{\mathrm{alg}}(\mathbb{R}^{2n+1} \times T^n_\theta)^{\sigma\times\tau^{-1}}$.

The above homomorphisms st pass to the quotient to define homomorphisms of unital $*$-algebras ($m = 2n, 2n+1$)

$$st : C_{\mathrm{alg}}(S^m_\theta)) \to C_{\mathrm{alg}}(S^m) \otimes C_{\mathrm{alg}}(T^n_\theta) = C_{\mathrm{alg}}(S^m \times T^n_\theta)$$

which are isomorphisms of $C_{\mathrm{alg}}(S^m_\theta)$ with $C_{\mathrm{alg}}(S^m \times T^n_\theta)^{\sigma\times\tau^{-1}}$, the fixed points of the diagonal action $\sigma \times \tau^{-1}$ of T^n (recall that σ was previously defined for any θ, in particular for $\theta = 0$).

We shall refer to the above homomorphisms st as the *splitting homomorphisms*. They satisfy $st \circ \sigma_s = (\sigma_s \otimes I) \circ st$ for any $s = (s_1, \dots, s_n) \in T^n$ and thus st induce isomorphisms

$$st : C_{\mathrm{alg}}(M_\theta)^\sigma \overset{\simeq}{\to} C_{\mathrm{alg}}(M)^\sigma \otimes \mathbb{1}(\subset C_{\mathrm{alg}}(M) \otimes C_{\mathrm{alg}}(T^n_\theta))$$

for $M = \mathbb{R}^m$ and S^m.

In a similar manner, with M as above, st extends to isomorphisms of unital graded-involutive differential algebras

$$st : \Omega_{\mathrm{alg}}(M_\theta) \to (\Omega_{\mathrm{alg}}(M) \otimes C_{\mathrm{alg}}(T^n_\theta))^{\sigma\times\tau^{-1}}$$

by setting

$$st(dz^\mu) = d{z^\mu}_{(0)} \otimes U^\mu \quad \text{and} \quad st(dx) = dx_{(0)} \otimes \mathbb{1}$$

using the previously defined action σ of T^n on $\Omega_{\mathrm{alg}}(M_\theta)$ for any θ (in particular $\theta = 0$).
The compatibility with the differential and the action of T^n is explicitly given by,

$$st \circ d \;=\; (d \otimes I) \circ st \tag{10.2}$$
$$st \circ \sigma_s \;=\; (\sigma_s \otimes I) \circ st \tag{10.3}$$

<u>Remark.</u> We shall use the splitting homomorphisms st to reduce computations involving θ-deformations to the classical case ($\theta = 0$). For instance we shall later define the Dirac operator, D_θ, on M_θ in such a way that is satisfies with obvious notations $st \circ ad(D_\theta) = (ad(D) \otimes I) \circ st$ where on the right-hand side D is the ordinary Dirac operator on the riemannian spin manifold M, ($M = \mathbb{R}^{2n}$, $\mathbb{R}^{2n+1}$, S^{2n-1}, S^{2n}); this will imply the first order condition, the reality condition and the identification of the differential algebra Ω_D with $\Omega_{\mathrm{alg}}(M_\theta)$, (see Section 13).

A similar discussion applies to the various θ-deformed groups mentionned above. To be specific, we introduce the n unitary elements U_μ with relations

$$U_\mu U_\nu = \lambda_{\nu\mu} U_\nu U_\mu \tag{10.4}$$

for $\mu, \nu = 1, \dots, n$, (recall that $\lambda_{\mu\nu} = e^{i\theta_{\mu\nu}} = \lambda^{\mu\nu}$, $\forall \mu, \nu$) which generate $C_{\mathrm{alg}}(T^n_{-\theta})$, the opposite algebra of $C_{\mathrm{alg}}(T^n_\theta)$.

Let us consider for $m = 2n$ or $m = 2n+1$ the homomorphism $r_{23} \circ (st \otimes st)$:

$$C_{\mathrm{alg}}(\mathbb{R}^m_\theta) \otimes C_{\mathrm{alg}}(\mathbb{R}^m_{-\theta}) \to C_{\mathrm{alg}}(\mathbb{R}^m) \otimes C_{\mathrm{alg}}(\mathbb{R}^m) \otimes C_{\mathrm{alg}}(T^n_\theta) \otimes C_{\mathrm{alg}}(T^n_{-\theta})$$

where r_{23} is the transposition of the second and the third factors in the tensor product, (i.e. $C_{\mathrm{alg}}(T^n_\theta) \otimes C_{\mathrm{alg}}(\mathbb{R}^m)$ is replaced by $C_{\mathrm{alg}}(\mathbb{R}^m) \otimes C_{\mathrm{alg}}(T^n_\theta)$ there). This $*$-homomorphism restricts to give a homomorphism, again denoted by st

$$st : M_\theta(m, \mathbb{R}) \to M(m, \mathbb{R}) \otimes C_{\mathrm{alg}}(T^n_\theta) \otimes C_{\mathrm{alg}}(T^n_{-\theta})$$

which is again a homomorphism of unital $*$-algebras and will be also refered to as splitting homomorphism. For instance, for $m = 2n$, it is the unique unital $*$-homomorphism such that

$$st(a^\mu_\nu) = \overset{(0)}{a^\mu_\nu} \otimes U^\mu \otimes U_\nu \tag{10.5}$$

$$st(b^\mu_\nu) = \overset{(0)}{b^\mu_\nu} \otimes U^\mu \otimes U^*_\nu \tag{10.6}$$

for $\mu, \nu = 1, \dots, n$ where $\overset{(0)}{a^\mu_\nu}$ and $\overset{(0)}{b^\mu_\nu}$ are the classical coordinates corresponding to a^μ_ν and b^μ_ν for $\theta = 0$. The counterpart of $b)$ in Theorem 6 is that st induces here an isomorphism of $M_\theta(m,\mathbb{R})$ onto the subalgebra of the elements x of $M(m,\mathbb{R})\otimes C_{\mathrm{alg}}(T^n_\theta)\otimes C_{\mathrm{alg}}(T^n_{-\theta})$ which are invariant by the diagonal action $(\sigma\otimes\sigma)\times(\tau\otimes\tau)^{-1}$ of $T^n\times T^n$ i.e. which satisfy $(\sigma_s\otimes\sigma_t)(\tau_{-s}\otimes\tau_{-t})(x)=x$, $\forall(s,t)\in T^n\times T^n$ (with the notations of the end of last section). One has

$$st\circ(\sigma_s\otimes\sigma_t)=((\sigma_s\otimes\sigma_t)\otimes I\otimes I)\circ st$$

which then implies that st induces an isomorphism of $M_\theta(m,\mathbb{R})^{\sigma\otimes\sigma}$ onto $M(m,\mathbb{R})^{\sigma\otimes\sigma}\otimes\mathbb{1}\otimes\mathbb{1}$ where $M_\theta(m,\mathbb{R})^{\sigma\otimes\sigma}$ denotes the subalgebra of elements which are invariant by the action of $T^n\times T^n$, (the same for $\theta=0$ on the right-hand side). This in particular implies that $st(\det_\theta)$ is in $M(2n,\mathbb{R})^{\sigma\otimes\sigma}\otimes\mathbb{1}\otimes\mathbb{1}$; in fact one has $st(\det_\theta)=\det\otimes\mathbb{1}\otimes\mathbb{1}$ where $\det=\det_{\theta=0}$ is the ordinary determinant.
The above homomorphism passes to the quotient to yield homomorphisms

$$st: C_{\mathrm{alg}}(G_\theta)\rightarrow C_{\mathrm{alg}}(G)\otimes C_{\mathrm{alg}}(T^n_\theta)\otimes C_{\mathrm{alg}}(T^n_{-\theta})$$

where G is any of the classical groups $GL(m,\mathbb{R})$, $SL(m,\mathbb{R})$, $O(m)$, $SO(m)$, $GL(n,\mathbb{C})$, $U(n)$ or the subgroup $GL_{(1)}(n,\mathbb{C})$ of $GL(n,\mathbb{C})$ consisting of matrices with determinants of modulus one, $m=2n$ or $m=2n+1$, and where G_θ denote the corresponding quantum groups defined in Section 7 and in Section 8. These homomorphisms st which will still be refered to as the splitting homomorphisms, have the property that they induce isomorphisms of $C_{\mathrm{alg}}(G_\theta)$ onto $(C_{\mathrm{alg}}(G)\otimes C_{\mathrm{alg}}(T^n_\theta)\otimes C(T^n_{-\theta}))^{(\sigma\otimes\sigma)\times(\tau\otimes\tau)^{-1}}$ for these groups G.

Thus, one sees that the situation is the same for the above quantum groups as for the noncommutative spaces M_θ with $M=\mathbb{R}^m$, S^m excepted that the action of T^n is replaced by an action of $T^n\times T^n=T^{2n}$ and that the noncommutative n-torus T^n_θ is replaced by the noncommutative $2n$-torus $T^{2n}_{\theta\times(-\theta)}$ where $\theta\times(-\theta)$ is the real antisymmetric $(2n,2n)$-matrix $\begin{pmatrix}\theta & 0\\ 0 & -\theta\end{pmatrix}\in M_{2n}(\mathbb{R})$; one has of course $C_{\mathrm{alg}}(T^{2n}_{\theta\times(-\theta)})=C_{\mathrm{alg}}(T^n_\theta)\otimes C_{\mathrm{alg}}(T^n_{-\theta})$.

11 Smoothness

Beside their usefulness for computations, the splitting homomorphisms give straightforward unambiguous notions of smooth functions on θ-deformations.

The locally convex $*$-algebra $C^\infty(T^n_\theta)$ of smooth functions on the noncommutative torus T^n_θ was defined in [9]. It is the completion of $C_{\mathrm{alg}}(T^n_\theta)$ endowed with the locally convex topology generated by the seminorms

$$|u|_r=\sup_{r_1+\dots+r_n\leq r}\parallel X^{r_1}_1\dots X^{r_n}_n(u)\parallel$$

where $\parallel\cdot\parallel$ is the C^*-norm (which is the sup of the C^*-seminorms) and where the X_μ are the infinitesimal generators of the action $s\mapsto\tau_s$ of T^n on T^n_θ. They are the unique derivations of $C_{\mathrm{alg}}(T^n_\theta)$ satisfying

$$X_\mu(U^\nu) = 2\pi i \delta^\nu_\mu U^\nu \tag{11.1}$$

for $\mu, \nu = 1, \dots, n$. Notice that these derivations are real and commute between themselves, i.e. $X_\mu(u^*) = (X_\mu(u))^*$ and $X_\mu X_\nu - X_\nu X_\mu = 0$. This locally convex $*$-algebra is a nuclear Fréchet space and it follows from the general theory of topological tensor products that the π-topology and ε-topology coincide [31] on any tensor product, [53] i.e.

$$E \otimes_\pi C^\infty(T^n_\theta) = E \otimes_\varepsilon C^\infty(T^n_\theta)$$

so that on $E \otimes C^\infty(T^n_\theta)$ there is essentially one reasonable locally convex topology and we denote by $E \widehat{\otimes} C^\infty(T^n_\theta)$ the corresponding completion.

It is then straightforward to define the function spaces $C^\infty(M_\theta)$ (of smooth functions) and $C^\infty_c(M_\theta)$ (of smooth functions with compact support) for any of the θ-deformed spaces mentionned above, as the fixed point algebra of the diagonal action of T^n on the completed tensor product $C^\infty(M) \widehat{\otimes} C^\infty(T^n_\theta)$ (and on $C^\infty_c(M) \widehat{\otimes} C^\infty(T^n_\theta)$).
Using the appropriate splitting homomorphisms, one defines in the same way the locally convex $*$-algebras $C^\infty(G_\theta)$ and $C^\infty_c(G_\theta)$ of smooth functions on the different quantum groups defined in Section 7 and in Section 8. The same discussion applies to the algebras $\Omega(M_\theta)$ and $\Omega_c(M_\theta)$of smooth differential forms.

12 Differential forms, self-duality, Hochschild cohomology for θ-deformations

Let M be a smooth m-dimensional manifold endowed with a smooth action $s \mapsto \sigma_s$ of the compact abelian Lie group T^n, (the n-torus). We also denote by $s \mapsto \sigma_s$ the corresponding group-homomorphism of T^n into the group $\mathrm{Aut}(C^\infty(M))$ (resp $\mathrm{Aut}(\Omega(M))$) of automorphisms of the unital $*$-algebra $C^\infty(M)$ of complex smooth functions on M with its standard topology (resp of the graded-involutive differential algebra $\Omega(M)$ of smooth differential forms).
Let $C^\infty(M_\theta)$ be the θ-deformation of the $*$-algebra $C^\infty(M)$ associated by [47] to the above data. We shall find it convenient to give the following (trivially equivalent) direct description of $C^\infty(M_\theta)$ as a fixed point algebra.
The completed tensor product $C^\infty(M) \widehat{\otimes} C^\infty(T^n_\theta)$ is unambiguously defined by nuclearity and is a unital locally convex $*$-algebra which is a complete nuclear space. We define by duality the noncommutative smooth manifold $M \times T^n_\theta$ by setting $C^\infty(M \times T^n_\theta) = C^\infty(M) \widehat{\otimes} C^\infty(T^n_\theta)$; elements of $C^\infty(M \times T^n_\theta)$ will be refered to as *the smooth functions on* $M \times T^n_\theta$. Let $C^\infty(M \times T^n_\theta)^{\sigma \times \tau^{-1}}$ be the subalgebra of the $f \in C^\infty(M \times T^n_\theta)$ which are invariant by the diagonal action $\sigma \times \tau^{-1}$ of T^n, that is such that $\sigma_s \otimes \tau_{-s}(f) = f$ for any $s \in T^n$. One defines by duality

the noncommutative manifold M_θ by setting $C^\infty(M_\theta) = C^\infty(M \times T^n_\theta)^{\sigma\times\tau^{-1}}$ and the elements of $C^\infty(M_\theta)$ will be refered to as *the smooth functions on* M_θ. This definition clearly coincides with the one used before for the examples of the previous sections once identified using the splitting homomorphisms.

Let us now give a first construction of smooth differential forms on M_θ generalizing the one given before in the examples. Let $\Omega(M_\theta)$ be the graded-involutive subalgebra $(\Omega(M)\widehat{\otimes}C^\infty(T^n_\theta))^{\sigma\times\tau^{-1}}$ of $\Omega(M)\widehat{\otimes}C^\infty(T^n_\theta)$ consisting of elements which are invariant by the diagonal action $\sigma\times\tau^{-1}$ of T^n. This subalgebra is stable by $d\otimes I$ so $\Omega(M_\theta)$ is a locally convex graded-involutive differential algebra which is a deformation of $\Omega(M)$ with $\Omega^0(M_\theta) = C^\infty(M_\theta)$ and which will be refered to as the *algebra of smooth differential forms on* M_θ. The action $s\mapsto \sigma_s$ of T^n on $\Omega(M)$ induces $s\mapsto \sigma_s\otimes I$ on $\Omega(M)\widehat{\otimes}C^\infty(T^n_\theta)$ which gives by restriction a group-homomorphism, again denoted $s\mapsto\sigma_s$, of T^n into the group $\mathrm{Aut}(\Omega(M_\theta))$ of automorphisms of the graded-involutive differential algebra $\Omega(M_\theta)$.

PROPOSITION 3 *The graded-involutive differential subalgebra $\Omega(M_\theta)^\sigma$ of σ-invariant elements of $\Omega(M_\theta)$ is in the graded center of $\Omega(M_\theta)$ and identifies canonically with the graded-involutive differential subalgebra $\Omega(M)^\sigma$ of σ-invariant elements of $\Omega(M)$.*

In other words the subalgebra of σ-invariant elements of $\Omega(M_\theta)$ is not deformed (i.e. independent of θ). One has $\Omega(M_\theta)^\sigma = \Omega(M)^\sigma\otimes \mathbb{1}$ $(\subset \Omega(M)\widehat{\otimes}C^\infty(T^n_\theta))$.
The notations M_θ, $C^\infty(M_\theta)$ introduced here are coherent with the standard ones T^n_θ, $C^\infty(T^n_\theta)$ used for the noncommutative torus. Indeed it is true that one has $C^\infty(T^n_\theta) = (C^\infty(T^n)\widehat{\otimes}C^\infty(T^n_\theta))^{\sigma\times\tau^{-1}}$ where σ is the canonical action of T^n on itself. Furthermore there is a natural definition of the graded differential algebra of smooth differential forms on the noncommutative n-torus T^n_θ [9] and it turns out that it coincides with the above one for $M = T^n$, that is with $\Omega(T^n_\theta)$, as easily verified.

Although simple and useful, the previous definition of smooth differential forms on M_θ is not the most natural one. Indeed the construction has the following geometric interpretation. The noncommutative manifold M_θ is the quotient of the product $M\times T^n_\theta$ by the diagonal action of T^n, and one has a noncommutative fibre bundle

$$M\times T^n_\theta \xrightarrow{T^n} M_\theta$$

with fibre T^n. In such a context it is natural to describe differential forms on M_θ as the basic forms on $M\times T^n_\theta$ for the operation of $\mathbf{Lie}(T^n)$ corresponding to the infinitesimal diagonal action of T^n. More precisely, let Y_μ, $\mu\in\{1,\dots,n\}$ be the vector fields on M corresponding to the infinitesimal action of T^n

$$Y_\mu(x) = \frac{\partial}{\partial s_\mu}\sigma_s(x)\mid_{s=0} \tag{12.1}$$

for $x \in M$. These vector fields are real and define n derivations of $C^\infty(M)$, again denoted by Y_μ, which are real and commute between themselves. The inner anti-derivations $Y_\mu \mapsto i_{Y_\mu}$ define *an operation of the (abelian) Lie algebra* $\mathbf{Lie}(T^n)$ *in the graded differential algebra* $\Omega(M)$ [6], [30] and the corresponding Lie derivatives $L_{Y_\mu} = di_{Y_\mu} + i_{Y_\mu}d$ are derivations of degree zero of $\Omega(M)$ which extend the Y_μ and correspond to the infinitesimal action of T^n on $\Omega(M)$. The natural graded differential algebra of smooth differential forms on $M \times T^n_\theta$ is $\Omega(M \times T^n_\theta) = \Omega(M)\widehat{\otimes}_{\mathrm{gr}}\Omega(T^n_\theta)$, and the operation [6], [30] of $\mathbf{Lie}(T^n)$ in $\Omega(M \times T^n_\theta)$ corresponding to the diagonal action of T^n is described by the antiderivations $i_\mu = i_{Y_\mu} \otimes I - (-I)^{\mathrm{gr}} \otimes i_{X_\mu}$ of $\Omega(M \times T^n_\theta)$ where i_{X_μ} is the antiderivation of degree -1 of $\Omega(T^n_\theta) = C^\infty(T^n_\theta)\otimes_{\mathbb{R}} \wedge\mathbb{R}^n$ [9] such that $i_{X_\mu}(\omega^\nu) = \delta^\nu_\mu$ with $\omega^\mu = \frac{1}{2\pi i}U^{\mu *}dU^\mu$. The infinitesimal diagonal action of T^n is described by the Lie derivatives $L_\mu = di_\mu + i_\mu d$ on $\Omega(M \times T^n_\theta)$ and the differential subalgebra $\Omega_B(M \times T^n_\theta)$ of the basic elements of $\Omega(M \times T^n_\theta)$, that is of the elements α satisfying $i_\mu(\alpha) = 0$ and $L_\mu(\alpha) = 0$ for $\mu \in \{1, \cdots, n\}$, is a natural candidate to be the algebra of smooth differential forms on M_θ. Fortunately, it is not hard to show that one has the following result which allows to use either point of view.

PROPOSITION 4 *As graded-involutive differential algebra* $\Omega_B(M \times T^n_\theta)$ *is isomorphic to* $\Omega(M_\theta)$.

The (first) construction of $\Omega(M_\theta)$ admits the following generalization. Let S be a smooth complex vector bundle of finite rank over M and let $C^\infty(M,S)$ be the $C^\infty(M)$-module of its smooth sections, endowed with its usual topology of complete nuclear space. The vector bundle S will be called *σ-equivariant* if it is endowed with a group-homomorphism $s \mapsto V_s$ of T^n into the group $\mathrm{Aut}(S)$ of automorphisms of S which covers the action $s \mapsto \sigma_s$ of T^n on M. In terms of smooth sections this means that one has

$$V_s(f\psi) = \sigma_s(f)V_s(\psi) \tag{12.2}$$

for $f \in C^\infty(M)$ and $\psi \in C^\infty(M,S)$ with an obvious abuse of notations. Let $C^\infty(M_\theta, S)$ be the closed subspace of $C^\infty(M,S)\widehat{\otimes}C^\infty(T^n_\theta)$ consisting of elements Ψ which are invariant by the diagonal action $V \times \tau^{-1}$ of T^n, i.e. which satisfy $V_s \otimes \tau_{-s}(\Psi) = \Psi$ for any $s \in T^n$. The locally convex space $C^\infty(M_\theta, S)$ is also canonically a topological bimodule over $C^\infty(M_\theta)$, or which is the same, a topological left module over $C^\infty(M_\theta)\widehat{\otimes}C^\infty(M_\theta)^{opp}$.

PROPOSITION 5 *The bimodule* $C^\infty(M_\theta, S)$ *is diagonal and (topologically) left and right finite projective over* $C^\infty(M_\theta)$.

The proof of this proposition uses the equivalence between the category of σ-equivariant finite projective modules over $C^\infty(M)$ (i.e. of σ-equivariant vector bundles over M) and the category of finite projective modules over the cross-product $C^\infty(M) \rtimes_\sigma T^n$, the fact that one has $C^\infty(M) \rtimes_\sigma T^n \simeq C^\infty(M_\theta) \rtimes_\sigma T^n$ and finally the equivalence between the category of finite projective modules over $C^\infty(M_\theta) \rtimes_\sigma T^n$ and the category of σ-equivariant finite projective modules over $C^\infty(M_\theta)$ [32].

Let D be a continuous $\mathbb{C}$-linear operator on $C^\infty(M,S)$ such that

$$DV_s = V_s D \tag{12.3}$$

for any $s \in T^n$. Then $C^\infty(M_\theta, S)$ $(\subset C^\infty(M,S)\widehat{\otimes} C^\infty(T^n_\theta))$ is stable by $D \otimes I$ which defines the operator D_θ $(= D \otimes I \restriction C^\infty(M_\theta, S))$ on $C^\infty(M_\theta, S)$. If D is a first-order differential operator it follows immediately from the definition that D_θ is a first-order operator of the bimodule $C^\infty(M_\theta, S)$ over $C^\infty(M_\theta)$ into itself, [11], [27]. If D is of order zero i. e. is a module homomorphism over $C^\infty(M)$ then it is obvious that D_θ is a bimodule homomorphism over $C^\infty(M_\theta)$.

We already met this construction in the case of $S = \wedge T^*M$ and $D = d$. There D_θ is the differential d of $\Omega(M_\theta)$ which is a first-order operator on the bimodule $\Omega(M_\theta)$ over $C^\infty(M_\theta)$. Let $\omega \mapsto *\omega$ be the Hodge operator on $\Omega(M)$ corresponding to a σ-invariant riemannian metric on M. One has $* \circ \sigma_s = \sigma_s \circ *$ thus $*$ satisfies (12.3) from which one obtains an endomorphism $*_\theta$ of $\Omega(M_\theta)$ considered as a bimodule over $C^\infty(M_\theta)$. We shall denote $*_\theta$ simply by $*$ in the following. One has $*\Omega^p(M_\theta) \subset \Omega^{m-p}(M_\theta)$.

THEOREM 7 *Let the $2n$-sphere S^{2n} be endowed with its usual metric, let $*$ be defined as above on $\Omega(S^{2n}_\theta)$ and let e be the hermitian projection of Theorem 4. Then e satisfies the self-duality equation $*e(de)^n = i^n e(de)^n$.*

Indeed using the splitting homomorphism, e identifies with

$$e = \frac{1}{2}(\mathbb{1} + \sum_{\mu=1}^{n} (u^\mu_{(0)} \tilde{\Gamma}^{\mu *} + u^{\mu *}_{(0)} \tilde{\Gamma}^\mu) + u\gamma)$$

where $u^\mu_{(0)}, \cdots, u$ are now the classical coordinates of $\mathbb{R}^{2n+1}$ for $S^{2n} \subset \mathbb{R}^{2n+1}$ and where $\tilde{\Gamma}^{\mu *} = \Gamma^{\mu *} \otimes U^\mu$, $\tilde{\Gamma}^\mu = \Gamma^\mu \otimes U^{\mu *}$ with γ identified with $\gamma \otimes \mathbb{1} \in M_{2^n}(C^\infty(T^n_\theta))$. Now one verifies easily that the $\tilde{\Gamma}^{\mu *}$, $\tilde{\Gamma}^\nu$ satisfy the relations of the usual Clifford algebra of $\mathbb{R}^{2n}$ so $*e(de)^n = i^n e(de)^n$ follows from the classical relation (5.3) for $P_+ = e\mid_{\theta=0}$ and from $* = * \otimes I$ where on the right-hand side $*$ is the classical one.

Similarly one has $*e_-(de_-)^n = -i^n e_-(de_-)^n$. Notice that if one replaces the usual metric of S^{2n} by another σ-invariant metric which is conformally equivalent, the same result holds but that σ-invariance is a priori necessary for this. Let us now compute the Hochschild dimension of M_θ. We first construct a continuous projective resolution of the left $C^\infty(M_\theta)\widehat{\otimes} C^\infty\ (M_\theta)^{opp}$-module $C^\infty(M_\theta)$.

LEMMA 6 *There are continuous homomorphisms of left modules*

$$i_p : \Omega^p(M_\theta)\widehat{\otimes} C^\infty(M_\theta) \to \Omega^{p-1}(M_\theta)\widehat{\otimes} C^\infty(M_\theta)$$

over $C^\infty(M_\theta)\widehat{\otimes}C^\infty(M_\theta)^{opp}$ *for* $p \in \{1, \cdots, m\}$ *such that the sequence*

$$0 \to \Omega^m(M_\theta)\widehat{\otimes}C^\infty(M_\theta) \stackrel{i_m}{\to} \cdots \stackrel{i_1}{\to} C^\infty(M_\theta)\widehat{\otimes}C^\infty(M_\theta) \stackrel{\mu}{\to} C^\infty(M_\theta) \to 0$$

is exact, where μ *is induced by the product of* $C^\infty(M_\theta)$.

In fact as was shown and used in [10] one has continuous projective resolutions of $C^\infty(M)$ and of $C^\infty(T^n_\theta)$ of the form

$$0 \to \Omega^m(M)\widehat{\otimes}C^\infty(M) \stackrel{i^0_m}{\to} \cdots \stackrel{i^0_1}{\to} C^\infty(M)\widehat{\otimes}C^\infty(M) \stackrel{\mu}{\to} C^\infty(M) \to 0$$

$$0 \to \Omega^n(T^n_\theta)\widehat{\otimes}C^\infty(T^n_\theta) \stackrel{j_n}{\to} \cdots \stackrel{j_1}{\to} C^\infty(T^n_\theta)\widehat{\otimes}C^\infty(T^n_\theta) \stackrel{\mu}{\to} C^\infty(T^n_\theta) \to 0$$

which combine to give a continuous projective resolution of

$$C^\infty(M)\widehat{\otimes}C^\infty(T^n_\theta) = C^\infty(M \times T^n_\theta)$$

of the form

$$\begin{array}{l} 0 \to \Omega^{m+n}(M \times T^n_\theta)\widehat{\otimes}C^\infty(M \times T^n_\theta) \stackrel{\tilde{i}_{m+n}}{\to} \cdots \\ \stackrel{\tilde{i}_1}{\to} C^\infty(M \times T^n_\theta)\widehat{\otimes}C^\infty(M \times T^n_\theta) \stackrel{\mu}{\to} C^\infty(M \times T^n_\theta) \to 0 \end{array}$$

where $\Omega^p(M \times T^n_\theta) = \oplus_{p \geq k \geq 0}\, \Omega^k(M)\widehat{\otimes}\Omega^{p-k}(T^n_\theta)$ and where

$$\tilde{i}_p = \sum_k (i^0_k \otimes I + (-I)^k \otimes j_{p-k}).$$

There is some freedom in the choice of the i^0_k, j_ℓ and one can choose them equivariant (by choosing a σ-invariant metric on M, etc.) in such a way that the $\tilde{i}_p$ restrict as continuous homomorphisms

$$i_p : \Omega^p_B(M \times T^n_\theta)\widehat{\otimes}C^\infty(M_\theta) \to \Omega^{p-1}_B(M \times T^n_\theta)\widehat{\otimes}C^\infty(M_\theta)$$

of $C^\infty(M_\theta)\widehat{\otimes}C^\infty(M_\theta)^{opp}$-modules which gives the desired resolution of $C^\infty(M_\theta)$ using Proposition 5.

This shows that the Hochschild dimension m_θ of M_θ is $\leq m$ where m is the dimension of M.
Let $w \in \Omega^m(M)$ be a non-zero σ-invariant form of degree m on M (obtained by a straightforward local averaging). In view of Proposition 3, $w \otimes \mathbb{1} = w_\theta$ is a $\sigma-$invariant element of $\Omega^m(M_\theta)$, i.e. $w_\theta \in \Omega^m(M_\theta)^\sigma$ which defines canonically a non-trivial invariant cycle v_θ in $Z_m(C^\infty(M_\theta), C^\infty(M_\theta))$. Thus one has $m_\theta \geq m$ and therefore the following result.

THEOREM 8 *Let* M_θ *be a* θ*-deformation of* M *then one has* $\dim(M_\theta) = \dim(M)$*, that is the Hochschild dimension* m_θ *of* $C^\infty(M_\theta)$ *coincides with the dimension* m *of* M.

Note that the conclusion of the theorem fails for general deformations by actions of $\mathbb{R}^d$ as described in [47]. Indeed, in the simplest case of the Moyal deformation of $\mathbb{R}^{2n}$ the Hochschild dimension drops down to zero for non-degenerate values of the deformation parameter. It is however easy to check that periodic cyclic cohomology (but not its natural filtration) is unaffected by the θ-deformation.

13 Metric aspect: The spectral triple

As in the last section we let M be a smooth m-dimensional manifold endowed with a smooth action $s \mapsto \sigma_s$ of T^n. It is well-known and easy to check that we can average any riemannian metric on M under the action of σ and obtain one for which the action $s \mapsto \sigma_s$ of T^n on M is isometric. Let us assume moreover that M is a spin manifold. Let S be the spin bundle over M and let D be the Dirac operator on $C^\infty(M,S)$. The bundle S is not σ-equivariant in the sense of the last section but is equivariant in a slightly generalized sense which we now explain. In fact the isometric action σ of T^n on M does not lift directly to S but lifts only modulo $\pm I$. More precisely one has a twofold covering $p : \tilde{T}^n \to T^n$ of the group T^n, and a group homomorphism $\tilde{s} \mapsto V_{\tilde{s}}$ of $\tilde{T}^n$ into the group $\mathrm{Aut}(S)$ which covers the action $s \mapsto \sigma_s$ of T^n on M. In terms of smooth sections, (12.2) generalizes here as

$$V_{\tilde{s}}(f\psi) = \sigma_s(f) V_{\tilde{s}}(\psi) \tag{13.1}$$

where $f \in C^\infty(M)$ and $\psi \in C^\infty(M,S)$ with $s = p(\tilde{s})$. The bundle S is also a hermitian vector bundle and one has

$$(V_{\tilde{s}}(\psi), V_{\tilde{s}}(\psi')) = \sigma_s((\psi,\psi')) \tag{13.2}$$

for $\psi, \psi' \in C^\infty(M,S)$, $\tilde{s} \in \tilde{T}^n$ and $s = p(\tilde{s})$ where $(.,.)$ denotes the hermitian scalar product. Furthermore, the Dirac operator D commutes with the $V_{\tilde{s}}$.

To the projection $p : \tilde{T}^n \to T^n$ corresponds an injective homomorphism of $C^\infty(T^n)$ into $C^\infty(\tilde{T}^n)$ which identifies $C^\infty(T^n)$ with the subalgebra $C^\infty(\tilde{T}^n)^{\mathrm{Ker}(p)}$ of $C^\infty(\tilde{T}^n)$ of elements which are invariant by the action of the subgroup $\mathrm{Ker}(p) \simeq \mathbb{Z}_2$ of $\tilde{T}^n$. Let $\tilde{T}^n_\theta$ be the noncommutative n-torus $T^n_{\frac{1}{2}\theta}$ and let $\tilde{s} \mapsto \tilde{\tau}_{\tilde{s}}$ be the canonical action of the n-torus $\tilde{T}^n$ that is the canonical group-homomorphism of $\tilde{T}^n$ into the group $\mathrm{Aut}(C^\infty(\tilde{T}^n_\theta))$. The very reason for these notations is that $C^\infty(T^n_\theta)$ identifies with the subalgebra $C^\infty(\tilde{T}^n_\theta)^{\mathrm{Ker}(p)}$ of $C^\infty(\tilde{T}^n_\theta)$ of elements which are invariant by the $\tilde{\tau}_{\tilde{s}}$ for $\tilde{s} \in \mathrm{Ker}(p) \simeq \mathbb{Z}_2$. Under this identification, one has $\tilde{\tau}_{\tilde{s}}(f) = \tau_s(f)$ for $f \in C^\infty(T^n_\theta)$ and $s = p(\tilde{s}) \in T^n$.

Define $C^\infty(M_\theta, S)$ to be the closed subspace of $C^\infty(M,S)\widehat{\otimes} C^\infty(\tilde{T}_\theta)$ consisting of elements Ψ which are invariant by the diagonal action $V \times \tilde{\tau}^{-1}$ of $\tilde{T}^n$; this is canonically a topological bimodule over $C^\infty(M_\theta)$. Since the Dirac operator commutes with the $V_{\tilde{s}}$, $C^\infty(M_\theta, S)$ is stable by $D \otimes I$ and we denote by D_θ the corresponding operator on $C^\infty(M_\theta, S)$. Again, D_θ is a first-order operator of the bimodule $C^\infty(M_\theta, S)$ over $C^\infty(M_\theta)$ into itself. The space $C^\infty(M,S)\widehat{\otimes} C^\infty(\tilde{T}^n_\theta)$ is canonically a bimodule over $C^\infty(M)\widehat{\otimes} C^\infty(\tilde{T}^n_\theta)$ (and therefore also on $C^\infty(M)\widehat{\otimes} C^\infty(T^n_\theta)$). One defines a hermitian structure on $C^\infty(M,S)\widehat{\otimes} C^\infty(\tilde{T}^n_\theta)$ for its right-module structure over $C^\infty(M)\widehat{\otimes} C^\infty(\tilde{T}^n_\theta)$ [9] by setting

$$(\psi \otimes t, \psi' \otimes t') = (\psi, \psi') \otimes t^* t'$$

for $\psi, \psi' \in C^\infty(M,S)$ and $t, t' \in C^\infty(\tilde{T}^n_\theta)$. This gives by restriction the hermitian structure of $C^\infty(M_\theta, S)$ considered as a right $C^\infty(M_\theta)$-module; that is one has

$$(\psi f, \psi' f') = f^*(\psi, \psi') f'$$

for any $\psi, \psi' \in C^\infty(M_\theta, S)$ and $f, f' \in C^\infty(M_\theta)$. Notice that when $\dim(M)$ is even, one has a $\mathbb{Z}_2$-grading γ of $C^\infty(M,S)$ as hermitian module which induces a $\mathbb{Z}_2$-grading, again denoted by γ, of $C^\infty(M_\theta, S)$ as hermitian right $C^\infty(M_\theta)$-module.

Let J denote the charge conjugation of S. This is an antilinear mapping of $C^\infty(M,S)$ into itself such that

$$\begin{aligned} (J\psi, J\psi) &= (\psi, \psi) & (13.3)\\ JfJ^{-1} &= f^* & (13.4) \end{aligned}$$

for any $\psi \in C^\infty(M,S)$ and for any $f \in C^\infty(M)$, $(f^*(x) = \overline{f(x)})$. Furthermore one has also

$$JV_{\tilde{s}} = V_{\tilde{s}} J \tag{13.5}$$

for any $\tilde{s} \in \tilde{T}^n$. Let us define $\tilde{J}$ to be the unique antilinear operator on $C^\infty(M,S) \widehat{\otimes} C^\infty(\tilde{T}^n_\theta)$ satisfying $\tilde{J}(\psi \otimes t) = J\psi \otimes t^*$ for $\psi \in C^\infty(M,S)$ and $\tilde{t} \in C^\infty(\tilde{T}^n_\theta)$. The subspace $C^\infty(M_\theta, S)$ is stable by $\tilde{J}$ and we define J_θ to be the induced antilinear mapping of $C^\infty(M_\theta, S)$ into itself. It follows from (13.3), (13.4) and from the definition that one has

$$\begin{aligned} (J_\theta\psi, J_\theta\psi) &= (\psi, \psi) & (13.6)\\ J_\theta f J_\theta^{-1}\psi &= \psi f^* & (13.7) \end{aligned}$$

for any $\psi \in C^\infty(M_\theta, S)$ and $f \in C^\infty(M_\theta)$. Thus left multiplication by $J_\theta f^* J_\theta^{-1}$ is the same as right multiplication by f. Obviously J_θ satisfies, in function of $\dim(M)$ modulo 8, the table of normalizations, commutations with D_θ and with γ in the even dimensional case which corresponds to the reality conditions 7) of [15]. This follows of course from the same properties of J, D, γ (i.e. the same properties for $\theta = 0$). So equipped $C^\infty(M_\theta, S)$ is in particular an involutive bimodule with a right-hermitian structure [46], [29].

Let us now investigate the symbol of D_θ. It is easy to see that the left universal symbol $\sigma_L(D_\theta)$ of D_θ (as defined in [27]) factorizes through a homomorphism

$$\hat{\sigma}_L(D_\theta) : \Omega^1(M_\theta) \underset{C^\infty(M_\theta)}{\otimes} C^\infty(M_\theta, S) \to C^\infty(M_\theta, S)$$

of bimodules over $C^\infty(M_\theta)$. By definition, one has

$$[D_\theta, f]\psi = \hat{\sigma}_L(D_\theta)(df \otimes \psi)$$

for $f \in C^\infty(M_\theta)$ and $\psi \in C^\infty(M_\theta, S)$ and $df \mapsto [D_\theta, f]$ extends as an injective linear mapping of $\Omega^1(M_\theta)$ into the continuous linear endomorphisms of $C^\infty(M_\theta, S)$.

LEMMA 7 *Let f_i, g_i be a finite family of elements of $C^\infty(M_\theta)$ such that $\sum_i f_i[D_\theta, g_i] = 0$. Then the endomorphism $\sum_i [D_\theta, f_i][D_\theta, g_i]$ is the left multiplication in $C^\infty(M_\theta, S)$ by an element of $C^\infty(M_\theta)$.*

When no confusion arises, we shall summarize this statement by writing $\sum_i [D_\theta, f_i][D_\theta, g_i] \in C^\infty(M_\theta)$ whenever $\sum_i f_i[D_\theta, g_i] = 0$. Indeed, using the fact that D_θ is the restriction of $D \otimes I$ where D is the classical Dirac operator on M one shows that

$$\sum_i [D_\theta, f_i][D_\theta, g_i] + \sum_i f_i \Delta_\theta(g_i) = [D_\theta, \sum f_i[D_\theta, g_i]] = 0$$

where Δ_θ is the restriction of $\Delta \otimes I$ to $C^\infty(M_\theta)$ with Δ being the ordinary Laplace operator on M which is σ-invariant. This implies that $\sum_i f_i \Delta_\theta(g_i)$ is in $C^\infty(M_\theta)$ and therefore the result.

Concerning the particular case $M = \mathbb{R}^{2n}$ one shows the following result using the splitting homomorphism.

PROPOSITION 6 *Let $z^\mu, \bar{z}^\nu \in C^\infty(\mathbb{R}^{2n}_\theta)$ be as in Section 4. Then the $\hat{\Gamma}^\mu = [D_\theta, z^\mu]$, $\hat{\bar{\Gamma}}^\nu = [D_\theta, \bar{z}^\nu]$ satisfy the relations*

$$\hat{\Gamma}^\mu \hat{\Gamma}^\nu + \lambda^{\mu\nu} \hat{\Gamma}^\nu \hat{\Gamma}^\mu = 0,\ \hat{\bar{\Gamma}}^\mu \hat{\bar{\Gamma}}^\nu + \lambda^{\mu\nu} \hat{\bar{\Gamma}}^\nu \hat{\bar{\Gamma}}^\mu = 0,\ \hat{\bar{\Gamma}}^\mu \hat{\Gamma}^\nu + \lambda^{\nu\mu} \hat{\Gamma}^\nu \hat{\bar{\Gamma}}^\mu = \delta^{\mu\nu} \mathbb{1}$$

where $\mathbb{1}$ is the identity mapping of $C^\infty(\mathbb{R}^{2n}_\theta, S)$ onto itself.

This θ-twisted version of the generators of the Clifford algebra connected with the symbol of D_θ differs from the one introduced in Section 4 by the replacement $\lambda^{\mu\nu} \mapsto \lambda^{\nu\mu}$ and is the version associated with the θ-twisted version $\wedge_c \mathbb{R}^{2n}_\theta$ of the exterior algebra which is itself behind the differential calculus $\Omega(\mathbb{R}^{2n}_\theta)$. This is a counterpart for this example of the fact that $\Omega_{D_\theta} = \Omega(M_\theta)$.

We now make contact with the axiomatic framework of [15]. To simplify the discussion we shall assume now that M is a compact oriented m-dimensional riemannian spin manifold endowed with an isometric action of T^n, (i.e. we add compactness). One defines a positive definite scalar product on $C^\infty(M, S)$ by setting

$$\langle \psi, \psi' \rangle = \int_M (\psi, \psi') \text{vol}$$

where vol is the riemannian volume m-form which is σ-invariant and we denote by $\mathcal{H} = L^2(M, S)$ the Hilbert space obtained by completion. As an unbounded operator in $\mathcal{H}$, the Dirac operator $D : C^\infty(M, S) \to C^\infty(M, S)$ is essentially self-adjoint on $C^\infty(M,S)$. We identify D with its closure that is with the corresponding self-adjoint operator in $\mathcal{H}$. The spectral triple $(C^\infty(M),\mathcal{H},D)$ together with the real structure J satisfy the axioms of [15]. The homomorphism $\tilde{s}\mapsto V_{\tilde{s}}$ uniquely extends as a unitary representation of the group $\tilde{T}^n$ in $\mathcal{H}$ which will be still denoted by $\tilde{s}\mapsto V_{\tilde{s}}$. On the other hand the action $\tilde{s}\mapsto\tilde{\tau}_{\tilde{s}}$ of $\tilde{T}^n$ on $C^\infty(\tilde{T}^n_\theta)$ extends as a unitary action again denoted by $\tilde{s}\mapsto\tilde{\tau}_{\tilde{s}}$ of $\tilde{T}^n$ on the Hilbert space $L^2(\tilde{T}^n_\theta)$ which is obtained from $C^\infty(\tilde{T}^n_\theta)$ by completion for the Hilbert norm $f\mapsto\parallel f\parallel=\mathrm{tr}(f^\ast f)^{1/2}$ where tr is the usual normalized trace of $C^\infty(\tilde{T}^n_\theta)=C^\infty(T^n_{\frac{1}{2}\theta})$. We now define the spectral triple $(C^\infty(M_\theta),\mathcal{H}_\theta,D_\theta)$ to be the following one. The Hilbert space $\mathcal{H}_\theta$ is the subspace of the Hilbert tensor product $\mathcal{H}\widehat{\otimes}L^2(\tilde{T}^n_\theta)$ which consists of elements Ψ which are invariant by the diagonal action of $\tilde{T}^n$, that is which satisfy $V_{\tilde{s}}\otimes\tilde{\tau}_{-\tilde{s}}(\Psi)=\Psi$, $\forall\tilde{s}\in\tilde{T}^n$. The operator D_θ identifies with an unbounded operator in $\mathcal{H}_\theta$ which is essentially self-adjoint on the dense subspace $C^\infty(M_\theta,S)$. We also identify D_θ with its closure that is with the self-adjoint operator which is also the restriction to $\mathcal{H}_\theta$ of $D\otimes I$. The antilinear operator J_θ canonically extends as anti-unitary operator in $\mathcal{H}_\theta$ (again denoted by J_θ).

THEOREM 9 *The spectral triple $(C^\infty(M_\theta),\mathcal{H}_\theta,D_\theta)$ together with the real structure J_θ satisfy all axioms of noncommutative geometry of [15].*

Notice that axiom 4) of orientability is directly connected to the σ-invariance of the m-form vol on M. Consequently this form defines a σ-invariant m-form on M_θ in view of Proposition 3 which corresponds to a σ-invariant Hochschild cycle in $Z_m(A,A)$ for both $A=C^\infty(M)$ and $A=C^\infty(M_\theta)$. The argument for Poincaré duality is the same as in [18]. Finally, the isospectral nature of the deformation $(C^\infty(M),\mathcal{H},D,J)\mapsto(C^\infty(M_\theta),\mathcal{H}_\theta,D_\theta,J_\theta)$ follows immediately from the fact that $D_\theta=D\otimes I$.

Coming back to the notations of sections 4 and 5, we can then return to the noncommutative geometry of S^m_θ.
This geometry (with variable metric) is entirely specified by the projection e, the matrix algebra (which together generate the algebra of coordinates) and the Dirac operator which fulfill a polynomial equation of degree m.

THEOREM 10 *Let g be any T^n-invariant Riemannian metric on S^m, $m=2n$ or $m=2n-1$, whose volume form is the same as for the round metric.* *(i) Let $e\in M_{2^n}(C^\infty(S^{2n}_\theta))$ be the projection of Theorem 4 . Then the Dirac operator D_θ of S^{2n}_θ associated to the metric g satisfies*

$$\langle(e-\frac{1}{2})[D_\theta,e]^{2n}\rangle=\gamma$$

where $\langle\,\rangle$ is the projection on the commutant of $M_{2^n}(\mathbb{C})$.
(ii) Let $U\in M_{2^{n-1}}(C^\infty(S^{2n-1}_\theta))$ be the unitary of Theorem 4. Then the Dirac operator D_θ of S^{2n-1}_θ associated to the metric g satisfies

$$\langle U[D_\theta,U^\ast]([D_\theta,U][D_\theta,U^\ast])^{n-1}\rangle=1$$

where $\langle\ \rangle$ *is the projection on the commutant of* $M_{2^{n-1}}(\mathbb{C})$.

Using the splitting homomorphism as for Theorem 7 it is enough to show that this holds for the classical case $\theta = 0$, i. e. when D is the classical Dirac operator associated to the metric g.

This result is of course a straightforward extension of results of [17], [18]. Since the deformed algebra $C^\infty(S^m_\theta)$ is highly nonabelian the inner fluctuations of the noncommutative metric ([15]) generate non-trivial internal gauge fields which compensate for the loss of gravitational degrees of freedom imposed by the T^n-invariance of the metric g.

14 Further prospect

We have shown that the basic K-theoretic equation defining spherical manifolds admits a complete solution in dimension 3 and that for generic values of the deformation parameters the obtained algebras of polynomials on the deformed $\mathbb{R}^4_{\mathbf{u}}$ only depend on two parameters and are isomorphic to the algebras introduced by Sklyanin in connection with the Yang-Baxter equation. The spheres themselves do depend on the three initial parameters and we postpone their analysis to part II.
We did concentrate here on the critical values of the deformation parameters i.e. on the subclass of θ-deformations and identified as m-dimensional noncommutative spherical manifolds the noncommutative m-sphere S^m_θ for any $m \in \mathbb{N}$. For this class we completed the path from the crudest level of the algebra $C_{\mathrm{alg}}(S)$ of polynomial functions on S to the full-fledged structure of noncommutative geometry [15], as exemplified in theorem 9. We showed that the basic polynomial equation fulfilled by the Dirac operator held unaltered in the noncommutative case. We also obtained the noncommutative analogue of the self-duality equations and described concretely the quantum symmetry groups.

Needless to say our goal in part II will be to analyse general spherical 3-manifolds including their smooth structure, their differential calculus and metric aspect. For these non-critical generic values the scale invariance inherited from criticality in the above examples will no longer hold. This will generate very interesting new phenomena. The analysis of the corresponding noncommutative spaces $S^3_{\mathbf{u}}$ is much more involved as we shall see in part II.

15 Appendix : Relations in the noncommutative Grassmannian

Let $\mathcal{A}$ be the universal Grassmannian generated by the 2^2 elements $\alpha, \beta, \gamma, \delta$ with the relations,

$$U\,U^* = U^*\,U = 1, \qquad U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{15.1}$$

In this appendix we shall show that the intersection $\mathcal{J}$ of the kernels of the representations ρ of $\mathcal{A}$ such that $\mathrm{ch}_{\frac{1}{2}}(\rho(U)) = 0$ is a non-trivial two sided ideal of $\mathcal{A}$. Thus the odd Grassmanian $\mathcal{B}$ which was introduced in [18] is a nontrivial quotient of $\mathcal{A}$.
Given an algebra $\mathcal{A}$ and elements $x_j \in \mathcal{A}$ we let,

$$[x_1, \ldots, x_n] = \Sigma\; \varepsilon(\sigma)\, x_{\sigma(1)} \ldots x_{\sigma(n)} \tag{15.2}$$

where the sum is over all permutations and $\varepsilon(\sigma)$ is the signature of the permutation.
With the above notations, let $\mu = [\alpha, \beta, \gamma, \delta]$. We shall check that,

LEMMA 8 *In any representation ρ of $\mathcal{A}$ for which $\mathrm{ch}_{\frac{1}{2}}(\rho(U)) = 0$ one has, $\rho([\mu, \mu^*]) = 0$. Moreover $[\mu, \mu^*] \neq 0$ in $\mathcal{A}$.*

Proof. For $y_i = \lambda_i^j\, x_j$, one has $[y_1, \ldots, y_n] = \det \lambda\, [x_1, \ldots, x_n]$. This allows to extend the map $a \otimes b \otimes c \otimes d \to [a, b, c, d]$ to a linear map c,

$$c : \wedge^4 \mathcal{A} \to \mathcal{A}\,. \tag{15.3}$$

Let us now show that, for any representation ρ of $\mathcal{A}$ for which $\mathrm{ch}_{\frac{1}{2}}(\rho(U)) = 0$, the following relation fulfilled by the matrix elements $\tilde{\alpha} = \rho(\alpha), \ldots, \tilde{\delta} = \rho(\delta)$,

$$\tilde{\alpha} \otimes \tilde{\alpha}^* + \tilde{\beta} \otimes \tilde{\beta}^* + \tilde{\gamma} \otimes \tilde{\gamma}^* + \tilde{\delta} \otimes \tilde{\delta}^* = \tilde{\alpha}^* \otimes \tilde{\alpha} + \tilde{\beta}^* \otimes \tilde{\beta} + \tilde{\gamma}^* \otimes \tilde{\gamma} + \tilde{\delta}^* \otimes \tilde{\delta} \tag{15.4}$$

implies,

$$[\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}]\, [\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}]^* = [\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}]^*\, [\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}]\,. \tag{15.5}$$

It follows from (15.4) that,

$$(\tilde{\alpha} \wedge \tilde{\beta} \wedge \tilde{\gamma} \wedge \tilde{\delta}) \otimes (\tilde{\alpha}^* \wedge \tilde{\beta}^* \wedge \tilde{\gamma}^* \wedge \tilde{\delta}^*) = (\tilde{\alpha}^* \wedge \tilde{\beta}^* \wedge \tilde{\gamma}^* \wedge \tilde{\delta}^*) \otimes (\tilde{\alpha} \wedge \tilde{\beta} \wedge \tilde{\gamma} \wedge \tilde{\delta})\,. \tag{15.6}$$

Indeed we view $\tilde{\mathcal{A}} = \rho(\mathcal{A})$ as a linear space and consider the tensor product of exterior algebras,

$$\wedge\tilde{\mathcal{A}} \otimes \wedge\tilde{\mathcal{A}} \qquad \text{(ungraded tensor product).} \tag{15.7}$$

We then take the 4^{th} power of (15.4) and get,

$$24\,(\tilde\alpha\wedge\tilde\beta\wedge\tilde\gamma\wedge\tilde\delta)\otimes(\tilde\alpha^*\wedge\tilde\beta^*\wedge\tilde\gamma^*\wedge\tilde\delta^*)=24\,(\tilde\alpha^*\wedge\tilde\beta^*\wedge\tilde\gamma^*\wedge\tilde\delta^*)\otimes(\tilde\alpha\wedge\tilde\beta\wedge\tilde\gamma\wedge\tilde\delta)\,. \tag{15.8}$$

We can then apply $c\otimes c$ on both sides and compose with $m:\tilde{\mathcal{A}}\otimes\tilde{\mathcal{A}}\to\tilde{\mathcal{A}}$, the product, to get (15.5), that is

$$\rho([\mu,\mu^*])=0 \tag{15.9}$$

It remains to check that $[\mu,\mu^*]\neq 0$ in $\mathcal{A}$.
One has

$$M_2(\mathbb{C})*\mathbb{C}\,\mathbb{Z}=M_2(\mathcal{A}) \tag{15.10}$$

where the free product in the left hand side is the free algebra generated by $M_2(\mathbb{C})$ and a unitary U, $U^*\,U=U\,U^*=1$. As above $\mathcal{A}$ is generated by the matrix elements of U,

$$U=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\ ;\ \alpha,\beta,\gamma,\delta\in\mathcal{A} \tag{15.11}$$

As a linear basis of $M_2(\mathbb{C})$ we use the Pauli spin matrices, which we view as a projective representation of $\Gamma=(\mathbb{Z}/2)^2$,

$$(0,0)\xrightarrow{\sigma}1\ ,\quad (0,1)\xrightarrow{\sigma}\sigma_1\ ,\quad (1,0)\xrightarrow{\sigma}\sigma_2\ ,\quad (1,1)\xrightarrow{\sigma}\sigma_3 \tag{15.12}$$

with $\sigma\,(a+b)=c\,(a,b)\ \sigma(a)\ \sigma(b)\qquad\forall\,a,b\in(\mathbb{Z}/2)^2$.
Since we are dealing with a free product, we have a natural basis of $M_2(\mathcal{A})$ given by the monomials,

$$\sigma_{i_1}\,U^{j_1}\ \sigma_{i_2}\,U^{j_2}\ldots\sigma_{i_k}\,U^{j_k} \tag{15.13}$$

where i_1 and j_k can be 0 but all other i_ℓ,j_ℓ are $\neq 0$. The projection to $\mathcal{A}$ is given by,

$$P(T)=\frac{1}{4}\sum_{\Gamma}\ \sigma(a)\,T\ \sigma^{-1}(a)\,. \tag{15.14}$$

In particular the matrix components $\alpha,\beta,\gamma,\delta$, of U are linear combinations of the four elements,

$$x_a=P\,(\sigma(a)\,U)\ ,\qquad a\in\Gamma=(\mathbb{Z}/2)^2\,. \tag{15.15}$$

We want to compute $[\alpha,\beta,\gamma,\delta]$ or equivalently $[x_0,x_1,x_2,x_3]$.
Let us first rewrite the product $x_{a_1}\,x_{a_2}\,x_{a_3}\,x_{a_4}$, which is up to an overall coefficient 4^{-4},

$$\sum_{b_i}\ \sigma(b_1)\,\sigma(a_1)\,U\,\sigma(b_1)^{-1}\,\sigma(b_2)\,\sigma(a_2)\,U\,\sigma(b_2)^{-1}\,\sigma(b_3)\,\sigma(a_3)\,U\,\sigma(b_3)^{-1}$$
$$\sigma(b_4)\,\sigma(a_4)\,U\,\sigma(b_4)^{-1} \tag{15.16}$$

as a sum of terms of the form,

$$\sigma(c_1)\,U\,\sigma(c_2)\,U\ldots U\,\sigma(c_4)\,U\,\sigma(c_4)^{-1}\,\sigma(c_3)^{-1}\,\sigma(c_2)^{-1}\,\sigma(c_1)^{-1}$$
$$\lambda\,(c_1,\ldots,c_4)\,\sigma(a_1)\,\sigma(a_2)\,\sigma(a_3)\,\sigma(a_4)\,. \tag{15.17}$$

where $c_1 = b_1 + a_1$, $c_2 = b_2 - b_1 + a_2$, $c_3 = b_3 - b_2 + a_3$, $c_4 = b_4 - b_3 + a_4$ vary independently in Γ and $\lambda\,(c_1,\ldots,c_4) \in U(1)$ can be computed using the trivial representation, $U \to 1$ by,

$$\sigma(b_1)\,\sigma(a_1)\,\sigma(b_1)^{-1}\,\sigma(b_2)\,\sigma(a_2)\sigma(b_2)^{-1}\,\sigma(b_2)\,\sigma(a_3)\,\sigma(b_3)^{-1}\,\sigma(b_4)\,\sigma(a_4)\,\sigma(b_4)^{-1}$$
$$= \lambda\,(c_1,\ldots,c_4)\,\sigma(a_1)\,\sigma(a_2)\,\sigma(a_3)\,\sigma(a_4). \tag{15.18}$$

Each term in the reduced expansion of $[x_0, x_1, x_2, x_3]$ is the sum of the above expressions multiplied by $\varepsilon(a) = \delta^{a_1\,a_2\,a_3\,a_4}_{0\,1\,2\,3}$ the signature of the permutation $\{0,1,2,3\} \to \{a_1, a_2, a_3, a_4\}$.
To see that $[x_0, x_1, x_2, x_3] \neq 0$ we compute the terms in

$$U^3\,\sigma(c)\,U\,\sigma(c)^{-1}\,. \tag{15.19}$$

Fixing c there is one contribution for each of the permutation of $\{0,1,2,3\}$ and in (15.16) we have

$$b_1 = a_1\,,\;\; b_2 = a_1 + a_2\,,\;\; b_3 = a_1 + a_2 + a_3\,,\;\; b_4 = c + a_1 + a_2 + a_3 + a_4\,. \tag{15.20}$$

In $(\mathbb{Z}/2)^2 = \Gamma$ one has $a_1 + a_2 + a_3 + a_4 = 0$ so that $b_4 = c$, $b_3 = a_4$. Since $\sigma(x)^2 = 1$ one can thus write (15.16) as

$$U\,\sigma(a_1)\,\sigma(a_1+a_2)\,\sigma(a_2)\,U\,\sigma(a_1+a_2)\,\sigma(a_4)\,\sigma(a_3)\,U\,\sigma(a_4)\,\sigma(c)\,\sigma(a_4)\,U\,\sigma(c) \tag{15.21}$$

which we should multiply by $\varepsilon(a)$ and sum over a.
It is clear here that $\sigma(a_1)\,\sigma(a_1+a_2)\,\sigma(a_2)$ and $\sigma(a_1+a_2)\,\sigma(a_4)\,\sigma(a_3)$ are *scalar* and thus commute with U which allows to write (15.21) as follows,

$$U^3\,\sigma(a_1)\,\sigma(a_1+a_2)\,\sigma(a_2)\,\sigma(a_1+a_2)\,\sigma(a_4)\,\sigma(a_3)\,\sigma(a_4)\,\sigma(c)\,\sigma(a_4)\,U\,\sigma(c)\,. \tag{15.22}$$

One has,

$$\sigma(a)\,\sigma(a')\,\sigma(a)^{-1}\,\sigma(a')^{-1} = (-1)^{\langle a,a'\rangle} \qquad \forall\, a, a' \in \Gamma \tag{15.23}$$

using the bilinear form with $\mathbb{Z}/2$ values on Γ given by,

$$\langle a, a'\rangle = \alpha\,\beta' - \alpha'\,\beta \quad \text{for} \quad a = (\alpha,\beta)\,,\; a' = (\alpha',\beta') \in (\mathbb{Z}/2)^2\,. \tag{15.24}$$

Permuting $\sigma(a_1+a_2)$ with $\sigma(a_2)$ and $\sigma(a_4)$ with $\sigma(a_3)$ introduces terms in $(-1)^n$ with $n = \langle a_1 + a_2, a_2\rangle + \langle a_3, a_4\rangle = \langle a_1, a_2\rangle + \langle a_3, a_4\rangle$. One has $0 \in \{a_1, a_2\}$ or $0 \in \{a_3, a_4\}$. In the first case $\langle a_1, a_2\rangle = 0$ and $\langle a_3, a_4\rangle = 1$ since they are distinct

$\neq 0$. Similarly if $0 \in \langle a_3, a_4\rangle$ we get $\langle a_1, a_2\rangle + \langle a_3, a_4\rangle = 1$ in all cases. We can thus replace (15.22) by

$$-\,U^3\,\sigma(a_1)\,\sigma(a_2)\,\sigma(a_3)\,\sigma(c)\,\sigma(a_4)\,U\,\sigma(c)\,. \tag{15.25}$$

Permuting c with a_4 gives a $(-1)^{\langle c, a_4\rangle}$. We have,

$$\sigma(a_1)\,\sigma(a_2)\,\sigma(a_3)\,\sigma(a_4) = (-1)^s\,\sigma_1\,\sigma_2\,\sigma_3 \;=\; i\,(-1)^s. \tag{15.26}$$

where $(-1)^s$ is the signature of the permutation of $\{1, 2, 3\}$ given by the *non zero* a_j's. The coefficient of $U^3\,\sigma(c)\,U\,\sigma(c)^{-1}$ is thus,

$$-4^{-4}\Sigma\, i\,\varepsilon(a)\,(-1)^s\,(-1)^{\langle c, a_4\rangle}\,. \tag{15.27}$$

Taking $c = (1, 0)$ we find 16 $-$ signs and 8 $+$ signs so that we get the term,

$$4^{-4}(-(-16+8)i)\,U^3\,\sigma_2\,U\,\sigma_2 = \frac{i}{32}\,U^3\,\sigma_2\,U\,\sigma_2\,. \tag{15.28}$$

Taking $c = (0, 1)$ we also find 16 $-$ signs and 8 $+$ signs which gives

$$\frac{i}{32}\,U^3\,\sigma_1\,U\,\sigma_1\,. \tag{15.29}$$

Thus if we let $\mu = [x_0, x_1, x_2, x_3]$ and compute $\mu\mu^*$ we get terms of the form,

$$\frac{-1}{(32)^2}\,U^3\,\sigma_1\,U\,\sigma_1\,\sigma_2\,U^{-1}\,\sigma_2\,U^{-3} \tag{15.30}$$

which cannot be simplified and do not appear in the product $\mu^*\mu$ where we always have negative powers for the first U's on the left followed by positive powers.
Thus we conclude that in the universal algebra $\mathcal{A}$ one has

$$[\mu, \mu^*] \neq 0\,. \tag{15.31}$$

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