Quantum Gravity Boundary Terms from Spectral Action

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We study the boundary terms of the spectral action of the noncommutative space, defined by the spectral triple dictated by the physical spectrum of the standard model, unifying gravity with all other fundamental interactions. We prove that the spectral action predicts uniquely the gravitational boundary term required for consistency of quantum gravity with the correct sign and coefficient. This is a remarkable result given the lack of freedom in the spectral action to tune this term.

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It has been known since the 1960's [1] that in the Hamiltonian quantization of gravity it is essential to include boundary terms in the action, as this allows to define consistently the momentum conjugate to the metric. This makes it necessary to modify the Einstein-Hilbert action by adding to it a surface integral term so that the variation of the action becomes well defined and yields the Einstein field equations. The reason for this manipulation is that the curvature scalar R contains second derivatives of the metric, which are removed after integrating by parts to obtain an action which is quadratic in first derivatives of the metric. These surface terms are canceled by modifying the Euclidean action to [2], [3]

$$I=-\frac{1}{16\pi}\int_M d^4x\sqrt{g}R-\frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{h}K,$$

where ∂M is the boundary of M, h_{ab} is the induced metric on ∂M and K is the trace of the second fundamental form on ∂M. We use the sign convention according to which R is positive for the sphere and K is positive for the ball. Notice that there is a relative factor of 2 and a fixed sign between the two terms, and that the boundary term has to be completely fixed. This is a delicate fine tuning and is not determined by any symmetry, but only by the consistency requirement. There is no known symmetry that predicts this combination and it is always added by hand.

In the noncommutative geometric approach to the formulation of a unified theory of all fundamental interactions including gravity, the starting point is the replacement of the Riemannian geometry of space-time with noncommutative geometry. The basic data of noncommutative geometry consists of an involutive algebra $\mathcal{A}$ of operators in Hilbert space $\mathcal{H}$, which plays the role of the algebra of coordinates, and a self-adjoint operator D in $\mathcal{H}$ [4] which plays the role of the inverse of the line element. The spectrum of the standard model indicates that the algebra is to be taken as $\mathcal{A}=C^{\infty}(M)\otimes\mathcal{A}_F$ where the algebra $\mathcal{A}_F$ is finite dimensional, $\mathcal{A}_F=\mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})$, and $\mathbb{H}\subset M_2(\mathbb{C})$ is the algebra of quaternions. The algebra $\mathcal{A}$ is a tensor product which geometrically corresponds to a product space. The spectral geometry of $\mathcal{A}$ is given by the product rule

$$\mathcal{H}=L^2(M,S)\otimes\mathcal{H}_F,\quad D=D_M\otimes 1+\gamma_5\otimes D_F,$$

where $L^2(M,S)$ is the Hilbert space of L^2 spinors, and D_M is the Dirac operator of the Levi-Civita spin connection on M. The Hilbert space of quarks and leptons fixes the choice of the Dirac operator D_F and the action of $\mathcal{A}_F$ in $\mathcal{H}_F$. The operator D_F anticommutes with the chirality operator γ_F on $\mathcal{H}_F$. The spectral geometry does not change if one replaces D by the equivalent operator

$$D=D_M\otimes\gamma_F+1\otimes D_F, \tag{1}$$

but this equivalence fails when M has a boundary and it is only the latter choice which has conceptual meaning since γ_5 no longer anticommutes with D_M when $\partial M\neq\emptyset$. The noncommutative space defined by a spectral triple has to satisfy the basic axioms of noncommutative geometry. This approach shares a common feature with Euclidean quantum gravity in that the Riemannian manifold is taken to be Euclidean in order for the line element, which is the inverse of the Dirac operator, to be compact. It is then assumed that one obtains the Lorentzian results by analytically continuing the expressions obtained by performing the path integral to Minkowski space. A fundamental principle in the noncommutative approach is that the usual emphasis on the points $x\in M$ of a geometric space is now replaced by the spectrum of the operator D. The spectral action principle states that the physical action depends only the spectrum of the Dirac operator, which is geometrical. Indeed, it was shown that all the fundamental interactions including gravity are unified in the spectral action [5]

$$I=\operatorname{Tr}f\left(\frac{D}{\Lambda}\right)+\langle\Psi,D\Psi\rangle,$$

where Tr is the usual trace of operators in the Hilbert space $\mathcal{H}$, Λ is a cut-off scale and f is a positive function. The action is then uniquely defined and the only arbitrariness one encounters is in the first few coefficients in the spectral expansion since higher coefficients are suppressed by the high-energy scale. This remarkable action includes the gravitational Einstein-Hilbert term with the square of the Weyl tensor, the $SU(3)_c \times SU(2)_w \times U(1)_Y$ gauge interactions, the Higgs couplings including the spontaneous symmetry breaking, all coming with the correct signs as well as a relation between the gauge couplings and Higgs couplings. It also accommodates small neutrino masses through the see-saw mechanism, thanks to a more subtle choice ([12]) of the chirality operator γ_F which gives to the geometry F a KO-dimension which is congruent to 6 modulo 8. The charge conjugation operator J for the product geometry (1) is then given by

$$J = J_M \ \gamma_5 \otimes J_F$$

which commutes with the operator D given by (1) since in even dimension J_M commutes with D_M while in dimension 6 modulo 8, J_F anticommutes with γ_F.

The results were derived for manifolds without boundary. We stress that definition of the noncommutative space corresponding to the physical space-time must satisfy the restrictive axioms of noncommutative geometry. Once this is done, there is essentially no freedom left in determining the spectral action, except for the three coefficients of the Mellin transform of the function f. These correspond to the cosmological constant, the Newton constant and the gauge couplings and where the dependence on the energy scale is governed by the renormalization group equations. Because of these constraints, it is essential to find out whether the boundary terms of the spectral action agree with the modifications dictated by the consistency of quantum gravity. This is a severe test of the spectral action principle as there is no freedom present in tuning the surface terms to reproduce the desired results with correct signs and numerical values. It is the purpose of this work to show that the spectral action does pass all tests predicting the correct modification of the boundary terms. We can go further and make the mass scale Λ appearing in the Dirac operator dynamical by replacing it with a dilaton field. We have recently shown that in this case the spectral action becomes almost scale invariant and gives the same low-energy limit as the Randall-Sundrum model as well as providing a model for extended inflation [6]. In other words, the simple form of the spectral action is capable of producing all the desirable features of unified theories including gravity with the correct physical predictions.

The Dirac operator in the spectral action must satisfy the hermiticity condition

$$\langle \Psi, D\Psi \rangle = \langle D\Psi, \Psi \rangle \, .$$

These are satisfied provided the following boundary condition is imposed [7], [8],[9]

$$\Pi_- \Psi|_{\partial M} = 0,$$

where the projection operator Π_- is given by $\Pi_- = \frac{1}{2}(1-\chi)$ where $\chi = \gamma_n \gamma_5$ satisfies $\chi^2 = 1$. The Clifford algebra is defined by $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$ and we denote by n the unit *inward* normal and γ_n the corresponding Clifford multiplication. Although one can keep the discussion general, it will be more transparent to specialize to the case where the dimensions of the continuous part of the noncommutative space is taken to be four. A local system of coordinates on M will be denoted by x^μ, $\mu = 1, \cdots 4$, and on ∂M will be denoted by y^a, $a = 1, 2, 3$. Let the functions $x^\mu(y^a)$ be given by the embedding of the hypersurface in M and let $e_a^\mu = \frac{\partial x^\mu}{\partial y^a}$, then the metric $g_{\mu\nu}$ on M induces a metric h_{ab} on the hypersurface such that $h_{ab} = g_{\mu\nu} e_a^\mu e_b^\nu$ and where n^μ is orthogonal to e_a^μ so that $g_{\mu\nu} n^\mu e_a^\nu = 0$. It is convenient to define $n_\mu = g_{\mu\nu} n^\nu$ so that $n_\mu e_a^\mu = 0$. We now define the inverse functions e_μ^a by $e_a^\mu e_\mu^b = \delta_a^b$ which satisfies the condition $e_a^\mu e_\nu^a = \delta_\nu^\mu - n^\mu n_\nu$ to be consistent with $n_\mu e_a^\mu = 0$. We therefore can write [10]

$$g_{\mu\nu} = h_{ab} e_\mu^a e_\nu^b + n_\mu n_\nu .$$

The inverse metric is also defined by $h^{ab} = g^{\mu\nu} e_\mu^a e_\nu^b$ and the inverse relation is

$$g^{\mu\nu} = h^{ab} e_a^\mu e_b^\nu + n^\mu n^\nu .$$

This shows that any tensor can be projected into the hypersurface using the completeness relations for the basis $\left\{ e_\mu^a, n_\mu \right\}$ .We finally define on ∂M,

$$\chi = -\frac{\sqrt{h}}{3!} \epsilon^{abc} \gamma_a \gamma_b \gamma_c, \quad \gamma_5 = \chi \gamma_n,$$

which satisfy $\chi^2 = 1$, $\chi\gamma^a = \gamma^a\chi$, $\chi\gamma^n = -\gamma^n\chi$, $\gamma_5^2 = 1$, $\chi\gamma_5 = -\gamma_5\chi$. The normal vector n^μ satisfies the properties

$$n_{\mu;\nu} = -K_{ab} e_\mu^a e_\nu^b, \qquad e_{a;\nu}^\mu e_b^\nu = \Gamma_{ab}^c e_c^\mu + K_{ab} n^\mu$$

where the covariant derivative $;\nu$ is the space-time covariant derivative and Γ_{ab}^c is the Christoffel connection of the metric h_{ab}, and K_{ab} is the extrinsic curvature whose symmetry follows from the relation $e_{a;b}^\mu = e_{b;a}^\mu$.

The bosonic part of the spectral action is then obtained by using the identity [5]

$$\mathrm{Tr}\left(f\left(D^2/m^2 \right) \right) \simeq \sum_{n \geq 0} f_{4-n} \, a_n \left(D^2/m^2 \right),$$

where f_n are related to the Mellin transforms of the function f. The Seeley-deWitt coefficients $a_n(P, \chi)$ are geometrical invariants. These were calculated for Laplacians which are the square of the Dirac operator, for manifolds with boundary. To evaluate these terms, we first write the Laplacian in the form

$$\begin{aligned} P = D^2 &= -\left(g^{\mu\nu} \partial_\mu \partial_\nu + \mathbb{A}^\mu + \mathbb{B} \right) \\ &= -\left(g^{\mu\nu} \nabla_\mu^{'} \nabla_\nu^{'} + E \right), \end{aligned}$$

where $\nabla_\mu^{'} = \partial_\mu + \omega_\mu^{'}$ and $\omega_\mu^{'} = \frac{1}{2} g_{\mu\nu} \left(\mathbb{A}^\nu + g^{\rho\sigma} \Gamma_{\rho\sigma}^\nu \right)$. It is convenient to write the Dirac operator in the form

$$D = \gamma^\mu \nabla_\mu - \Phi,$$

where $\nabla_\mu = \partial_\mu + \omega_\mu$ and ω_μ is the torsion free spin-connection. The boundary conditions for D^2 are then equivalent to [8], [9]

$$\mathcal{B}_\chi \Psi = \Pi_- (\Psi)|_{\partial M} \oplus \Pi_+ \left(\nabla_n^{'} + S\right) \Pi_+ (\Psi)|_{\partial M} = 0,$$

where

$$S = \Pi_+ \left(\gamma_n \Phi - \frac{1}{2}\gamma_n \gamma^a \nabla_a^{'} \chi\right) \Pi_+,$$
$$\nabla_a^{'} \chi = \partial_a \chi + \left[\omega_a^{'}, \chi\right] = K_{ab}\chi\gamma^n\gamma^b + [\theta_a, \chi],$$

and where $\theta_a = \omega_a' - \omega_a$. We then have the relations

$$E = \gamma^\mu \nabla_\mu \Phi - \Phi^2 - \frac{1}{2}\gamma^{\mu\nu}\Omega_{\mu\nu},$$
$$\Omega_{\mu\nu} = \partial_\mu \omega_\nu^{'} - \partial_\nu \omega_\mu^{'} + \omega_\mu^{'}\omega_\nu^{'} - \omega_\nu^{'}\omega_\mu^{'}.$$

We list the first relevant Seeley-deWitt coefficients for Laplacians which are square of Dirac operators [11]

$$a_0 (P, \chi) = \frac{1}{16\pi^2} \int_M d^4x \sqrt{g} \mathrm{Tr}\,(1),$$

$$a_1 (P, \chi) = 0,$$

$$a_2 (P, \chi) = \frac{1}{96\pi^2} \left(\int_M d^4x \sqrt{g} \mathrm{Tr}\,(6E + R) + \int_{\partial M} d^3x \sqrt{h} \mathrm{Tr}\,(2K + 12S) \right),$$

$$a_3 (P, \chi) = \frac{1}{384(4\pi)^{\frac{3}{2}}} \int_{\partial M} d^3x \sqrt{h} \mathrm{Tr}\,\big(96\chi E + 3K^2 + 6K_{ab}K^{ab} + 96SK + 192S^2 - 12\nabla_a^{'}\chi\nabla^{'a}\chi\big),$$

As a warm up, these results could be applied to the simple case of an ordinary Dirac operator

$$D = \gamma^\mu (\partial_\mu + \omega_\mu).$$

Therefore, in the above formulas we have

$$\omega_\mu^{'} = \omega_\mu, \quad E = -\frac{1}{4}R, \quad \Phi = 0,$$
$$S = -\frac{1}{2}K\Pi_+, \quad \nabla_a^{'}\chi = K_{ab}\chi\gamma^n\gamma^b$$

Substituting $\mathrm{Tr}(1) = 4$ and $\mathrm{Tr}(S) = -K$ we have for the first few terms

$$a_0 (P, \chi) = \frac{1}{4\pi^2} \int_M d^4x \sqrt{g}$$

$$a_2 (P, \chi) = -\frac{1}{24\pi^2} \left(\int_M d^4x \frac{1}{2}\sqrt{g} R + \int_{\partial M} d^3x \sqrt{h} K \right)$$

$$a_3 (P, \chi) = \frac{1}{32(4\pi)^{\frac{3}{2}}} \int_{\partial M} d^3x \sqrt{h} \left(K^2 - 2K_{ab}K^{ab}\right)$$

The important point in the above result is the emergence of the combination [2]

$$-\int_M d^4x \sqrt{g} R - 2 \int_{\partial M} d^3x \sqrt{h} K$$

as the lowest term of the gravitational action which is known to be the required correction to the Einstein action involving the surface term so as to make the Hamiltonian formalism consistent. This is remarkable because both the sign and the coefficient are correct. The only assumption made is that the boundary conditions are local and enforce the hermiticity of the Dirac operator. This is yet another miracle concerning correct signs obtained in the spectral action of the Dirac operator. We also notice that the relative coefficient between R and K depends, in general, on the nature of the Laplacian. The desired answer is true for the square of the Dirac operator, but *not* for a general Laplacian.

This is a general result and applies to all noncommutative models based on spaces which are the tensor product of the spectral triple of a Riemannian manifold by that of a discrete space. In particular the above feature also works for the spectral action of the standard model. Indeed by applying the above formulas to the Dirac operators in the quarks and leptonic sectors with the corresponding boundary conditions one derives the full spectral action with boundary terms included. It is given by the following expression (note that in [12] we use the opposite sign convention for the scalar R):

$$I = \frac{48\Lambda^4}{\pi^2} f_4 \int_M d^4x \sqrt{g} + \frac{8\Lambda^2}{\pi^2} f_2 \left\{ \int_M d^4x \sqrt{g} \left(-\frac{1}{2}R - \frac{1}{4}\left(a\,|\varphi|^2 + \frac{1}{2}c \right) \right) - \int_{\partial M} d^3x \sqrt{h} K \right\} + \frac{2\Lambda}{(4\pi)^{\frac{3}{2}}} f_1 \int_{\partial M} d^3x \sqrt{h} \left(3\left(K^2 - 2K_{ab}K^{ab}\right)\right)$$

$$+\frac{f_0}{2\pi^2}\left\{\int_M d^4x\sqrt{g}\left(-\frac{3}{5}C^2_{\mu\nu\rho\sigma}+\frac{11}{30}R^*R^*-\frac{2}{5}R_{;\mu}^{\ \mu}\right.\right.$$
$$+a\left|D_\mu\varphi\right|^2+\frac{1}{6}R\left(a\left|\varphi\right|^2+\frac{1}{2}c\right)$$
$$\left.+g_3^2\left(G^i_{\mu\nu}\right)^2+g_2^2\left(F^\alpha_{\mu\nu}\right)^2+\frac{5}{3}g_1^2\left(B_{\mu\nu}\right)^2\right)$$
$$\left.+b\left|\varphi\right|^4+2e\left|\varphi\right|^2+\frac{1}{2}d-\frac{1}{3}a\left(\left|\varphi\right|^2\right)_{;\mu}^{\ \mu}\right\}$$
$$+\frac{f_0}{2\pi^2}\left\{\int_{\partial M} d^3x\sqrt{h}\left(\frac{1}{3}K\left(a\left|\varphi\right|^2+\frac{1}{2}c\right)\right.\right.$$
$$\left.+\frac{2}{15}\left(5RK+4KR^a_{\ nan}+4K_{ab}R^c_{\ acb}+18R_{anbn}K^{ab}\right)\right)$$
$$\left.+\frac{4}{315}\left(17K^3+39KK_{ab}K^{ab}-116K_a^{\ b}K_b^{\ c}K_c^{\ a}\right)\right\},$$

where $f_n=\int\limits_0^\infty v^{n-1}f(v)dv$, and

$$a=\text{tr}\left(3\left|k^u\right|^2+3\left|k^d\right|^2+\left|k^e\right|^2+\left|k^\nu\right|^2\right),$$
$$b=\text{tr}\left(3\left|k^u\right|^4+3\left|k^d\right|^4+\left|k^e\right|^4+\left|k^\nu\right|^4\right),$$
$$c=\text{tr}\left(\left|k^{\nu_R}\right|^2\right),\qquad d=\text{tr}\left(\left|k^{\nu_R}\right|^4\right),$$
$$e=\text{tr}\left(\left|k^{\nu_R}\right|^2\left|k^\nu\right|^2\right)$$

In the above expression, g_1, g_2, and g_3 are the $U(1)$, $SU(2)$ and $SU(3)$ gauge couplings with the corresponding gauge field strengths $B_{\mu\nu}$, $F^\alpha_{\mu\nu}$ and $G^i_{\mu\nu}$, and where the Higgs doublet is φ and the Yukawa fermionic couplings are given by the 3×3 matrices k^u, k^d, k^e, k^ν and k^{ν_R}. The boundary part of the a_4 term will be discussed in details in the expanded version of this paper [13]. Remarkably, the boundary term $\frac{1}{3}K\left(a\left|\varphi\right|^2+\frac{1}{2}c\right)$ exactly compensates for the presence of the term $\frac{1}{6}R\left(a\left|\varphi\right|^2+\frac{1}{2}c\right)$ appearing again with the same sign and the same relative factor of 2. This is a proof that the spectral action takes care of its self consistency.

From all these considerations we deduce that the simple requirement of having local boundary conditions consistent with the hermiticity of the Dirac operator, is enough to guarantee that the spectral action has all the correct features and expected terms, including correct signs and coefficients.

Finally we note that we can include the effects of introducing a dilaton field to make the mass scale dynamical and obtain an almost scale invariant action. The main results obtained recently [6] where it was shown that the dilaton interacts only through its kinetic term with a potential generated at the quantum level. The model has the same low-energy sector as the Randall-Sundrum model and the model of extended inflation. In the case of manifolds without boundary, the only modifications needed in the spectral action is the addition of the dilaton term $\frac{8}{3\pi^2}f_2\int_M d^4x\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ and the boundary term $\frac{2}{\pi^2}f_0\int\limits_{\partial M}d^3x\sqrt{h}h^{ab}K\left(\nabla^h_a\nabla^h_b\phi-\nabla_a\phi\nabla_b\phi\right)$ but with rescaled metric, and these could play some role in cosmological considerations.

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