ON THE FOUNDATION OF NONCOMMUTATIVE GEOMETRY

ISRAEL GELFAND is one of the handful of mathematicians who really shaped the mathematics of the twentieth century. Even among them he does stand out by the fecundity of the concepts he created and the astonishing number of new fields he originated.

One characteristic feature of his mathematics is that, while working at a high level of conceptual breadth, it never looses contact with concrete computations and applications, including those to theoretical physics, a subject in which his influence is hard to match.

Although mathematicians before Gelfand had studied normed rings, it was he who created the tools that got the theory really started. In his thesis he brought to light the fundamental concept of maximal ideal and proved that the quotient of a commutative Banach algebra by a maximal ideal is always the field $\mathbb{C}$ of complex numbers. This easily implied, for instance, Wiener's well known result that the inverse of a function with no zeros and absolutely convergent Fourier expansion also has absolutely convergent Fourier expansion. The fundamental result in the commutative case characterized the rings of continuous functions on a (locally) compact space in a purely algebraic manner. Dropping the commutativity assumption led Gelfand and Naimark to the theory of C^*-algebras, again proving the fundamental result that any such ring can be realized as an involutive norm closed subalgebra of the algebra of operators in Hilbert space. The key step, known as the "Gelfand-Naimark-Segal" construction, plays a basic role in quantum field theory, and was used early on by Gelfand and Raikov to show that any locally compact group admits enough irreducible Hilbert space representations. These and many others of Gelfand's results were so influent that it is hard for us to imagine mathematics without them.

They played a decisive role in the foundation of Noncommutative Geometry, a subject to which I devoted most of my mathematical work. I refer to the survey [35] for a thorough presentation of the subject and will describe here, after a brief introduction, a few of the open frontiers and problems, which are actively explored at this point.

As long as we consider geometry as intimately related to our model of space-time, Einstein's general relativity clearly vindicated the ideas of Gauss and Riemann, allowing for variable curvature, and formulating the intrinsic geometry of a curved space independently of its embedding in Euclidean space. The two key notions are those of manifold of arbitrary dimension, whose points are locally labeled by finitely many real numbers x^μ, and that of the line element, i.e. the infinitesimal unit of length, which, when carried around, allows one to measure distances. The infinitesimal calculus encodes the geometry by the formula for the line element ds in local terms

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu ,$$

and allows one to generalize most of the concepts, which were present either in euclidean or noneuclidean geometry, while considerably enhancing the number of available interesting examples.

Riemann was sufficiently cautious in his lecture on the foundation of geometry to question the validity of his hypotheses in the infinitely small. He explicitly proposed to "gradually modify the foundations under the

compulsion of facts which cannot be explained by it" in case physics would find new unexplained phenomena in the exploration of smaller scales.

The origin of noncommutative geometry can be traced back to the discovery of such unexplained phenomena in the phase space of the microscopic mechanical system describing an atom. This system manifests itself through its interaction with radiation and the basic laws of spectroscopy, as found in particular by Ritz and Rydberg, are in contradiction with the "manifold" picture of the phase space.

The very bare fact, which came directly from experimental findings in spectroscopy and was unveiled by Heisenberg (and then understood at a more mathematical level by Born, Jordan, Dirac and the physicists of the late 1920's), is the following. Whereas when you are dealing with a manifold you can parameterize (locally) its points x by real numbers $x_1, x_2, \ldots$, which specify completely the situation of the system, when you turn to the phase space of a microscopic mechanical system, even of the simplest kind, the coordinates, namely the real numbers $x_1, x_2, \ldots$ that you would like to use to parameterize points, actually do not commute.

This means that the classical geometrical framework is too narrow to describe in a faithful manner many physical spaces of great interest. In noncommutative geometry one replaces the usual notion of manifold formed of points labeled by coordinates with spaces of a more general nature, as we shall see shortly. Usual geometry is just a particular case of this new theory, in the same way as Euclidean and non Euclidean geometry are particular cases of Riemannian geometry. Many of the familiar geometrical concepts do survive in the new theory, but they carry also a new unexpected meaning.

Before describing the novel notion of space, it is worthwhile to explain in simple terms how noncommutative geometry modifies the measurement of distances. Such a simple description is possible because the evolution between the Riemannian way of measuring distances and the new (noncommutative) way exactly parallels the improvement of the standard of length* in the metric system. The original definition of the meter at the end of the 18th century was based on a small portion (one forty millionth part) of

*or equivalently of time using the speed of light as a conversion factor

the size of the largest available macroscopic object (here the earth circumference). Moreover this "unit of length" became concretely represented in 1799 as "mètre des archives" by a platinum bar localized near Paris. The international prototype was a more stable copy of the "mètre des archives" which served to define the meter. The most drastic change in the definition of the meter occurred in 1960 when it was redefined as a multiple of the wavelength of a certain orange spectral line in the light emitted by isotope 86 of krypton. This definition was then replaced in 1983 by the current definition which using the speed of light as a conversion factor is expressed in terms of inverse-frequencies rather than wavelength, and is based on a hyperfine transition in the caesium atom. The advantages of the new standard are obvious. No comparison to a localized "mètre des archives" is necessary, the uncertainties are estimated as 10^{-15} and for most applications a commercial caesium beam is sufficiently accurate. Also we could (if any communication were possible) communicate our choice of unit of length to aliens, and uniformize length units in the galaxy without having to send out material copies of the "mètre des archives"!
As we shall see below, the concept of "metric" in noncommutative geometry is precisely based on such a spectral data.

Let us now come to "spaces". What the discovery of Heisenberg showed is that the familiar duality of algebraic geometry

between a space and its algebra of coordinates (i.e. the algebra of functions on that space) is too restrictive to model the phase space of microscopic physical systems. The basic idea then is to stretch this duality, so that the algebra of coordinates on a space is no longer required to be commutative. Gelfand's work on C^*-algebras provides the right framework to define noncommutative topological spaces. They are given by their algebra of continuous functions which can be an arbitrary, not necessarily commutative, C^*-algebra.
It turns out that there is a wealth of examples of spaces, which have obvious geometric meaning but which are best described by a noncommutative algebra of coordinates. The first examples came, as we saw above, from phase space in quantum mechanics but there are many others, such as

- Space of leaves of foliations
- Space of irreducible representations of discrete groups
- Space of Penrose tilings of the plane
- Brillouin zone in the quantum Hall effect
- Phase space in quantum mechanics
- Space time
- Space of $\mathbb{Q}$-lattices in $\mathbb{R}^n$

This last class of examples [44] [45] appears to be of great relevance in number theory and will be discussed at the end of this short survey. The space of $\mathbb{Q}$-lattices [45] is a natural geometric space, with an action of the scaling group providing a spectral interpretation of the zeros of the L-functions of number theory and an interpretation of the Riemann explicit formulas as a trace formula [31]. Another rich class of examples arises from deformation theory, such as deformation of Poisson manifolds, quantum groups and their homogeneous spaces. Moduli spaces also generate very interesting new examples as in [32] [72] as well as the fiber at ∞ in arithmetic geometry [46]. Thus, there is no shortage of examples of noncommutative spaces that beg our understanding but which are very difficult to comprehend. Among them the noncommutative tori were fully analyzed at a very early stage of the theory in 1980 ([15]) and a beginner might be tempted to be happy with the understanding of such simple examples ignoring the wild diversity of the general landscape. The common feature of many of these spaces is that, when one tries to analyze them from the usual set theoretic point of view, the usual tools break down for the following simple reason. Even though as a set they have the cardinality of the continuum, it is impossible to tell apart their points by a finite (or countable) set of explicit functions. In other words, any explicit countable family of *invariants* fails to separate points.

Here is the general principle that allows one to nevertheless encode them by a function algebra, which will no longer be commutative. The above spaces are obtained as quotients from a larger classical space Y gifted with an equivalence relation $\mathcal{R}$. The usual algebra of functions associated to the quotient is

$$\mathcal{A} = \{f \mid f(a) = f(b)\,, \quad \forall (a,b) \in \mathcal{R}\}\,. \tag{1}$$

This algebra is, by construction, a *subalgebra* of the original function algebra on Y and remains commutative. There is, however, a much better way to encode in an algebraic manner the above quotient operation. It consists, instead of taking the subalgebra given by (1), of adjoining to the algebra of functions the identification of a with b, whenever $(a,b) \in \mathcal{R}$. The algebra obtained this way,

$$\mathcal{B} = \{f = [f_{ab}]\,,\ (a,b) \in \mathcal{R}\}\ , \tag{2}$$

is the convolution algebra of the groupoid associated to $\mathcal{R}$ and is of course no longer commutative in general, nor Morita equivalent to a commutative algebra. By encoding the *dynamics* underlying the identification of points by the relation $\mathcal{R}$, it bypasses the problem created by the lack of constructible *static* invariants labeling points in the quotient.

The first operation (1) is of a cohomological flavor, while the second (2) always gives a satisfactory answer, which keeps a close contact with the quotient space. One then recovers the "naive" function spaces attached by the first operation (1) from the cyclic cohomology of the noncommutative algebra obtained from the second operation (2).

The second vital ingredient of the theory is the extension of geometric ideas to the noncommutative framework. It may seem at first sight that it is a simple matter to rewrite algebraically the usual geometric concepts but, in fact, the stretching of geometric thinking imposed by passing to noncommutative spaces forces one to rethink about most of our familiar notions. The most interesting part comes from totally unexpected new features, such as the canonical dynamics of noncommutative measure spaces, which have no counterpart in the classical geometric set up.

As we shall see below, far reaching extensions of classical concepts have been obtained, with variable degrees of perfection, for measure theory, topology, differential geometry, and Riemannian geometry.

- Metric geometry
- Differential geometry
- Topology
- Measure theory

II Measure Theory

One compelling reason to start working in noncommutative geometry is that, even at the very coarse level of measure theory, the general noncommutative case is becoming highly non trivial. When one looks at an ordinary space and does measure theory, one uses the Lebesgue theory, which is a beautiful theory, but all spaces are the same. There is nothing really happening as far as classification is concerned. This is not at all the case in noncommutative measure theory. What happens there is very surprising. It is an absolutely fascinating fact that, when one takes a non commutative algebra M from the measure theory point of view, such an algebra evolves with time !
More precisely, it admits a god-given time evolution, given by a canonical group homomorphism ([10] [11])

$$\delta : \mathbb{R} \to \mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Int}(M) \tag{1}$$

from the additive group $\mathbb{R}$ to the center of the group of automorphism classes of M modulo inner automorphisms.
This homomorphism is provided by the uniqueness of the, a priori state dependent, modular automorphism group of a state. Together with the earlier work of Powers, Araki-Woods and Krieger, it was the beginning of a long story that eventually led to the complete classification ([11], [86], [87], [68], [29], [12], [13], [19], [57]) of approximately finite dimensional factors (also called hyperfinite).
They are classified by their module,

$$\mathrm{Mod}(M) \underset{\sim}{\subset} \mathbb{R}_+^* \,, \tag{2}$$

which is a virtual closed subgroup of $\mathbb{R}_+^*$ in the sense of G. Mackey, i.e. an ergodic action of $\mathbb{R}_+^*$, called the flow of weights [29]. This invariant was first defined and used in my thesis [11], to show in particular the existence of hyperfinite factors which are not isomorphic to Araki-Woods factors.
There is a striking analogy, which I described in [27], between the above classification and the Brauer theory of central simple algebras. It has taken new important steps recently, since noncommutative manifolds ([40] [41]) give examples of construction of the hyperfinite II_1 factor as the crossed product of the field K_q of elliptic functions by a subgroup of its Galois group, in perfect analogy with the Brauer theory.

So, we see that noncommutative measure theory is already highly non trivial, hence we have all reasons to believe that, if one goes further in the natural hierarchy of features of a space, one will discover really interesting new phenomena.

III Topology

The development of the topological ideas was prompted by the work of Israel Gelfand, whose C* algebras give the required framework for noncommutative topology. The two main driving forces in the development of noncommutative topology were the Novikov conjecture on homotopy invariance of higher signatures of ordinary manifolds as well as the Atiyah-Singer Index theorem. It has led, through the work of Atiyah, Singer, Brown, Douglas, Fillmore, Miscenko and Kasparov ([1] [83] [5] [75] [62]), to the realization that not only the Atiyah-Hirzebruch K-theory but more importantly the dual K-homology admit Hilbert space techniques and functional analysis as their natural framework. The cycles in the K-homology group $K_*(X)$ of a compact space X are indeed given by Fredholm representations of the C* algebra A of continuous functions on X. The central tool is the Kasparov bivariant K-theory. A basic example of C* algebra, to which the theory applies, is the group ring of a discrete group, and restricting oneself to commutative algebras is an obviously undesirable assumption.

For a C^* algebra A, let $K_0(A)$, $K_1(A)$ be its K theory groups. Thus, $K_0(A)$ is the algebraic K_0 theory of the ring A and $K_1(A)$ is the algebraic K_0 theory of the ring $A \otimes C_0(\mathbb{R}) = C_0(\mathbb{R}, A)$. If $A \to B$ is a morphism of C^* algebras, then there are induced homomorphisms of abelian groups $K_i(A) \to K_i(B)$. Bott periodicity provides a six term K theory exact sequence for each exact sequence $0 \to J \to A \to B \to 0$ of C^* algebras, and excision shows that the K groups involved in the exact sequence only depend on the respective C^* algebras.

Discrete groups, Lie groups, group actions, and foliations give rise, through their convolution algebra, to a canonical C^* algebra, hence to K theory groups. The analytical meaning of these K theory groups is clear as a receptacle for indices of elliptic operators. However, these groups are difficult to compute. For instance, in the case of semi-simple Lie groups, the free abelian group with one generator for each irreducible discrete series representation is contained in $K_0\, C_r^*G$ where C_r^*G is the reduced C^* algebra of G. Thus, an explicit determination of the K theory in this case in particular involves an enumeration of the discrete series.

We introduced with P. Baum ([3]) a geometrically defined K theory, which specializes to discrete groups, Lie groups, group actions, and foliations. Its main features are its computability and the simplicity of its definition. In essence, it is the group of topological data labeling the symbols of elliptic operators. It does not involve the difficult quotient spaces, but replaces them (up to homotopy) by the familiar homotopy quotient (replacing free actions by proper actions). In the case of semi-simple Lie groups, it elucidates the role of the homogeneous space G/K (K the maximal compact subgroup of G) in the Atiyah-Schmid geometric construction of the discrete series [2]. Using elliptic operators, we constructed a natural map μ from our geometrically defined K theory groups to the above analytic (i.e. C^* algebra) K theory groups. Much progress has been made in the past years to determine the range of validity of the isomorphism between the geometrically defined K theory groups and the above analytic (i.e. C^* algebra) K theory groups. We refer to the three Bourbaki seminars ([84]) for an update on this topic and for a precise account of the various contributions. Among the most important contributions are those of Kasparov and Higson, who showed that the conjectured isomorphism holds for amenable groups. It also holds for real semi-simple Lie groups, thanks in particular to the work of A. Wassermann. Moreover, the recent work of V. Lafforgue ([69]) crossed the barrier of property T, showing that it holds for cocompact subgroups of rank one Lie groups and also of $SL(3,\mathbb{R})$ or of p-adic Lie groups. He also gave the first general conceptual proof of the isomorphism for real or p-adic semi-simple Lie groups. The proof of the isomorphism, for all connected locally compact groups, based on Lafforgue's work, has been obtained by J. Chabert, S. Echterhoff and R. Nest ([7]). The proof by G. Yu of the analog (due to J. Roe) of the conjecture in the context of coarse geometry for metric spaces that are uniformly embeddable in Hilbert space and the work of G. Skandalis, J. L. Tu, J. Roe, and N. Higson on the groupoid case got very striking consequences, such as the injectivity of the map μ for exact $C_r^*(\Gamma)$, due to Kaminker, Guentner and Ozawa. Finally, the independent results of Lafforgue and Mineyev-Yu ([74]) show that the conjecture holds for arbitrary hyperbolic groups (most of which have property T), and P. Julg was even able to prove the conjecture with coefficients for rank one groups. This was the strongest existing positive result until its extension to arbitrary hyperbolic groups which has recently been achieved by Vincent Lafforgue. On the negative side, recent progress due to Gromov, Higson, Lafforgue and Skandalis gives counterexamples to the general conjecture for locally compact groupoids, for the simple reason that the functor $G \to K_0(C_r^*(G))$ is not half exact, unlike the functor given by the geometric group. This makes the general problem of computing $K(C_r^*(G))$ really interesting. It shows that, besides determining the large class of locally compact groups, for which the original conjecture is valid, one should understand how to take homological algebra into account to deal with the correct general formulation.

It also raises many integrality questions in cyclic cohomology of both discrete groups and foliations since a number of natural cyclic cocycles take integral values on the range of the map μ from the geometric group to the analytic group [20].

In summary, the above gives, in any of the listed examples, a natural construction, based on index problems, of K-theory classes in the relevant algebra, and tools to decide if this construction does exhaust all the K-theory. It also provides a classifying space, which gives a rough approximation "up to homotopy" of the singular quotient encoded by the noncommutative geometric description.

IV Differential Geometry

The development of differential geometric ideas, including de Rham homology, connections and curvature of vector bundles, etc... took place during the eighties thanks to cyclic cohomology which I introduced in 1981, including the spectral sequence relating it to Hochschild cohomology [16]. This led Loday and Quillen to their interpretation of cyclic homology in terms of the homology of the Lie algebra of matrices, which was also obtained independently by Tsygan in [88]. My papers appeared in preprint form in 1982 [17] and were quoted by Loday and Quillen [71] (see also [18] and [6]).

The first role of cyclic cohomology was to obtain index formulas computing an index, by the pairing of a K-theory class with a cyclic cocycle (see [15] for a typical example with a cyclic 2-cocycle on the algebra $C^\infty(\mathbb{T}^2_\theta)$ of smooth functions on the noncommutative 2-torus). This pairing is a simple extension of the Chern Weil theory of characteristic classes, using the following dictionary to relate geometrical notions to their algebraic counterpart, in such a way that the latter is meaningful in the general noncommutative situation.

Space	Algebra
Vector bundle	Finite projective module
Differential form	(Class of) Hochschild cycle
DeRham current	(Class of) Hochschild cocycle
DeRham homology	Cyclic cohomology
Chern Weil theory	Pairing $\langle K(\mathcal{A}), HC(\mathcal{A})\rangle$

The pairing $\langle K(\mathcal{A}), HC(\mathcal{A})\rangle$ has a very concrete form and we urge the reader to prove the following simple lemma to get the general flavor of these computations of differential geometric nature.

Lemma 1 *Let $\mathcal{A}$ be an algebra and φ a trilinear form on $\mathcal{A}$ such that*

- $\varphi(a_0, a_1, a_2) = \varphi(a_1, a_2, a_0) \quad \forall a_j \in \mathcal{A}$
- $\varphi(a_0 a_1, a_2, a_3) - \varphi(a_0, a_1 a_2, a_3) + \varphi(a_0, a_1, a_2 a_3) - \varphi(a_2 a_0, a_1, a_2) = 0 \quad \forall a_j \in \mathcal{A}$

Then the scalar $\varphi_n(E, E, E)^\dagger$ is invariant under homotopy for projectors (idempotents) $E \in M_n(\mathcal{A})$.

In the example of the noncommutative torus, the cyclic 2-cocycle representing the fundamental class ([15]) gives an integrality theorem, which J. Bellissard showed to be the integrality of the Hall conductivity in the quantum Hall effect, when applied to a specific spectral projection of the Hamiltonian (see [24] for an account of the work of J. Bellissard).
Basically, by extending the Chern-Weil characteristic classes to the general framework, the theory allows for many concrete computations of differential geometric nature on noncommutative spaces. Indeed, the purely K-theoretic description of the Atiyah-Singer index formula would be of little practical use, if it were not supplemented by the explicit local formulas in terms of characteristic classes and of the Chern character. This is achieved in the general noncommutative framework by the "Local Index Formula", which will be described below in more detail.
Cyclic cohomology was used at a very early stage [20] to obtain index theorems, whose implications could be formulated independently of the whole framework of noncommutative geometry. A typical example is the following strengthening of a well known result of A. Lichnerowicz [70].

Theorem 2 *[20] Let M be a compact oriented manifold and assume that the $\hat{A}$-genus $\hat{A}(M)$ is non-zero (since M is not assumed to be a spin manifold $\hat{A}(M)$ need not be an integer). Let then F be an integrable Spin sub-bundle of TM. There exists no metric on F for which the scalar curvature (of the leaves) is strictly positive ($\geq \varepsilon > 0$) on M.*

†Note that φ has been uniquely extended to $M_n(\mathcal{A})$ using the trace on $M_n(\mathbb{C})$, i.e. $\varphi_n = \varphi \otimes \text{Trace}$.

The proof is based on the construction of cyclic cohomology classes (on the algebra of the foliation) associated to Gelfand-Fuchs cohomology. The main difficulty is in extending the cocycles to a subalgebra stable under the holomorphic functional calculus. The reason for working with cyclic cocycles rather than with the (obviously dual) cyclic homology can be understood easily in the above example. Cyclic cocycles are functionals of differentiable geometric nature and as such are of course not everywhere defined on the algebra of all continuous functions. There is, however, in general a strong compatibility between differentiability and continuity, which reflects itself in the *closability* of the densely defined operators of differential geometry. It is precisely this *closability* that is exploited in [20] to construct the *smooth* domain of the cocycles. In that way cyclic cocycles are closely related to unbounded operators in Hilbert space, each defining its own independent *smooth* domain.
The theory also showed, early on, the depth of the relation between the above classification of factors and the geometry of foliations. In a remarkable series of papers (see [60] for references), J. Heitsch and S. Hurder have analyzed the interplay between the vanishing of the Godbillon-Vey invariant of a compact foliated manifold (V, F) and the type of the von Neumann algebra of the foliation. Their work culminates in the following beautiful result of S. Hurder ([60]). If the von Neumann algebra is *semi-finite*, then the Godbillon-Vey invariant *vanishes*. We have shown, in fact, that cyclic cohomology yields a stronger result, proving that, if $\mathrm{GV} \neq 0$, then the central decomposition of M contains necessarily factors M, whose virtual modular spectrum is of finite covolume in $\mathbb{R}_+^*$.

Theorem 3 *[20] Let (V, F) be an oriented, transversally oriented, compact, foliated manifold,* $(\mathrm{codim}\, F = 1)$*. Let M be the associated von Neumann algebra, and* $\mathrm{Mod}(M)$ *be its flow of weights. Then, if the Godbillon-Vey class of (V, F) is different from 0, there exists an invariant probability measure for the flow* $\mathrm{Mod}(M)$*.*

One actually constructs an invariant measure for the flow $\mathrm{Mod}(M)$, exploiting the following remarkable property of the natural cyclic 1-cocycle τ on the algebra $\mathcal{A}$ of the transverse 1-jet bundle for the foliation. When viewed as a linear map δ from $\mathcal{A}$ to its dual, δ is an unbounded derivation, which is *closable*, and whose domain extends to the center Z of the von-Neumann algebra generated by $\mathcal{A}$. Moreover, δ vanishes on this center, whose elements $h \in Z$ can then be used to obtain new cyclic cocycles τ_h on $\mathcal{A}$. The pairing

$$L(h) = < \tau_h,\ \mu(x) >$$

with the K-theory classes $\mu(x)$ obtained from the assembly map μ, which we had constructed with P. Baum (cf. the topology section), then gives a measure on Z, whose invariance under the flow of weights follows from the discreteness of the K-group. To show that it is non-zero, one uses an index formula that evaluates the cyclic cocycles, associated as above to the Gelfand-Fuchs classes, on the range of the assembly map μ.
Cyclic cohomology led H. Moscovici and myself to the first proof of the Novikov conjecture for hyperbolic groups

Theorem 4 *[36] Any hyperbolic discrete group satisfies the Novikov conjecture.*

The proof is based on the higher index theorem for discrete groups and the analysis ‡ of a natural dense subalgebra stable under holomorphic functional calculus in the reduced

‡due to Haagerup and Jolissaint

group C^*-algebra. Cyclic cohomology got many other applications, which I will not describe here. The very important general result of excision was obtained in the work of Cuntz and Quillen [48], [49].

In summary, the basic notions of differential geometry extend to the noncommutative framework and, starting from the noncommutative algebra $\mathcal{A}$ of "coordinates", the first task is to compute both its Hochschild and Cyclic cohomologies, in order to get the relevant tools, before proceeding further to its "geometric" structure.
One should keep in mind the new subtleties that arise from noncommutativity. For instance, unlike the de Rham cohomology, its noncommutative replacement, which is cyclic cohomology, is not graded but filtered. Also it inherits from the Chern character map a natural integral lattice. These two features play a basic role in the description of the natural moduli space (or more precisely, its covering Teichmüller space, together with a natural action of $SL(2, \mathbb{Z})$ on this space) for the noncommutative tori $\mathbb{T}^2_\theta$. The discussion parallels the description of the moduli space of elliptic curves, but involves the even cohomology instead of the odd one ([32]).

V Quantized Calculus

The infinitesimal calculus is built on the tension expressed in the basic formula

$$\int_a^b df \; = \; f(b) - f(a)$$

between the integral and the infinitesimal variation df. One gets to terms with this tension by developing the Lebesgue integral and the notion of differential form. At the intuitive level, the naive picture of the "infinitesimal variation" df as the increment of f for very nearby values of the variable is good enough for most purposes, so that there is no need for trying to create a theory of infinitesimals.

The scenery is different in noncommutative geometry, where quantum mechanics provides a natural stage for the calculus [24] [35]. It is of course a bit hard to pass from the classical stage, where one just deals with functions, to the new one, in which operators in Hilbert space $\mathcal{H}$ play the central role, but one basic input of quantum mechanics is precisely that the intuitive notion of a real variable quantity should be modeled as a self-adjoint operator in $\mathcal{H}$. One gains a lot in doing so. The set of values of the variable is the spectrum of the operator, and the number of times a value is reached is the spectral multiplicity. Continuous variables (operators with continuous spectrum) coexist happily with discrete variables precisely because of noncommutativity of operators. Furthermore, we now have a perfect home for infinitesimals, namely for variables that are smaller than ϵ for any ϵ, without being zero. Of course, requiring that the operator norm is smaller than ϵ for any ϵ is too strong, but one can be more subtle and ask that, for any ϵ positive, one can condition the operator by a finite number of linear conditions, so that its norm becomes less than ϵ. This is a well known characterization of compact operators in Hilbert space, and they are the obvious candidates for infinitesimals. The basic rules of infinitesimals are easy to check, for instance the sum of two compact operators is compact, the product compact times bounded is compact and they form a two sided ideal $\mathcal{K}$ in the algebra of bounded operators in $\mathcal{H}$.

The size of the infinitesimal $\epsilon \in \mathcal{K}$ is governed by the rate of decay of the decreasing sequence of its characteristic values $\mu_n = \mu_n(\epsilon)$ as $n \to \infty$. (By definition $\mu_n(\epsilon)$ is the n'th eigenvalue of the absolute value $|\epsilon| = \sqrt{\epsilon^*\epsilon}$). In particular, for all real positive α, the following condition defines infinitesimals of order α:

$$\mu_n(\epsilon) = O(n^{-\alpha}) \qquad \text{when } n \to \infty. \tag{1}$$

Infinitesimals of order α also form a two–sided ideal and, moreover,

$$\epsilon_j \text{ of order } \alpha_j \Rightarrow \epsilon_1\epsilon_2 \text{ of order } \alpha_1 + \alpha_2\,. \tag{2}$$

The other key ingredient in the new calculus is the integral

$$⨍$$

It has the usual properties of additivity and positivity of the ordinary integral, but it allows one to recover the power of the usual infinitesimal calculus, by automatically neglecting the ideal of infinitesimals of order > 1

$$⨍ \epsilon = 0\,, \quad \forall \epsilon \quad \mu_n(\epsilon) = o(n^{-1})\,. \tag{3}$$

By filtering out these operators, one passes from the original stage of the quantized calculus described above to a *classical* stage where, as we shall see later, the notion of locality finds its correct place.
Using (3), one recovers the above mentioned tension of the ordinary differential calculus, which allows one to neglect infinitesimals of higher order (such as $(df)^2$) in an integral expression.
We refer to [24] for the construction of the integral in the required generality, obtained by the analysis, mainly due to Dixmier ([51]), of the logarithmic divergence of the ordinary trace for an infinitesimal of order one.

The first interesting concrete example is provided by pseudodifferential operators k on a differentiable manifold M. When k is of order 1 in the above sense, it is measurable and $⨍ k$ is the non-commutative residue of k ([89]). It has a local expression in terms of the distribution kernel $k(x,y)$, $x,y \in M$. For k of order 1 in the above sense, the kernel $k(x,y)$ diverges logarithmically near the diagonal,

$$k(x,y) = -a(x)\log|x-y| + 0(1) \;\; (\text{for} \;\; y \to x), \tag{4}$$

where $a(x)$ is a 1–density independent of the choice of Riemannian distance $|x-y|$. Then one has (up to normalization),

$$⨍ k = \int_M a(x). \tag{5}$$

The right hand side of this formula makes sense for all pseudodifferential operators (cf. [89]), since one can easily see that the kernel of such an operator is asymptotically of the form

$$k(x,y) = \sum a_n(x, x-y) - a(x)\log|x-y| + 0(1) \tag{6}$$

where $a_n(x,\xi)$ is homogeneous of degree $-n$ in ξ, and the 1–density $a(x)$ is defined intrinsically, since the logarithm does not mix with rational terms under a change of local coordinates.

What is quite remarkable is that this allows one to extend the domain of the integral $⨍$ to infinitesimals that are of order < 1, hence to obtain a computable answer to questions that would be meaningless in the ordinary calculus – the prototype being "what is the area of a four manifold?" – which we shall discuss below. The same principle of extension of $⨍$ to infinitesimals of order < 1 turns out to work in much greater generality. It works, for instance, for hypoelliptic operators and more generally for spectral triples, whose dimension spectrum is simple, as we shall see below.

VI Metric Geometry and Spectral Action

With the above calculus as a tool, we now have a home for infinitesimals and can come back to the two basic notions introduced by Riemann in the classical framework, those of *manifold* and of *line element* ([82]). We have shown that both of these notions adapt remarkably well to the noncommutative framework and lead to the notion of spectral triple, on which noncommutative geometry is based (cf. [35] for an overall presentation and [26], [25], [55] for the more technical aspects). This definition is entirely spectral: the elements of the algebra are operators, the points, if they exist, come from the joint spectrum of operators, and the line element is an operator. In a *spectral triple*

$$(\mathcal{A}, \mathcal{H}, D) \tag{1}$$

the algebra $\mathcal{A}$ of coordinates is concretely represented on the Hilbert space $\mathcal{H}$ and the operator D is an unbounded self-adjoint operator, which is the inverse of the line element,

$$ds = 1/D. \tag{2}$$

The basic properties of such spectral triples are easy to formulate and do not make any reference to the commutativity of the algebra $\mathcal{A}$. They are

$$[D,a] \text{ is bounded for any } a \in \mathcal{A}\,, \tag{3}$$

$$D = D^* \text{ and } (D+\lambda)^{-1} \text{ is a compact operator } \forall\, \lambda \notin \mathbb{C}\,. \tag{4}$$

There is a simple formula for the distance in the general noncommutative case. It measures the distance between states[§],

$$d(\varphi,\psi) = \mathrm{Sup}\,\{|\varphi(a) - \psi(a)|\ ;\ a \in \mathcal{A}\ ,\ \|[D,a]\| \leq 1\}\,. \tag{5}$$

The significance of D is two-fold. On the one hand it defines the metric by the above equation, on the other hand its homotopy class represents the K-homology fundamental class of the space under consideration. In the classical geometric case, both the fundamental cycle in K-homology and the metric are encoded in the *spectral triple*

[§]Recall that a state is a normalized positive linear form on $\mathcal{A}$

$(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is the algebra of functions acting in the Hilbert space $\mathcal{H}$ of spinors, while D is the Dirac operator. In some sense this encoding of Riemannian geometry takes a square root of the usual ansatz giving ds^2 as $g_{\mu\nu}\, dx^\mu\, dx^\nu$, the point being that the spin structure allows for the extraction of the square root of ds^2 (as is well known, Dirac found the corresponding operator as a differential square root of a Laplacian).

The first thing one checks is that, in the classical Riemannian case, the geodesic distance $d(x, y)$ between two points is re-obtained by

$$d(x,y) = \mathrm{Sup}\,\{|f(x) - f(y)|\,;\ f \in \mathcal{A}\,,\ \|[D, f]\| \leq 1\}, \tag{6}$$

with $D = ds^{-1}$ as above, and where $\mathcal{A}$ is the algebra of smooth functions. Note that ds has the dimension of a length L, D has dimension L^{-1}, and the above expression for $d(x, y)$ also has the dimension of a length. It is also important to notice that we do not have to give the algebra of smooth functions. Indeed, imagine we are just given the von Neumann algebra $\mathcal{A}''$ in $\mathcal{H}$, weak closure of $\mathcal{A}$. How do we recover the subalgebra $C^\infty(M)$ of smooth functions ? This is hopeless without using D, since the pair $(\mathcal{A}'', \mathcal{H})$ contains no more information than the multiplicity of this representation of a Lebesgue measure space (recall that they are all isomorphic in the measure category). Using D, however, the answer is as follows. We shall say that an operator T in $\mathcal{H}$ is *smooth* iff the following map is smooth,

$$t \to F_t(T) = e^{it|D|}\, T\, e^{-it|D|} \in C^\infty(\mathbb{R}, \mathcal{L}(\mathcal{H}))\,. \tag{7}$$

We let

$$OP^0 = \{T \in \mathcal{L}(\mathcal{H})\,;\ T \text{ is smooth}\}\,. \tag{8}$$

It is then an exercise to show, in the Riemannian context, that

$$C^\infty(M) = \,OP^0 \cap L^\infty(M)\,,$$

where $L^\infty(M) = \mathcal{A}''$ is the von Neumann algebra weak closure in $\mathcal{H}$.
In the general context, the flow 7 plays the role of the geodesic flow, assuming the following regularity hypothesis on $(\mathcal{A}, \mathcal{H}, D)$:

$$a \text{ and } \ [D, a] \ \in\ \cap\, \mathrm{Dom}\, \delta^k,\ \forall\, a \in \mathcal{A}, \tag{9}$$

where δ is the derivation $\delta(T) = [|D|, T]$, for any operator T. This derivation is the generator of the geodesic flow.
The usual notion of *dimension* of a space is replaced by the *dimension spectrum* , which is the subset Σ of $\{z \in \mathbb{C}, \mathrm{Re}(z) \geq 0\}$ of singularities of the analytic functions

$$\zeta_b(z) = \mathrm{Trace}\,(b|D|^{-z}) \qquad \mathrm{Re}\, z > p\ ,\ \ b \in \mathcal{B}\,, \tag{10}$$

where we let $\mathcal{B}$ denote the algebra generated by $\delta^k(a)$ and $\delta^k([D, a])$, for $a \in \mathcal{A}$. The dimension spectrum Σ is of course bounded above by the crude dimension provided by the growth of eigenvalues of D, or equivalently by the order of the infinitesimal ds. In essence, the dimension spectrum is the set of complex numbers where the space under consideration becomes visible from the classical standpoint of the integral $⨍$.
The dimension spectrum of an ordinary manifold M is the set $\{0, 1, \ldots, n\}$, $n = \dim M$; it is simple. Multiplicities appear for singular manifolds. Cantor sets provide examples of complex points $z \notin \mathbb{R}$ in the dimension spectrum.

Going back to the usual Riemannian case, one checks that one recovers the volume form of the Riemannian metric by the equality (valid up to a normalization constant [24])

$$ {\int\!\!\!\!\!\!-} f\,|ds|^n = \int_{M_n} f\,\sqrt{g}\;d^n x, \tag{11} $$

but the first interesting point is that, besides this coherence with the usual computations, there are new simple questions we can ask now, such as "what is the two-dimensional measure of a four manifold?" or, in other words, "what is its area ?". Thus one should compute

$$ {\int\!\!\!\!\!\!-} ds^2. \tag{12} $$

From invariant theory, this should be proportional to the Hilbert–Einstein action. The direct computation has been done in [63], the result being

$$ {\int\!\!\!\!\!\!-} ds^2 = \frac{-1}{24\,\pi^2} \int_{M_4} r\,\sqrt{g}\;d^4 x, \tag{13} $$

where, as above, $dv = \sqrt{g}\;d^4x$ is the volume form, $ds = D^{-1}$ the length element, *i.e.* the inverse of the Dirac operator, and r is the scalar curvature.

A spectral triple is, in effect, a fairly minimal set of data allowing one to start doing quantum field theory. First the inverse

$$ ds = D^{-1}\,. \tag{14} $$

plays the role of the propagator for Euclidean fermions and allows one to start writing the contributions of Feynman graphs whose internal lines are fermionic. The gauge bosons then appear as derived objects, through the simple issue of Morita equivalence. Indeed, to define the analog of the operator D for the algebra of endomorphisms of a finite projective module over $\mathcal{A}$

$$ \mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E}), \tag{15} $$

where $\mathcal{E}$ is a finite, projective, hermitian right $\mathcal{A}$–module, requires the choice of a *hermitian connection* on $\mathcal{E}$. Such a connection ∇ is a linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$, satisfying the rules ([24])

$$ \nabla(\xi a) = (\nabla \xi)a + \xi \otimes da \qquad \forall\, \xi \in \mathcal{E}\;,\; a \in \mathcal{A} \tag{16} $$

$$ (\xi, \nabla \eta) - (\nabla \xi, \eta) = d(\xi, \eta) \qquad \forall\, \xi, \eta \in \mathcal{E}, \tag{17} $$

where $da = [D, a]$ and where $\Omega^1_D \subset \mathcal{L}(\mathcal{H})$ is the $\mathcal{A}$–bimodule of operators of the form

$$ A = \Sigma\, a_i [D, b_i]\;,\; a_i, b_i \in \mathcal{A}\,. \tag{18} $$

Any algebra $\mathcal{A}$ is Morita equivalent to itself (with $\mathcal{E} = \mathcal{A}$) and when one applies the construction above in the this context one gets the inner deformations of the spectral geometry (we ignore the real structure and refer to [26] for the full story). These replace the operator D by

$$ D \to D + A \tag{19} $$

where $A = A^*$ is an arbitrary self adjoint operator of the form 18, where we disregard the real structure for simplicity. Analyzing the divergences of the simplest diagrams with fermionic internal lines, as proposed early on in [23], provides perfect candidates for the counter-terms, and hence the bosonic self-interactions. Such terms are readily expressible as residues or Dixmier traces and are gauge invariant by construction. The basic results are that

- In the above general context of NCG and in dimension 4, the obtained counter-terms are a sum of a Chern-Simons action associated to a cyclic 3-cocycle on the algebra $\mathcal{A}$ and a Yang-Mills action expressed from a Dixmier trace, along the lines of [23] and [24]. The main additional hypothesis is the vanishing of the "tadpole", which expresses that one expands around an extremum.

- In the above generality exactly the same terms appear in the spectral action $\langle N(\Lambda)\rangle$ as the terms independent of the cutoff parameter Λ.

The spectral action was defined in [9] and computed there for the natural spectral triple describing the standard model. We refer to [64] for the detailed calculation. The overall idea of the approach is to use the above more flexible geometric framework to model the geometry of space-time, starting from the observed Lagrangian of gravity coupled with matter. The usual paradigm guesses space-time from the Maxwell part of the Lagrangian, concludes that it is Minkowski space and then adds more and more particles to account for new terms in the Lagrangian. We start instead from the full Lagrangian and derive the geometry of space-time directly from this *empirical* data. The only rule is that we want a theory that is pure gravity, with action functional given by the spectral action $\langle N(\Lambda)\rangle$ explained below, with an added fermionic term

$$S = \langle N(\Lambda)\rangle + \; < \psi, D\,\psi > .$$

Note that D here stands for the Dirac operator with all its decorations, such as the inner fluctuations A explained above. Thus, D stands for $D + A$, but this decomposition is an artifact of the standard distinction between gravity and matter, which is irrelevant in our framework[¶]. The gauge bosons appear as the *inner* part of the metric, in the same way as the invariance group, which is the noncommutative geometry analog of the group of diffeomorphisms, contains inner automorphisms as a normal subgroup (corresponding to the internal symmetries in physics).
The phenomenological Lagrangian of physics is the Einstein Lagrangian plus the minimally coupled standard model Lagrangian. The Fermionic part of this action is used to determine a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where the algebra $\mathcal{A}$ determines the effective space-time from the internal symmetries and yields an answer, which differs from the usual space-time (coming from QED). The Hilbert space $\mathcal{H}$ encodes not only the ordinary spinors (coming from QED) but all quarks and leptons, and the operator D encodes not only the ordinary Dirac operator but also the Yukawa coupling matrix. Then one recovers the bosonic part as follows. Both the Hilbert–Einstein action functional for the Riemannian metric, the Yang–Mills action for the vector potentials, the self interaction and the minimal coupling for the Higgs fields all appear with the correct

[¶]The spectral action is clearly superior to the Dixmier trace (residue) version of the Yang-Mills action in that it does not use this artificial splitting.

signs in the asymptotic expansion for large Λ of the number $N(\Lambda)$ of eigenvalues of D that are $\leq \Lambda$ (cf. [9]),

$$N(\Lambda) = \#\text{ eigenvalues of } D \text{ in } [-\Lambda, \Lambda]. \tag{20}$$

This step function $N(\Lambda)$ is the superposition of two terms,

$$N(\Lambda) = \langle N(\Lambda) \rangle + N_{\text{osc}}(\Lambda).$$

The oscillatory part $N_{\text{osc}}(\Lambda)$ is the same as for a random matrix, governed by the statistic dictated by the symmetries of the system, and does not concern us here. The average part $\langle N(\Lambda) \rangle$ is computed by a semiclassical approximation from local expressions involving the familiar heat equation expansion and delivers the correct terms. Other nonzero terms in the asymptotic expansion are cosmological, Weyl gravity and topological terms. In general, the average part $\langle N(\Lambda) \rangle$ is given as a sum of residues. Assuming that the dimension spectrum Σ is simple, it is given by

$$\langle N(\Lambda) \rangle := \sum_{k>0} \frac{\Lambda^k}{k} \operatorname*{Res}_{s=k} \zeta_D(s) + \zeta_D(0)\,, \tag{21}$$

where the sum is over $k \in \Sigma$ and

$$\zeta_D(s) = \text{Trace}(\,|D|^{-s})\,.$$

For instance, for Spec $D \subset \mathbb{Z}$ and $P(n)$ the total multiplicity of $\{\pm n\}$, for a polynomial P, one has

$$\langle N(\Lambda) \rangle = \int_0^\Lambda P(u)\, du\, + \text{cst},$$

which smoothly interpolates through the irregular step function

$$N(\Lambda) = \sum_0^\Lambda P(n)\,.$$

As explained above, the Yang-Mills action appears in general as a part of the spectral action, but can also be defined directly using the calculus. This analog of the Yang-Mills action functional and the classification of Yang-Mills connections on the noncommutative tori were developed in [30], with the primary goal of finding a "manifold shadow" for these noncommutative spaces. These moduli spaces turned out indeed to fit this purpose perfectly, allowing us, for instance, to find the usual Riemannian space of gauge equivalence classes of Yang-Mills connections as an invariant of the noncommutative metric. We refer to [24] for the construction of the metrics on noncommutative tori from the conceptual point of view and to [25] for the verification that all natural axioms of noncommutative geometry are fulfilled in that case.
Gauge theory on noncommutative tori was shown to be relevant in string theory compactifications in [32]. Indeed, both the noncommutative tori and the components ∇_j of the Yang-Mills connections occur naturally in the classification of the BPS states in M-theory [32]. In the matrix formulation of M-theory, the basic equations to obtain periodicity of two of the basic coordinates X_i turn out to be the following:

$$U_i X_j U_i^{-1} = X_j + a\delta_i^j, i = 1, 2, \tag{22}$$

where the U_i are unitary gauge transformations. The multiplicative commutator

$$U_1 U_2 U_1^{-1} U_2^{-1}$$

is then central and, in the irreducible case, its scalar value $\lambda = \exp 2\pi i\theta$ brings in the algebra of coordinates on the noncommutative torus. The X_j are then the components of the Yang-Mills connections. It is quite remarkable that the same picture emerged from the other information one has about M-theory, concerning its relation with 11 dimensional supergravity, and that string theory dualities could be interpreted using Morita equivalence. The latter [79] relates the values of θ on an orbit of $SL(2,\mathbb{Z})$, and this type of relation, which is obvious from the foliation point of view [15], would be invisible in a purely deformation theoretic perturbative expansion like the one given by the Moyal product. The gauge theories on noncommutative 4-space were used very successfully in [76] to give a conceptual meaning to the compactifications of moduli spaces of instantons on $\mathbb{R}^4$ in terms of instantons on noncommutative $\mathbb{R}^4$. The corresponding spectral triple has been shown to fit in the general framework of NCG in [52] and the spectral action has been computed in [53]. These constructions apply to flat spaces, but were greatly generalized to isospectral deformations of Riemannian geometries of rank > 1 in ([39], [40]). We shall not review here the renormalization of QFT on noncommutative spaces, but we simply refer to a recent remarkable positive result by H. Grosse and R. Wulkenhaar [56].

In summary, we now have at our disposal an operator theoretic analog of the "calculus" of infinitesimals and a general framework of geometry.

In general, given a noncommutative space with coordinate algebra $\mathcal{A}$, the determination of corresponding geometries $(\mathcal{A}, \mathcal{H}, D)$ is obtained in two independent steps.

1) The first consists of presenting the algebraic relations between coordinates $x \in \mathcal{A}$ and the inverse line element D, the simplest instance being the equation

$$U^{-1}\,[D, U] = 1$$

fixing the geometry of the one dimensional circle. We refer to ([39], [40]) for noncommutative versions of this equation in dimension 3 and 4. Basically, this fixes the volume form v as a Hochschild cocycle and then allows for arbitrary metrics with v as volume form as solutions.

2) Once the algebraic relations between coordinates and the inverse line element have been determined, the second step is to find irreducible representations of these relations in Hilbert space. Different metrics will correspond to the various inequivalent irreducible representations of the pair $(\mathcal{A}, D)$, fulfilling the prescribed relations. One guiding principle is that the homotopy class of such a representation should yield a non-trivial K-homology class on $\mathcal{A}$, playing the role of the "fundamental class". The corresponding index problem yields the analog of the Pontrjagin classes and of "curvature", as discussed below. In order to compare noncommutative metrics, it is very natural to use spectral invariants, such as the spectral action mentioned above.

In many ways, the above two steps parallel the description of particles as irreducible representations of the Poincaré group. We thus view a given geometry as an irreducible representation of the algebraic relations between the coordinates and the line element, while the choice of such representations breaks the natural invariance group of the theory. The simplest instance of this view of geometry as a symmetry breaking phenomenon is what happens in the Higgs sector of the standard model.

VII Metric Geometry, The Local Index Formula

In the spectral noncommutative framework, the next appearance of the notion of curvature (besides the above spectral one) comes from the local computation of the analog of Pontrjagin classes, i.e. of the components of the cyclic cocycle that is the Chern character of the K-homology class of D, which make sense in general. This result allows us, using the infinitesimal calculus, to go from local to global in the general framework of spectral triples $(\mathcal{A}, \mathcal{H}, D)$.
The Fredholm index of the operator D determines (we only look at the odd case for simplicity but there are similar formulas in the even case) an additive map $K_1(\mathcal{A}) \xrightarrow{\varphi} \mathbb{Z}$, given by the equality

$$\varphi([u]) = \text{Index}\,(PuP)\ ,\ u \in GL_1(\mathcal{A}), \tag{1}$$

where P is the projector $P = \frac{1+F}{2}$, $F = \text{Sign}\,(D)$.
It is an easy fact that this map is computed by the pairing of $K_1(\mathcal{A})$ with the following cyclic cocycle

$$\tau(a^0, \ldots, a^n) = \text{Trace}\,(a^0[F, a^1] \ldots [F, a^n]) \qquad \forall\ a^j \in \mathcal{A}, \tag{2}$$

where $F = \text{Sign}\ D$, and we assume that the dimension p of our space is finite, which means that $(D+i)^{-1}$ is of order $1/p$, also $n \geq p$ is an odd integer. There are similar formulas involving the grading γ in the even case, and it is quite satisfactory ([21] [61]) that both cyclic cohomology and the Chern character formula adapt to the infinite dimensional case, in which the only hypothesis is that $\exp(-D^2)$ is a trace class operator.

The cocycle τ is, however, nonlocal in general, because the formula (2) involves the ordinary trace instead of the local trace $⨍$ and it is crucial to obtain a local form of the above cocycle.

This problem is solved by the "Local Index Formula" ([37]), under the regularity hypothesis (9) on $(\mathcal{A}, \mathcal{H}, D)$.
We assume that the dimension spectrum Σ is discrete and simple and refer to [37] for the case of a spectrum with multiplicities. Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular spectral triple with simple dimension spectrum, the local index theorem is the following, [37]:

Theorem 5 • *The equality*

$$⨍ P = \text{Res}_{z=0}\ \text{Trace}\,(P|D|^{-z})$$

defines a trace on the algebra generated by $\mathcal{A}$, $[D, \mathcal{A}]$ *and* $|D|^z$, *where* $z \in \mathbb{C}$.

- *There is only a finite number of non–zero terms in the following formula defining the odd components $(\varphi_n)_{n=1,3,\ldots}$ of a cocycle in the bicomplex (b,B) of $\mathcal{A}$,*

$$\varphi_n(a^0,\ldots,a^n)=\sum_k c_{n,k} ⨍ a^0[D,a^1]^{(k_1)}\ldots[D,a^n]^{(k_n)}\,|D|^{-n-2|k|}\qquad \forall\ a^j\in\mathcal{A}$$

where the following notations are used: $T^{(k)}=\nabla^k(T)$ and $\nabla(T)=D^2T-TD^2$, k is a multi-index, $|k|=k_1+\ldots+k_n$,

$$c_{n,k}=(-1)^{|k|}\sqrt{2i}(k_1!\ldots k_n!)^{-1}\left((k_1+1)\ldots(k_1+k_2+\ldots+k_n+n)\right)^{-1}\Gamma\left(|k|+\frac{n}{2}\right).$$

- *The pairing of the cyclic cohomology class $(\varphi_n)\in HC^*(\mathcal{A})$ with $K_1(\mathcal{A})$ gives the Fredholm index of D with coefficients in $K_1(\mathcal{A})$.*

We refer to [59] for a user friendly account of the proof. The first test of this general local index formula was the computation of the local Pontrjagin classes in the case of foliations. Their transverse geometry is, as explained in [37], encoded by a spectral triple. The answer for the general case [38] was obtained thanks to a Hopf algebra $\mathcal{H}(n)$ only depending on the codimension n of the foliation. It allowed us to organize the computation and to encode algebraically the noncommutative curvature. It also dictated the correct generalization of cyclic cohomology for Hopf algebras ([38], [42]). This extension of cyclic cohomology has been pursued with great success recently ([65], [66], [58]) and we shall use it later. In the above context of foliations, the index computation transits through the cyclic cohomology of the Hopf algebra $\mathcal{H}(n)$ and we showed that it coincides with the Gelfand-Fuchs cohomology [54].
Another case of great interest came recently from quantum groups, where the local index formula works fine and yields quite remarkable formulas involving a sequence of rational approximations to the logarithmic derivative of the Dedekind eta function, even in the simplest case of $SU_q(2)$ (cf. [8], [43]). What is very striking in this example is that it displays the meaning of *locality* in the noncommutative framework. This notion requires no definition in the usual topological framework but would appear far more elusive in the noncommutative case without such concrete examples. What it means is that one works at ∞ in momentum space, but with very precise rules that allow one to strip all formulas from irrelevant details that wont have any effect in the computation of residues. We urge the reader to look at the concrete computations of [43] to really appreciate this point.

After the breakthrough of [50] showing that, contrary to a well installed negative belief, one could get non-trivial spectral geometries associated to quantum homogeneous spaces, it was shown in [77] how the above local index formula should be adapted to deal with the situation when the *principal symbol* of D^2 is no longer scalar because of a q-twist. Finally, recent work in the general case of quantum flag manifolds opens the way to a large class of examples, in which the above machinery should be tested and improved [67].

An open question of great relevance in the general framework of the analysis of spectral triples is to associate to any spectral triple $(\mathcal{A},\mathcal{H},D)$ the coarse geometry of its "momentum space" P. This space should be an ordinary metric space, with growth exactly governed by the spectrum of $|D|$ (which would give the set of distances to the origin). When viewed in the sense of the coarse geometry of John Roe, the space P should give an accurate description of the "infinitesimal" structure of the noncommutative space given by the spectral triple $(\mathcal{A}, \mathcal{H}, D)$. In particular, the classification of M. Gromov of discrete groups with polynomial growth should be extended to show that the "local structure" of noncommutative finite dimensional manifolds is essentially of nilpotent nature. (The map from discrete groups G to spectral triples is the action of the group ring in $l^2(\Gamma)$. With $|D|$ being the multiplication by the word metric, it only takes care of the absolute value of $|D|$.)

VIII Renormalization, Residues and Locality

At about the same time as the Hopf algebra $\mathcal{H}(n)$, another Hopf algebra was independently discovered by Dirk Kreimer, as the organizing concept in the computations of renormalization in quantum field theory.
His Hopf algebra is commutative as an algebra, and we showed in [33] that it is the dual Hopf algebra of the enveloping algebra of a Lie algebra $\underline{G}$, whose basis is labeled by the one particle irreducible Feynman graphs. The Lie bracket of two such graphs is computed from insertions of one graph in the other and vice versa. The corresponding Lie group G is the group of characters of $\mathcal{H}$.
We showed that the group G is a semi-direct product of an easily understood abelian group by a highly non-trivial group closely tied up with groups of diffeomorphisms.

Our joint work shows that the essence of the concrete computations performed by physicists in the renormalization technique is conceptually understood as a special case of a general principle of multiplicative extraction of finite values coming from the Birkhoff decomposition in the Riemann-Hilbert problem. The Birkhoff decomposition is the factorization

$$\gamma(z) = \gamma_-(z)^{-1}\,\gamma_+(z) \qquad z \in C, \tag{1}$$

where we let $C \subset P_1(\mathbb{C})$ be a smooth simple curve, C_- the component of the complement of C containing $\infty \notin C$ and C_+ the other component. Both γ and $\gamma_\pm$ are loops with values in G,

$$\gamma(z) \in G \qquad \forall\ z \in \mathbb{C}$$

and $\gamma_\pm$ are boundary values of holomorphic maps (still denoted by the same symbol),

$$\gamma_\pm : C_\pm \to G\,. \tag{2}$$

The normalization condition $\gamma_-(\infty) = 1$ ensures that, if it exists, the decomposition (2) is unique (under suitable regularity conditions).
When G is a simply connected nilpotent complex Lie group, the existence (and uniqueness) of the Birkhoff decomposition (2) is valid for any γ. When the loop $\gamma : C \to G$ extends to a holomorphic loop: $C_+ \to G$, the Birkhoff decomposition is given by $\gamma_+ = \gamma$, $\gamma_- = 1$. In general, for $z \in C_+$, the evaluation

$$\gamma \to \gamma_+(z) \in G \tag{3}$$

is a natural principle to extract a finite value from the singular expression $\gamma(z)$. This extraction of finite values coincides with the removal of the pole part when G is the

additive group $\mathbb{C}$ of complex numbers and the loop γ is meromorphic inside C_+ with z as its only singularity. It is convenient, in fact, to use the decomposition relative to an infinitesimal circle C_+ around z.
The main result of our joint work ([34]) is that the renormalized theory is just the evaluation at $z = D$ of the holomorphic part γ_+ of the Birkhoff decomposition of the loop γ with values in G provided by the dimensional regularization.

In fact, the relation that we uncovered in [34] between the Hopf algebra of Feynman graphs and the Hopf algebra of coordinates on the group of formal diffeomorphisms of the dimensionless coupling constants of the theory allows us to prove the following result, which for simplicity deals with the case of a single dimensionless coupling constant.

Theorem 6 *[34] Let the unrenormalized effective coupling constant $g_{\rm eff}(\varepsilon)$ be viewed as a formal power series in g and let $g_{\rm eff}(\varepsilon) = g_{{\rm eff}_+}(\varepsilon)\,(g_{{\rm eff}_-}(\varepsilon))^{-1}$ be its (opposite) Birkhoff decomposition in the group of formal diffeomorphisms. Then the loop $g_{{\rm eff}_-}(\varepsilon)$ is the bare coupling constant and $g_{{\rm eff}_+}(0)$ is the renormalized effective coupling.*

This allows us, using the relation between the Birkhoff decomposition and the classification of holomorphic bundles, to encode geometrically the operation of renormalization. It also signals a very clear analogy between the renormalization group as an "ambiguity" group of physical theories and the missing Galois theory at Archimedean places alluded to above. We refer to [28] for more information.

Note also that the residue, which is the corner stone of our integral calculus (section V), plays a key role in the Birkhoff decomposition. Indeed, we showed in ([34]) that the negative part in the Birkhoff decomposition (the part coming from divergencies and giving the counter-terms) is entirely determined by its residue (the term in $1/\epsilon$) in the dimensional regularization, a strong form of the t'Hooft relations in QFT. This then gives all its strength to the idea that *local functionals* are best expressed in the general context of NCG as noncommutative integrals, i.e. residues.

With this conceptual understanding of renormalization at hand, the next obvious question is to match it with the above framework of NCG and apply it to the spectral action. One very nice feature of the framework of NCG is that it allows for the *dressing* of the geometry. Indeed in quantum field theory the Dirac propagator undergoes a whole series of quantum corrections that provide a formal power series in powers of $\hbar$. In our framework, these corrections mean that the whole geometry is affected by the quantum field theory. This directly fine tunes its fundamental ingredient, which is the "line element" $ds = \times\!\!\longrightarrow\!\!\times$. Of course it is necessary, in order to really formulate things coherently, to pass to the 2nd quantized Hilbert space, rather than staying in the one particle Hilbert space. This remains to be done in the general framework of NCG, but one can already appreciate a direct benefit of passing to the 2nd quantized Hilbert space. Indeed, if we concentrate on space (versus space-time), its "line element" $ds = \times\!\!\longrightarrow\!\!\times$ becomes positive, as the inverse of the Dirac Hamiltonian, thanks to the "Dirac sea" construction, which makes a preferred choice of the spin representation of the infinite dimensional Clifford algebra. What remains at the second quantized level of the cohomological significance of ds (as a generator of Poincaré duality in K-homology) should be captured by a "regulator" pairing with algebraic K-theory along the lines of [22].

IX Modular Forms and the space of $\mathbb{Q}$-Lattices

We shall end this short presentation of the subject with examples of noncommutative spaces, which appeared in our joint work with H. Moscovici [44] and M. Marcolli [45] and have obvious relevance in number theory. Modular forms already appeared in noncommutative geometry in the classification of noncommutative three spheres, [40] [41], where hard computations with the noncommutative analog of the Jacobian, involving the ninth power of the Dedekind eta function, were necessary in order to analyze the relation between such spheres and noncommutative nilmanifolds.
The coexistence of two *a priori* unrelated structures on modular forms, namely the algebra structure given by the pointwise product on one hand and the action of the Hecke operators on the other, led us in [44] to associate to any congruence subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$ a crossed product algebra $\mathcal{A}(\Gamma)$, the *modular Hecke algebra* of level Γ, which is a direct extension of both the ring of classical Hecke operators and of the algebra $\mathcal{M}(\Gamma)$ of Γ-modular forms. With $\mathcal{M}$ denoting the algebra of modular forms of arbitrary level, the elements of $\mathcal{A}(\Gamma)$ are maps with finite support

$$F : \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q}) \to \mathcal{M}\,, \qquad \alpha \mapsto F_\alpha \in \mathcal{M}\,,$$

satisfying the covariance condition

$$F_{\alpha\gamma} \;=\; F_\alpha|\gamma\,, \qquad \forall\,\alpha \in \mathrm{GL}_2^+(\mathbb{Q})\,, \gamma \in \Gamma \tag{1}$$

and their product is given by convolution

$$(F^1 * F^2)_\alpha \;:=\; \sum_{\beta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} F^1_\beta \cdot F^2_{\alpha\beta^{-1}}|\beta.$$

In the simplest case $\Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$, the elements of $\mathcal{A}(\Gamma(1))$ are encoded by a finite number of modular forms $f_N \in \mathcal{M}(\Gamma_0(N))$ of arbitrary high level and the product operation is non-trivial.
Our starting point is the basic observation that the Hopf algebra $\mathcal{H}_1 = \mathcal{H}(1)$ of transverse geometry in codimension 1 mentioned above admits a natural action on the modular Hecke algebras $\mathcal{A}(\Gamma)$. As an algebra, $\mathcal{H}_1$ coincides with the universal enveloping algebra of the Lie algebra with basis $\{X, Y, \delta_n\,; n \geq 1\}$ and brackets

$$[Y,X] = X\,,\ [Y,\delta_n] = n\,\delta_n\,,\ [X,\delta_n] = \delta_{n+1}\,,\ [\delta_k,\delta_\ell] = 0\,, \quad n,k,\ell \geq 1\,, \tag{2}$$

while the coproduct, which confers it the Hopf algebra structure, is determined by the identities

$$\begin{aligned} \Delta\,Y &= Y\otimes 1 + 1\otimes Y\,, \quad \Delta\,\delta_1 = \delta_1\otimes 1 + 1\otimes \delta_1\,, \\ \Delta\,X &= X\otimes 1 + 1\otimes X + \delta_1\otimes Y\,, \end{aligned}$$

together with the property that $\Delta : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_1$ is an algebra homomorphism. The action of X on $\mathcal{A}(\Gamma)$ is given by a classical operator going back to Ramanujan, which corrects the usual differentiation by the logarithmic derivative of the Dedekind eta function $\eta(z)$. The action of Y is given by the standard grading by the weight (the Euler operator) on modular forms. Finally, δ_1 and its higher 'derivatives' δ_n act by generalized cocycles on $\mathrm{GL}_2^+(\mathbb{Q})$ with values in modular forms.

One can then analyze this action of $\mathcal{H}_1$ through its cyclic cohomology. The latter admits three basic generators, with one of them cobounding in the periodized theory. We describe them and their meaning for foliations.

- *Godbillon-Vey Cocycle*

  Its class is represented by δ_1 which is a primitive element of $\mathcal{H}_1$, i.e. fulfills

  $$\Delta\,\delta_1 = \delta_1 \otimes 1 + 1 \otimes \delta_1 .$$

  It is cyclic, so that $B(\delta_1) = 0$, where B is the boundary operator in cyclic cohomology, whose definition involves the antipode, the product and the coproduct. One shows that the class of δ_1

  $$[\delta_1] \in HC^1_{\mathrm{Hopf}}(\mathcal{H}_1) .$$

  is the generator of $PHC^{\mathrm{odd}}_{\mathrm{Hopf}}(\mathcal{H}_1)$ and corresponds to the Godbillon-Vey class in the isomorphism with Gelfand-Fuchs cohomology. It is this class, which is responsible for the type III property of codimension one foliations explained above.

- *Schwarzian Derivative*

  The element $\delta_2' := \delta_2 - \frac{1}{2}\delta_1^2 \in \mathcal{H}_1$ is a Hopf cyclic cocycle, whose action in the foliation case corresponds to the multiplication by the Schwarzian derivative of the holonomy and whose class

  $$[\delta_2'] \in HC^1_{\mathrm{Hopf}}(\mathcal{H}_1)$$

  is equal to $B(c)$, where c is the Hochschild 2-cocycle

  $$c := \delta_1 \otimes X + \frac{1}{2}\,\delta_1^2 \otimes Y .$$

- *Fundamental Class*

  The generator of the even group $PHC^{\mathrm{even}}_{\mathrm{Hopf}}(\mathcal{H}_1)$ is the class of the cyclic 2-cocycle

  $$F := X \otimes Y - Y \otimes X - \delta_1\, Y \otimes Y ,$$

  which for foliations represents the transverse fundamental class.

We showed in [44] that each of the above cocycles admits a beautiful interpretation in its action on the modular Hecke algebras. First the action of δ_1', coupled with modular symbols, yields a rational representative for the Euler class of $\mathrm{GL}_2^+(\mathbb{Q})$. Next the action of δ_2' is an inner derivation implemented by the modular form ω_4,

$$\omega_4 = -\frac{E_4}{72}, \qquad E_4(q) := 1 + 240 \sum_1^\infty n^3 \frac{q^n}{1-q^n} .$$

Moreover, there is no way to perturb the action of the Hopf algebra $\mathcal{H}_1$ on the modular Hecke algebras so that δ_2' vanishes, and the obstruction exactly agrees with the obstruction found by D. Zagier in his work on Rankin-Cohen algebras [90].
Next the action of the fundamental class gives a Hochschild 2-cocycle which is the natural extension to modular Hecke algebras of the first Rankin-Cohen bracket. This led us to the following general result, which provides the correct notion of one dimensional projective structure for noncommutative spaces and extends the Rankin-Cohen deformation in that generality. Let the Hopf algebra $\mathcal{H}_1$ act on an algebra $\mathcal{A}$, in such a way that the derivation δ_2' is inner, implemented by an element $\Omega \in \mathcal{A}$,

$$\delta_2'(a) = \Omega\, a - a\, \Omega\,,\ \forall a \in \mathcal{A} \tag{3}$$

with

$$\delta_k(\Omega) =\ 0\,,\ \forall k \in \mathbb{N}. \tag{4}$$

Such an action of $\mathcal{H}_1$ on an algebra $\mathcal{A}$ will be said to define a *projective structure* on $\mathcal{A}$, and the element $\Omega \in \mathcal{A}$ implementing the inner derivation δ_2' will be called its *quadratic differential*.
Our main result in [44] is the construction of a sequence of brackets RC_*, which, applied to any algebra $\mathcal{A}$ endowed with a projective structure, yields a family of formal associative deformations of $\mathcal{A}$. The formulas for RC_n are completely explicit, but they become rapidly quite involved, as witnessed by the complexity of the following formula for RC_3, which gives the *bidifferential operator* expression for the third bracket. (The expression of RC_4 is much longer, it would occupy several pages.)

$$\begin{aligned}
RC_3 \ =\ & -2\,X \otimes X^2 - 2\,X \otimes X^2.Y + 2\,X \otimes \alpha[\Omega].Y + 2\,X \otimes \alpha[\Omega].Y^2 + 2\,X^2 \otimes X + 6\,X^2 \otimes X.Y + \\
& +4\,X^2 \otimes X.Y^2 - \frac{2\,X^3 \otimes Y}{3} - 2\,X^3 \otimes Y^2 - \frac{4\,X^3 \otimes Y^3}{3} + \frac{2\,Y \otimes X^3}{3} - \frac{2}{3}\,Y \otimes \alpha[\Omega].X \\
& -\frac{2}{3}\,Y \otimes \alpha[X[\Omega]].Y - 2\,Y \otimes \alpha[\Omega].X.Y + 2\,Y^2 \otimes X^3 - 2\,Y^2 \otimes \alpha[\Omega].X - 2\,Y^2 \otimes \alpha[X[\Omega]].Y \\
& -6\,Y^2 \otimes \alpha[\Omega].X.Y + \frac{4\,Y^3 \otimes X^3}{3} - \frac{4}{3}\,Y^3 \otimes \alpha[\Omega].X - \frac{4}{3}\,Y^3 \otimes \alpha[X[\Omega]].Y - 4\,Y^3 \otimes \alpha[\Omega].X.Y \\
& -6\,X.Y \otimes X^2 - 6\,X.Y \otimes X^2.Y + 6\,X.Y \otimes \alpha[\Omega].Y + 6\,X.Y \otimes \alpha[\Omega].Y^2 - 4\,X.Y^2 \otimes X^2 \\
& -4\,X.Y^2 \otimes X^2.Y + 4\,X.Y^2 \otimes \alpha[\Omega].Y + 4\,X.Y^2 \otimes \alpha[\Omega].Y^2 + 2\,X^2.Y \otimes X + 6\,X^2.Y \otimes X.Y \\
& +4\,X^2.Y \otimes X.Y^2 - 2\,\delta_1.X \otimes X - 6\,\delta_1.X \otimes X.Y - 4\,\delta_1.X \otimes X.Y^2 + 2\,\delta_1.X^2 \otimes Y + 6\,\delta_1.X^2 \otimes Y^2 \\
& +4\,\delta_1.X^2 \otimes Y^3 + 2\,\delta_1.Y \otimes X^2 + 2\,\delta_1.Y \otimes X^2.Y - 2\,\delta_1.Y \otimes \alpha[\Omega].Y - 2\,\delta_1.Y \otimes \alpha[\Omega].Y^2 \\
& +6\,\delta_1.Y^2 \otimes X^2 + 6\,\delta_1.Y^2 \otimes X^2.Y - 6\,\delta_1.Y^2 \otimes \alpha[\Omega].Y - 6\,\delta_1.Y^2 \otimes \alpha[\Omega].Y^2 + 4\,\delta_1.Y^3 \otimes X^2 \\
& +4\,\delta_1.Y^3 \otimes X^2.Y - 4\,\delta_1.Y^3 \otimes \alpha[\Omega].Y - 4\,\delta_1.Y^3 \otimes \alpha[\Omega].Y^2 - \delta_1^2.X \otimes Y - 3\,\delta_1^2.X \otimes Y^2 \\
& -2\,\delta_1^2.X \otimes Y^3 + \delta_1^2.Y \otimes X + 3\,\delta_1^2.Y \otimes X.Y + 2\,\delta_1^2.Y \otimes X.Y^2 + 3\,\delta_1^2.Y^2 \otimes X + 9\,\delta_1^2.Y^2 \otimes X.Y \\
& +6\,\delta_1^2.Y^2 \otimes X.Y^2 + 2\,\delta_1^2.Y^3 \otimes X + 6\,\delta_1^2.Y^3 \otimes X.Y + 4\,\delta_1^2.Y^3 \otimes X.Y^2 + \frac{1}{3}\,\delta_1^3.Y \otimes Y + \delta_1^3.Y \otimes Y^2 \\
& +\frac{2}{3}\,\delta_1^3.Y \otimes Y^3 + \delta_1^3.Y^2 \otimes Y + 3\,\delta_1^3.Y^2 \otimes Y^2 + 2\,\delta_1^3.Y^2 \otimes Y^3 + \frac{2}{3}\,\delta_1^3.Y^3 \otimes Y + 2\,\delta_1^3.Y^3 \otimes Y^2 \\
& +\frac{4}{3}\,\delta_1^3.Y^3 \otimes Y^3 + \frac{2}{3}\,\alpha[\Omega].X \otimes Y + 2\,\alpha[\Omega].X \otimes Y^2 + \frac{4}{3}\,\alpha[\Omega].X \otimes Y^3 - 2\,\alpha[\Omega].Y \otimes X \\
& -6\,\alpha[\Omega].Y \otimes X.Y - 4\,\alpha[\Omega].Y \otimes X.Y^2 - 2\,\alpha[\Omega].Y^2 \otimes X - 6\,\alpha[\Omega].Y^2 \otimes X.Y - 4\,\alpha[\Omega].Y^2 \otimes X.Y^2 \\
& +\frac{2}{3}\,\alpha[X[\Omega]].Y \otimes Y + 2\,\alpha[X[\Omega]].Y \otimes Y^2 + \frac{4}{3}\,\alpha[X[\Omega]].Y \otimes Y^3 - 6\,\delta_1.X.Y \otimes X - 18\,\delta_1.X.Y \otimes X.Y \\
& -12\,\delta_1.X.Y \otimes X.Y^2 - 4\,\delta_1.X.Y^2 \otimes X - 12\,\delta_1.X.Y^2 \otimes X.Y - 8\,\delta_1.X.Y^2 \otimes X.Y^2 + 2\,\delta_1.X^2.Y \otimes Y
\end{aligned}$$

$$
\begin{aligned}
&+6\,\delta_1.X^2.Y\otimes Y^2+4\,\delta_1.X^2.Y\otimes Y^3-3\,\delta_1^2.X.Y\otimes Y-9\,\delta_1^2.X.Y\otimes Y^2-6\,\delta_1^2.X.Y\otimes Y^3\\
&-2\,\delta_1^2.X.Y^2\otimes Y-6\,\delta_1^2.X.Y^2\otimes Y^2-4\,\delta_1^2.X.Y^2\otimes Y^3+2\,\alpha[\Omega].X.Y\otimes Y+6\,\alpha[\Omega].X.Y\otimes Y^2\\
&+4\,\alpha[\Omega].X.Y\otimes Y^3-2\,\alpha[\Omega].\delta_1.Y\otimes Y-6\,\alpha[\Omega].\delta_1.Y\otimes Y^2-4\,\alpha[\Omega].\delta_1.Y\otimes Y^3-2\,\alpha[\Omega].\delta_1.Y^2\otimes Y\\
&-6\,\alpha[\Omega].\delta_1.Y^2\otimes Y^2-4\,\alpha[\Omega].\delta_1.Y^2\otimes Y^3\;.
\end{aligned}
$$

The modular Hecke algebras turn out to be intimately related to the analysis of a very natural noncommutative space, which arose in a completely different context [45], having to do with the interplay between number theory and phase transitions with spontaneous symmetry breaking in quantum statistical mechanics, as initiated in [4]. The search for a two dimensional analog of the statistical system of [4] was obtained in [45], by first reinterpreting the latter from the geometry of the space of $\mathbb{Q}$-lattices in dimension one and then passing to two dimensions.

An n-dimensional $\mathbb{Q}$-lattice consists of an ordinary lattice Λ in $\mathbb{R}^n$ and a homomorphism

$$\phi : \mathbb{Q}^n/\mathbb{Z}^n \to \mathbb{Q}\Lambda/\Lambda.$$

Two such $\mathbb{Q}$-lattices are called *commensurable* if and only if the corresponding lattices are commensurable and the maps agree modulo the sum of the lattices.

The space $\mathcal{L}_n$ of commensurability classes of $\mathbb{Q}$-lattices in $\mathbb{R}^n$ turns out to be a very involved noncommutative space, which appears to be of great number theoretical significance because of its relation to both the Riemann zeta function (for $n = 1$) and modular forms (for $n = 2$). In physics language, what emerges is that the zeros of zeta appear as an absorption spectrum in the L^2 space of the space of commensurability classes of one dimensional $\mathbb{Q}$-lattices as in [31]. The noncommutative geometry description of the space of one-dimensional $\mathbb{Q}$-lattices modulo scaling recovers the quantum statistical mechanical system of [4], which exhibits a phase transition with spontaneous symmetry breaking. In a similar manner, we showed in [45] that the space of $\mathbb{Q}$-lattices in $\mathbb{C}$ modulo scaling generates a very interesting quantum statistical mechanical system whose ground states are parameterized by

$$\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{C}^*,$$

while the natural symmetry group of the system is the quotient

$$S = \mathbb{Q}^*\backslash \mathrm{GL}_2(\mathbb{A}_f)\,.$$

The values of a ground state φ on the natural *rational* subalgebra $\mathcal{A}_{\mathbb{Q}}$ of rational observables generates, in the generic case, a specialization $F_\varphi \subset \mathbb{C}$ of the modular field F. The state φ then intertwines the symmetry group S of the system with the Galois group of the modular field and there exists an isomorphism θ of S with $\mathrm{Gal}(F_\varphi/\mathbb{Q})$, such that

$$\alpha\circ\varphi = \varphi\circ\theta^{-1}(\alpha)\,,\quad \forall\alpha\in \mathrm{Gal}(F_\varphi/\mathbb{Q})\,.$$

In general, while the zeros of zeta and L-functions appear at the critical temperature, the analysis of the low temperature equilibrium states concentrates on the Langlands space

$$\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A}).$$

The subalgebra $\mathcal{A}_{\mathbb{Q}}$ of rational observables turns out to be intimately related to the modular Hecke algebras, which leads us to suspect that many of the results of [44] will survive in the context of [45], and will therefore be relevant in the analysis of the higher dimensional analog of the trace formula of [31].

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