Transgressions of the Godbillon-Vey class and Rademacher functions

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Introduction

In earlier work [8, 9] we investigated a surprising interconnection between the transverse geometry of codimension 1 foliations and modular forms. At the core of this interplay lies the Hopf algebra $\mathcal{H}_1$, the first in a series of Hopf algebras $\mathcal{H}_n$ that were found [6] to determine the affine transverse geometry of codimension n foliations. The periodic Hopf-cyclic cohomology of $\mathcal{H}_1$ is generated by two classes, $[\delta_1]$ for the odd component and $[RC_1]$ for the even component. The tautological action of $\mathcal{H}_1$ on the étale groupoid algebra $\mathcal{A}_{\mathcal{G}}$ associated to the frame bundle of a codimension 1 foliation preserves (up to a character) the canonical trace on $\mathcal{A}_{\mathcal{G}}$, and thus gives rise to a characteristic homomorphism in cyclic cohomology. This homomorphism maps $[\delta_1]$ to the Godbillon-Vey class and $[RC_1]$ to the transverse fundamental class.

The starting point of the investigation in [8] was the realization that the Hopf algebra $\mathcal{H}_1$ can be made to act on the crossed product $\mathcal{A}_{\mathbb{Q}}$ of the algebra of modular forms of all levels by $\mathrm{GL}^+(2,\mathbb{Q})$, via a natural connection provided by the Ramanujan operator on modular forms, thus conferring a symmetry structure to the space of lattices modulo the action of the Hecke correspondences. Although the algebra $\mathcal{A}_{\mathbb{Q}}$ no longer has an invariant trace,

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we used an ad hoc pairing with modular symbols to convert the Godbillon-Vey class $[\delta_1] \in HC^1(\mathcal{H}_1)$ into the Euler class of $\mathrm{GL}^+(2,\mathbb{Q})$.

In this paper we provide a completely conceptual explanation for the above pairing and at the same time extend it to the higher weight case. This is achieved by constructing out of modular symbols $\mathcal{H}_1$-invariant 1-traces that support characteristic maps for certain actions of $\mathcal{H}_1$ on $\mathcal{A}_\mathbb{Q}$, canonically associated to modular forms. Moreover, we show that the image of the Godbillon-Vey class through these characteristic homomorphisms, obtained by the cup product between $[\delta_1]$ and the invariant 1-traces, transgresses to secondary data. For the projective action determined by the Ramanujan connection the transgression takes place within the Euler class, in a manner that ressembles the K-homological transgression in the context of $SU_q(2)$ [5], and leads to the classical Rademacher function [23]. For the actions associated to cusp forms of higher weight the transgressed classes implement the Eichler-Shimura isomorphism. The actions corresponding to Eisenstein series give rise by transgression to higher Dedekind sums and generalized Rademacher functions ([28, 20]), or equivalently to the Eisenstein cocycle of [28].

Generalized Dedekind sums have been related to special values of L-functions in the work of C. Meyer on the class-number formula [18, 19], and higher Dedekind sums appear in the work of Siegel [25, 26] and Zagier [29] on the values at non-positive integers of partial zeta functions over real quadratic fields. Eisenstein cocycles were employed by Stevens [28] and by Sczech [24] in order to compute these values more efficiently. The fact that these notions can be interpreted as secondary invariants is reminiscent of the secondary nature of the regulator invariants (cf. e.g. [10]) that are involved in the expression of the special values at non-critical points of L-functions associated to number fields (see e.g. [16], [30]).

Contents

1 The standard modular Hopf action

In this preliminary section, we briefly review the basic facts (cf. [6], [8]) concerning the Hopf algebra $\mathcal{H}_1$ and its standard Hopf action on the crossed product $\mathcal{A}_{\mathbb{Q}}$ of the algebra of modular forms of all levels by $\mathrm{GL}^+(2,\mathbb{Q})$, associated to the Ramanujan connection.

1.1 The Hopf algebra $\mathcal{H}_1$ and its cyclic classes

We start by recalling the definition of the Hopf algebra $\mathcal{H}_1$. As an algebra, it coincides with the universal enveloping algebra of the Lie algebra with basis $\{X, Y, \delta_n\,; n \geq 1\}$ and brackets

$$[Y,X] = X\ ,\ [Y,\delta_n] = n\,\delta_n\ ,\ [X,\delta_n] = \delta_{n+1}\ ,\ [\delta_k,\delta_\ell] = 0\,, \quad n,k,\ell \geq 1\,.$$

As a Hopf algebra, the coproduct $\Delta : \mathcal{H}_1 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_1$ is determined by

$$\begin{aligned} \Delta\,Y &= Y \otimes 1 + 1 \otimes Y\,, \qquad\qquad \Delta\,\delta_1 = \delta_1 \otimes 1 + 1 \otimes \delta_1 \\ \Delta\,X &= X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y \end{aligned}$$

and the multiplicativity property

$$\Delta(h^1\,h^2) = \Delta h^1 \cdot \Delta h^2\,, \quad h^1, h^2 \in \mathcal{H}_1\ ;$$

the antipode is determined by

$$S(Y) = -Y, \quad S(X) = -X + \delta_1 Y, \quad S(\delta_1) = -\delta_1$$

and the anti-isomorphism property

$$S(h^1\, h^2) = S(h^2)\, S(h^1)\,, \quad h^1, h^2 \in \mathcal{H}_1\ ;$$

finally, the counit is

$$\varepsilon(h) = \text{constant term of} \quad h \in \mathcal{H}_1\,.$$

The modular character $\delta \in \mathcal{H}_1^*$, determined by

$$\delta(Y) = 1, \quad \delta(X) = 0, \quad \delta(\delta_n) = 0\,,$$

together with the unit of $1 \in \mathcal{H}_1$ forms a modular pair in involution $(\delta, 1)$, and thus the Hopf-cyclic cohomology $HC^1_{(\delta,1)}(\mathcal{H}_1)$ is well-defined (for definitions, see [6, 7]).

The element $\delta_1 \in \mathcal{H}_1$ is a Hopf-cyclic cocycle, which gives a nontrivial class

$$[\delta_1] \,\in\, HC^1_{(\delta,1)}(\mathcal{H}_1)$$

in the Hopf-cyclic cohomology of $\mathcal{H}_1$ with respect to the modular pair $(\delta, 1)$. Its periodic image generates the periodic group $HP^1(\mathcal{H}_1; \delta, 1)$, and represents the universal Godbillon-Vey class (cf. [8, Prop. 3]).

The even component of the periodic cyclic cohomology group $HP^0(\mathcal{H}_1; \delta, 1)$ is generated by the "transverse fundamental class", represented by the Hopf-cyclic 2-cocycle

$$RC_1 \,:=\, X \otimes Y - Y \otimes X - \delta_1\, Y \otimes Y\,.$$

(See [9] for the explanation of the notation.)

There is one other Hopf-cyclic 1-cocycle, intimately related to the classical Schwarzian, which plays a prominent role in the transverse geometry of modular Hecke algebras (cf. [8, 9]). It is given by the primitive element

$$\delta_2' := \delta_2 - \frac{1}{2}\delta_1^2 \,\in \mathcal{H}_1\,.$$

Its periodic class vanishes because $\delta_2' = B(c)$, where c is the following Hochschild 2-cocycle:

$$c \,:=\, \delta_1 \otimes X + \frac{1}{2}\,\delta_1^2 \otimes Y\,.$$

1.2 Standard modular action of $\mathcal{H}_1$

The notation being as in [8, §1], we form the crossed product algebra

$$\mathcal{A}_{G^+(\mathbb{Q})} \;=\; \mathcal{M} \ltimes \mathrm{GL}^+(2,\mathbb{Q})\,,$$

where $\mathcal{M}$ is the algebra of (holomorphic) modular forms of all levels. The product of two elements in $\mathcal{A}_{G^+(\mathbb{Q})}$,

$$a^0 = \sum_{\alpha} f^0_\alpha U_\alpha \quad \text{and} \quad a^1 = \sum_{\beta} f^1_\beta U_\beta\,,$$

is given by the convolution rule

$$a^0\, a^1 \;=\; \sum_{\alpha,\beta} f^0_\alpha\, f^1_\beta|\alpha^{-1}\, U_{\alpha\beta}\,.$$

We recall (see [8, Prop. 7]) that there is a unique Hopf action of the Hopf algebra $\mathcal{H}_1$ on $\mathcal{A}_{G^+(\mathbb{Q})}$ determined by letting the generators $\{Y, X, \delta_1\}$ of $\mathcal{H}_1$ act on monomials $fU^*_\gamma \in \mathcal{A}_{G^+(\mathbb{Q})}$ as follows:

$$Y(fU^*_\gamma) = Y(f)\, U^*_\gamma, \quad \text{where} \quad Y(f) = \frac{w(f)}{2}\, f, \quad w(f) = \mathrm{weight}(f); \quad (1.1)$$

$$X(fU^*_\gamma) = X(f)\, U^*_\gamma, \quad \text{where} \quad X = \frac{1}{2\pi i}\,\frac{d}{dz} - \frac{1}{2\pi i}\,\frac{d}{dz}(\log \eta^4)\, Y \qquad (1.2)$$

and η stands for the Dedekind η-function,

$$\eta^{24}(z) \;=\; q\, \prod_{n=1}^{\infty} (1-q^n)^{24}\,, \qquad q = e^{2\pi i z};$$

lastly,

$$\delta_1(fU^*_\gamma) = \mu_\gamma\, fU^*_\gamma\,, \qquad (1.3)$$

with the factor μ_γ given by the expression

$$\mu_\gamma\,(z) \;=\; \frac{1}{12\pi i}\,\frac{d}{dz}\log\frac{\Delta|\gamma}{\Delta} \;=\; \frac{1}{2\pi i}\,\frac{d}{dz}\log\frac{\eta^4|\gamma}{\eta^4}\,; \qquad (1.4)$$

equivalently,

$$\mu_\gamma\,(z) = \frac{1}{2\,\pi^2}\left(G_2|\gamma\,(z) - G_2(z) + \frac{2\pi i\, c}{cz+d}\right), \qquad (1.5)$$

where

$$G_2(z) \;=\; 2\zeta(2) \,+\, 2\sum_{m\geq 1}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^2} \;=\; \frac{\pi^2}{3} \,-\, 8\pi^2 \sum_{m,n\geq 1} m e^{2\pi i m n z}$$

is the quasimodular holomorphic Eisenstein series of weight 2. The factor μ_γ can further be expressed as the difference

$$\mu_\gamma \;=\; 2(\phi_{\mathbf{0}}|\gamma \,-\, \phi_{\mathbf{0}})\,, \quad \phi_{\mathbf{0}} = \frac{1}{4\pi^2} G_{\mathbf{0}}\,, \tag{1.6}$$

where $G_{\mathbf{0}}$ is the modular (but nonholomorphic) weight 2 Eisenstein series

$$G_{\mathbf{0}}(z) \;=\; G_{\mathbf{0}}(z,0)$$

obtained by taking the value at $s = 0$ of the analytic continuation of the series

$$\begin{aligned} G_{\mathbf{0}}(z,s) \;&=\; \sum_{(m,n)\in\mathbb{Z}^2\setminus 0} (mz+n)^{-2}\,|mz+n|^{-s} \\ &=\; 2\zeta(2+s) \,+\, 2\sum_{m\geq 1}\sum_{n\in\mathbb{Z}} (mz+n)^{-2}\,|mz+n|^{-s}\,, \quad \operatorname{Re} s > 0\,. \end{aligned}$$

It is related to G_2 by the identity

$$G_2(z) \;=\; G_{\mathbf{0}}(z) \,+\, \frac{2\pi i}{z-\bar{z}}\,.$$

The equation (1.6) shows that the range of μ is contained in the space $\mathcal{E}_2(\mathbb{Q})$ of weight 2 Eisenstein series whose constant term in the q-expansion at each cusp is rational. We recall (following [27, §2.4]) Hecke's construction [14] of a lattice of generators for the $\mathbb{Q}$-vector space $\mathcal{E}_2(\mathbb{Q})$.
For $\mathbf{a} = (a_1, a_2) \in (\mathbb{Q}/\mathbb{Z})^2$ and $z \in \mathbb{H}$ fixed, the series

$$G_{\mathbf{a}}(z,s) := \sum_{\mathbf{m}\neq\mathbf{0},\, \mathbf{m}\equiv\mathbf{a}\,(\mathrm{mod}\,\mathbf{1})} (m_1 z + m_2)^{-2}|m_1 z + m_2|^{-s}\,, \quad \operatorname{Re} s > 0\,;$$

defines a function that can be analytically continued beyond $\operatorname{Re} s = 0$, which allows to define

$$G_{\mathbf{a}}(z) := G_{\mathbf{a}}(z,0)\,.$$

Furthermore, one has

$$G_{\mathbf{a}}|\gamma \;=\; G_{\mathbf{a}\cdot\gamma}\,, \quad \forall\, \gamma \in \Gamma(1)\,,$$

which shows that $G_{\mathbf{a}}(z)$ behaves like a weight 2 modular form of some level N. However it is only quasi-holomorphic, in the sense that the function

$$z \mapsto G_{\mathbf{a}}(z) \;+\; \frac{2\pi i}{z-\bar{z}}$$

is holomorphic in $z \in \mathbb{H}$. Moreover, the difference

$$\wp_{\mathbf{a}}(z) \;=\; G_{\mathbf{a}}(z) \;-\; G_{\mathbf{0}}(z)$$

is precisely the $\mathbf{a}-$division value of the Weierstrass $\wp$-function, and the collection of functions

$$\left\{ \wp_{\mathbf{a}}\;;\; \mathbf{a} \in \left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^2 \setminus \mathbf{0} \right\}$$

generates the space of weight 2 Eisenstein series of level N.

In order to obtain a set of generators for $\mathcal{E}_2(\mathbb{Q})$, one considers the additive characters $\chi_{\mathbf{x}} : \left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^2 \to \mathbb{C}^\times$ defined by

$$\chi_{\mathbf{x}}\left(\frac{\mathbf{a}}{N}\right) \;:=\; e^{2\pi i\,(a_2x_1-a_1x_2)}\,, \tag{1.7}$$

for each $\mathbf{x} = (x_1, x_2) \in \left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^2$, and one forms the series

$$\phi_{\mathbf{x}}(z) \;:=\; (2\pi N)^{-2} \sum_{\mathbf{a}\in\left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^2} \chi_{\mathbf{x}}(\mathbf{a}) \cdot G_{\mathbf{a}}(z)\,. \tag{1.8}$$

The definition is independent of N and, for each $\mathbf{x} = (x_1, x_2) \in \left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^2 \setminus \mathbf{0}$ then $\phi_{\mathbf{x}}$, gives a weight 2 Eisenstein series of level N.

To account for the special case when $\mathbf{x} = \mathbf{0}$, one adjoins the non-holomorphic but modular function $\phi_{\mathbf{0}}$ defined in (1.6).

All the linear relations among the functions $\phi_{\mathbf{x}}$, $\mathbf{x} \in (\mathbb{Q}/\mathbb{Z})^2$ are encoded in the *distribution property*

$$\phi_{\mathbf{x}} \;=\; \sum_{\mathbf{y}\cdot\tilde{\gamma}=\mathbf{x}} \phi_{\mathbf{y}}|\gamma\,. \tag{1.9}$$

where

$$\check{\gamma} \,=\, \det \gamma \cdot \gamma^{-1} \,.$$

This allows to equip the extended Eisenstein space

$$\mathcal{E}_2^*(\mathbb{Q}) \,=\, \mathcal{E}_2(\mathbb{Q}) \,\oplus\, \mathbb{Q} \cdot \phi_0 \,,$$

with a linear $\mathrm{PGL}^+(2,\mathbb{Q})$-action, as follows. Denoting

$$\mathcal{S} \,:=\, (\mathbb{Q}/\mathbb{Z})^2 \,, \quad \text{resp.} \quad \mathcal{S}' \,:=\, \mathcal{S} \setminus \mathbf{0}$$

and identifying in the obvious way

$$\mathrm{PGL}^+(2,\mathbb{Q}) \,\cong\, M_2^+(\mathbb{Z})/\{\text{scalars}\} \,,$$

where $M_2^+(\mathbb{Z})$ stands for the set of integral 2×2-matrices of determinant > 0, one defines the action of $\gamma \in M_2^+(\mathbb{Z})$ by:

$$\mathbf{x}|\gamma \,:=\, \sum_{\mathbf{y}\cdot\check{\gamma}=\mathbf{x}} \mathbf{y} \,\in\, \mathbb{Q}[\mathcal{S}] \,.$$

With this definition one has

$$\phi_{\mathbf{x}}|\gamma \,=\, \phi_{\mathbf{x}|\gamma} \,, \qquad \gamma \in M_2^+(\mathbb{Z}) \,.$$

Modulo the subspace of 'distribution relations'

$$\mathcal{R} \,:=\, \mathbb{Q} - \text{span of} \quad \left\{ \mathbf{x} \,-\, \mathbf{x}| \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \; ; \mathbf{x} \in \mathcal{S} \,, n \in \mathbb{Z} \setminus 0 \right\} ,$$

the assignment $\mathbf{x} \in \mathcal{S} \longmapsto \phi_{\mathbf{x}}$ induces an isomorphism of $\mathrm{PGL}_2^+(2,\mathbb{Q})$-modules

$$\mathbb{Q}[\mathcal{S}]/\mathcal{R} \,\cong\, \mathcal{E}_2^*(\mathbb{Q}) \,.$$

In view of the above the identity (1.6) can be completed as follows:

$$\mu_\gamma \,=\, 2(\phi_{\mathbf{0}}|\gamma \,-\, \phi_{\mathbf{0}}) \,=\, 2 \left(\sum_{\mathbf{y}\cdot\check{\gamma}=\mathbf{0}} \phi_{\mathbf{y}} \,-\, \phi_{\mathbf{0}} \right) , \quad \forall\, \gamma \in M_2^+(\mathbb{Z}). \qquad (1.10)$$

2 Characteristic map for the standard action

In this section we provide the conceptual explanation for the period pairing which was employed in [8] to obtain the Euler class out of the universal Godbillon-Vey class $[\delta_1] \in HC^1_{(\delta,1)}(\mathcal{H}_1)$, by showing that it is in fact the by-product of a characteristic map associated to a 1-trace which is invariant with respect to the standard action of $\mathcal{H}_1$ on $\mathcal{A}$.
Extending a classical 'splitting formula' for the restriction to $\mathrm{SL}(2,\mathbb{Z})$ of the 2-cocycle that gives the universal cover of $\mathrm{SL}(2,\mathbb{R})$, we shall then obtain a new formula for the rational 2-cocycle representing the Euler class found in [8], showing that it differs from the Petersson cocycle ([1], [21]) by precisely the coboundary of the classical Rademacher function.

2.1 Characteristic map and cup products

In [6] (see also [7]) we defined a characteristic map associated to a Hopf module algebra with invariant trace. The construction has been subsequently extended to higher traces (cf. [11]) and turned into a cup product in Hopf-cyclic cohomology (cf. [15]). A predecessor of these constructions is the contraction of a cyclic n-cocycle by the generator of a 1-parameter group of automorphisms that fixes the cocycle, cf. [4, $Chap.III.\,6.\beta$]. We shall apply the latter to a specific 1-trace $\tau_0 \in ZC^1(\mathcal{A})$, which will be described in details in the next subsection. Further on, it will also be applied in the context of cyclic cohomology with coefficients, to 1-traces $\tau_W \in ZC^1(\mathcal{A}, W)$, where W denotes an algebraically irreducible $\mathrm{GL}(2,\mathbb{Q})$-module.

Let us assume that $\tau \in C^1(\mathcal{A})$ is a cyclic cocycle which satisfies, with respect to a given Hopf action of $\mathcal{H}_1$ on $\mathcal{A}$, the invariance property

$$\tau(h_{(1)}(a^0), h_{(2)}(a^1)) \,=\, \delta(h)\,\tau(a^0,a^1), \qquad \forall\, h \in \mathcal{H}_1,\ a^0,a^1 \in \mathcal{A}\,. \tag{2.1}$$

The simplest expression for the cup product

$$gv \,=\, \delta_1 \# \tau \in ZC^2(\mathcal{A})$$

is given by the contraction formula in [4, $Chap.III.\,6.\beta$] mentioned above, which (in the non-normalized form, cf. [11, §3]) takes the expression:

$$gv\,(a^0,a^1,a^2) \,=\, \tau(a^0\,\delta_1(a^1),a^2)\,, \qquad a^0,a^1,a^2 \in \mathcal{A}\,. \tag{2.2}$$

For the convenience of the reader, let us check directly that this formula gives a cocycle in the (b,B)-bicomplex of the algebra $\mathcal{A}$.

Lemma 1. *Let $\tau \in ZC^1(\mathcal{A})$ be a cyclic cocycle satisfying the $\mathcal{H}_1$-invariance property (2.1). Then* $\quad b(gv) = 0 \quad$ *and* $\quad B(gv) = 0\,.$

Proof. Using the fact that δ_1 acts as a derivation, one has

$$b(gv)\,(a^0,a^1,a^2,a^3) \,=\, \tau\,(a^0a^1\delta_1(a^2),a^3) \,-\, \tau\,(a^0\delta_1(a^1a^2),a^3)$$

$$+ \;\; \tau\,(a^0\delta_1(a^1),a^2a^3) \,-\, \tau\,(a^3a^0\delta_1(a^1),a^2)$$

$$= \;\; -\tau\,(a^0\delta_1(a^1)a^2,a^3) \,+\, \tau\,(a^0\delta_1(a^1),a^2a^3)$$

$$- \;\; \tau\,(a^3a^0\delta_1(a^1),a^2) \,=\, -b\tau\,(a^0\delta_1(a^1),a^2,a^3) \,=\, 0\,.$$

Passing to B, since $\tau(a,1) \,=\, 0$ for any $a \in \mathcal{A}$, one has

$$B(gv)\,(a^0,a^1) \,=\, gv\,(1,a^0,a^1) \,-\, gv\,(1,a^1,a^0)$$

$$= \;\; \tau\,(\delta_1(a^0),a^1) \,-\, \tau\,(\delta_1(a^1),a^0) \,=\, \tau\,(\delta_1(a^0),a^1) \,+\, \tau\,(a^0,\delta_1(a^1)) \,=\, 0\,,$$

the vanishing taking place because $\tau \in ZC^1(\mathcal{A})$ is $\mathcal{H}_1$-invariant. ∎

2.2 The basic invariant 1-cocycles

The modular symbol cocycle of weight 2 associated to a base point $z_0 \in \mathbb{H}$, $\tau_0 \in ZC^1(\mathcal{A})$, is defined as follows. For monomials $f^0U_{\gamma_0}, f^1U_{\gamma_1} \in \mathcal{A}$, $\tau_0(f^0U_{\gamma_0}, f^1U_{\gamma_1}) \,=\, 0$, unless they satisfy the condition

$$w(f^0)+w(f^1)=2 \qquad \text{and} \qquad \gamma_0\gamma_1 = 1\,, \tag{2.3}$$

in which case it is given by the integral

$$\tau_0(f^0U_{\gamma_0}, f^1U_{\gamma_1}) \,=\, \int_{z_0}^{\gamma_0\,z_0} f^0\,f^1|\gamma_1\,dz\,. \tag{2.4}$$

The fact that $\tau_0 \in C^1(\mathcal{A})$ is indeed an $\mathcal{H}_1$-invariant cyclic cocycle is the content of the following result.

Proposition 2. *The cochain $\tau_0 \in C^1(\mathcal{A})$ is a cyclic cocycle which satisfies the $\mathcal{H}_1$-invariance property (2.1) with respect to the standard action.*

Proof. Let $f^0 U_{\gamma_0}, f^1 U_{\gamma_1}, f^2 U_{\gamma_2} \in \mathcal{A}$ be such that

$$w(f^0) + w(f^1) + w(f^2) = 2 \qquad \text{and} \qquad \gamma_0\gamma_1\gamma_2 = 1 \tag{2.5}$$

One has

$$\begin{aligned}
& b\tau_0(f^0 U_{\gamma_0}, f^1 U_{\gamma_1}, f^2 U_{\gamma_2}) \; = \\
= \; & \tau_0(f^0 f^1|\gamma_0{}^{-1} U_{\gamma_0\gamma_1}, f^2 U_{\gamma_2}) - \tau_0(f^0 U_{\gamma_0}, f^1 f^2|\gamma_1{}^{-1} U_{\gamma_1\gamma_2}) \\
+ \; & \tau_0(f^2 f^0|\gamma_2{}^{-1} U_{\gamma_2\gamma_0}, f^1 U_{\gamma_1}) \\
= \; & \int_{z_0}^{\gamma_0\gamma_1 z_0} f^0 f^1|\gamma_0{}^{-1} f^2|\gamma_1{}^{-1}\gamma_0{}^{-1} dz - \int_{z_0}^{\gamma_0 z_0} f^0 f^1|\gamma_0{}^{-1} f^2|\gamma_1{}^{-1}\gamma_0{}^{-1} dz \\
+ \; & \int_{z_0}^{\gamma_2\gamma_0 z_0} f^2 f^0|\gamma_2{}^{-1} f^1|\gamma_0{}^{-1}\gamma_2{}^{-1} dz \\
= \; & \int_{\gamma_0 z_0}^{\gamma_0\gamma_1 z_0} f^0 f^1|\gamma_0{}^{-1} f^2|\gamma_2 dz + \int_{z_0}^{\gamma_2\gamma_0 z_0} f^2 f^0|\gamma_2{}^{-1} f^1|\gamma_0{}^{-1}\gamma_2{}^{-1} dz \\
= \; & \int_{\gamma_0 z_0}^{\gamma_2{}^{-1} z_0} f^0 f^1|\gamma_0{}^{-1} f^2|\gamma_2 dz + \int_{\gamma_2{}^{-1} z_0}^{\gamma_0 z_0} f^2|\gamma_2 f^0 f^1|\gamma_0{}^{-1} dz \quad = \quad 0\,,
\end{aligned}$$

and so τ_0 is a Hochschild cocycle.
It is also cyclic, because for $f^0 U_{\gamma_0}, f^1 U_{\gamma_1} \in \mathcal{A}$ satisfying (2.3) one has

$$\begin{aligned}
\lambda_1\tau_0(f^0 U_{\gamma_0}, f^1 U_{\gamma_1}) \; &= \; -\tau_0(f^1 U_{\gamma_1}, f^0 U_{\gamma_0}) = -\int_{z_0}^{\gamma_1 z_0} f^1 f^0|\gamma_1{}^{-1} dz \\
&= \; \int_{z_0}^{\gamma_0 z_0} f^1|\gamma_1 f^0 dz \; = \; \tau_0(f^0 U_{\gamma_0}, f^1 U_{\gamma_1})\,.
\end{aligned}$$

In view of its multiplicative nature, it suffices to check the $\mathcal{H}_1$-invariance property (2.1) on the algebra generators $\{Y, X, \delta_1\}$. Starting with Y, and with $f^0 U_{\gamma_0}, f^1 U_{\gamma_1}$ satisfying (2.3), one has

$$\tau_0(Y(f^0 U_{\gamma_0}), f^1 U_{\gamma_1}) + \tau_0(f^0 U_{\gamma_0}, Y(f^1 U_{\gamma_1})) =$$

$$= \int_{z_0}^{\gamma_0\, z_0} Y(f^0 f^1|{\gamma_0}^{-1}) dz = \frac{w(f^0)+w(f^1)}{2} \int_{z_0}^{\gamma_0\, z_0} f^0 f^1|{\gamma_0}^{-1} dz$$
$$= \int_{z_0}^{\gamma_0\, z_0} f^0 f^1|{\gamma_0}^{-1} dz \quad = \quad \delta(Y)\, \tau_0(f^0 U_{\gamma_0}, f^1 U_{\gamma_1})\,.$$

Passing to X, the identity (2.1) is nontrivial only if $f^0 U_{\gamma_0}, f^1 U_{\gamma_1} \in \mathcal{A}$ satisfy

$$w(f^0) + w(f^1) = 0 \qquad \text{and} \qquad \gamma_0 \gamma_1 = 1\,, \tag{2.6}$$

which actually implies that f^0 and f^1 are constants. One gets

$$\tau(X(f^0)U_{\gamma_0}, f^1 U_{\gamma_1}) + \tau(f^0 U_{\gamma_0}, X(f^1)U_{\gamma_1}) + \tau(\delta_1(f^0 U_{\gamma_0}), Y(f^1)U_{\gamma_1}) = 0$$

since $X(f^j) = 0$ and $Y(f^1) = 0$.
Finally, with $f^0 U_{\gamma_0}, f^1 U_{\gamma_1}$ as above, one has

$$\tau(\delta_1(f^0 U_{\gamma_0}), f^1 U_{\gamma_1}) + \tau(f^0 U_{\gamma_0}, \delta_1(f^1 U_{\gamma_1})) =$$
$$= \int_{z_0}^{\gamma_0\, z_0} (\mu_{\gamma_1} + \mu_{\gamma_0}|\gamma_1)\, f^0 f^1|\gamma_1 dz \;=\; 0\,,$$

because of the cocycle property of μ. □

Allowing the base point z_0 to belong to the 'arithmetic' boundary of the upper half plane $P^1(\mathbb{Q})$ requires some regularization of the integral. This can be achieved by the standard procedure of removing the poles of Eisenstein series (cf. [27]). In the case at hand, it amounts to a coboundary modification which we proceed now to describe.
To obtain it, we start from the observation that the derivative of τ_0 with respect to the base point $z_0 \in \mathbb{H}$ is a coboundary:

$$\frac{d}{dz_0}\tau_0(f^0 U_{\gamma_0}, f^1 U_{\gamma_1}) = (f^0|\gamma_0\, f^1)(z_0) - (f^0\, f^1|\gamma_1)(z_0) = -b\epsilon(f^0 U_{\gamma_0}, f^1 U_{\gamma_1}),$$

where ϵ is the evaluation map at z_0:

$$\epsilon(f\, U_\gamma) = \begin{cases} f(z_0) & \text{if} \quad \gamma = 1\,,\ f \in \mathcal{M}_2 \\ \\ 0 & \text{otherwise}\,. \end{cases}$$

Taking the base point at the cusp ∞, we split the evaluation functional ϵ into two other functionals, the constant term at ∞

$$\mathbf{a}_0(f\,U_\gamma) = \begin{cases} a_0 & \text{if} \quad \gamma = 1\,,\ f \in \mathcal{M}_2 \\ \\ \ \ 0 & \qquad\text{otherwise} \end{cases}$$

and the evaluation of the remainder

$$\widetilde{\epsilon}(f\,U_\gamma) = \begin{cases} \widetilde{f}(z_0) & \text{if} \quad \gamma = 1\,,\ f \in \mathcal{M}_2 \\ \\ \ \ 0 & \qquad\text{otherwise}\,, \end{cases}$$

where for $f \in \mathcal{M}_2$ of level N,

$$f(z) \;=\; \sum_{n=0}^{\infty} a_n\, e^{\frac{2\pi i n z}{N}}$$

represents its Fourier expansion at ∞, and

$$\widetilde{f}(z) \;:=\; f(z) \;-\; \mathbf{a}_0(f)\ .$$

Both functionals are well-defined, because a_0 is independent of the level. To obtain a cohomologous cocycle independent of z_0, it suffices to add to τ_0 the sum of coboundaries of suitable anti-derivatives for the two components. We therefore define

$$\widetilde{\tau_0} \;=\; \tau_0 \;+\; z_0\, b\, \mathbf{a}_0 \;-\; \int_{z_0}^{i\infty} b\, \widetilde{\epsilon}\, dz\,,$$

that is, for $f^0 U_{\gamma_0}, f^1 U_{\gamma_1} \in \mathcal{A}$ as in (2.3)

$$\begin{aligned} \widetilde{\tau_0}(f^0 U_{\gamma_0}, f^1 U_{\gamma_1}) \;\; &= \;\; \int_{z_0}^{\gamma_0\, z_0} f^0\, f^1|\gamma_1\, dz \;+\; z_0\, \mathbf{a}_0(f^0\, f^1|\gamma_1 - f^0|\gamma_0\, f^1) \\ &- \;\; \int_{z_0}^{i\infty} \widetilde{\epsilon}(f^0\, f^1|\gamma_1 - f^0|\gamma_0\, f^1)\, dz\,. \end{aligned} \tag{2.7}$$

The fact that $\widetilde{\tau_0} \in C^1(\mathcal{A})$ still satisfies Proposition 2 can be checked by an obvious adaptation of its proof.

2.3 Euler class and the transgression formula

As a first example, we now specialize the construction of the cup product to the invariant cocycle $\tau_0 \in ZC^1(\mathcal{A})$. Let $f^0 U_{\gamma_0}, f^1 U_{\gamma_1}, f^2 U_{\gamma_2} \in \mathcal{A}$ be such that $\gamma_0 \gamma_1 \gamma_2 = 1$. Then

$$
\begin{aligned}
& gv\,(f^0 U_{\gamma_0}, f^1 U_{\gamma_1}, f^2 U_{\gamma_2}) \;=\; \tau_0\,(f^0 U_{\gamma_0} \mu_{\gamma_1^{-1}} f^1 U_{\gamma_1}, f^2 U_{\gamma_2}) = \\
= \;& \tau_0 (f^0 f^1 | {\gamma_0}^{-1} \mu_{\gamma_1^{-1}} | {\gamma_0}^{-1} U_{\gamma_0 \gamma_1}, f^2 U_{\gamma_2}) \\
= \;& \int_{z_0}^{\gamma_0 \gamma_1 z_0} f^0 f^1 | {\gamma_0}^{-1} \mu_{\gamma_1^{-1}} | {\gamma_0}^{-1} f^2 | {\gamma_1}^{-1} {\gamma_0}^{-1} \, dz \\
= \;& \int_{{\gamma_0}^{-1} z_0}^{\gamma_1 z_0} f^0 | \gamma_0 \, f^1 \, \mu_{\gamma_1^{-1}} \, f^2 | {\gamma_1}^{-1} \, dz = \int_{\gamma_2 z_0}^{z_0} f^0 | \gamma_0 \gamma_1 \, f^1 | \gamma_1 \, \mu_{\gamma_1^{-1}} | \gamma_1 \, f^2 \, dz \\
= \;& \int_{z_0}^{\gamma_2 z_0} f^0 | \gamma_0 \gamma_1 \, f^1 | \gamma_1 \, f^2 \, \mu_{\gamma_1} \, dz \,. \qquad (2.8)
\end{aligned}
$$

In particular, the restriction to $\mathcal{A}_0 = \mathbb{C}[\mathrm{GL}^+(2,\mathbb{Q})]$ is the group cocycle

$$
GV\,(\gamma_1, \gamma_2) := \; gv\,(U_{\gamma_0}, U_{\gamma_1}, U_{\gamma_2}) \;=\; \int_{z_0}^{\gamma_2 z_0} \mu_{\gamma_1} \, dz \,, \qquad (2.9)
$$

whose real part

$$
\mathrm{Re}\, GV\,(\gamma_1, \gamma_2) := \; \mathrm{Re} \int_{z_0}^{\gamma_2 z_0} \mu_{\gamma_1} \, dz \,, \qquad \gamma_1, \gamma_2 \in \mathrm{GL}^+(2,\mathbb{Q}) \qquad (2.10)
$$

represents a generator of $H^2(\mathrm{SL}(2,\mathbb{Q}), \mathbb{R})$, hence a multiple of the Euler class (cf. [8, Thm. 16]).

For a more precise identification, we shall be very specific about the choice of the Euler class. Namely, we take it as the class $\mathbf{e} \in H^2_{\mathrm{bor}}(\mathbb{T}, \mathbb{Z})$ defined by the extension

$$
0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 1 \,;
$$

via the canonical isomorphisms

$$
H^2_{\mathrm{bor}}(\mathbb{T}, \mathbb{Z}) \simeq H^2(B\mathbb{T}, \mathbb{Z}) \simeq H^2(B\,\mathrm{SL}(2,\mathbb{R}), \mathbb{Z}) \simeq H^2_{\mathrm{bor}}(\mathrm{SL}(2,\mathbb{R}), \mathbb{Z})
$$

followed by the succession of natural map

$$
H^2_{\mathrm{bor}}(\mathrm{SL}(2,\mathbb{R}), \mathbb{Z}) \to H^2(\mathrm{SL}(2,\mathbb{Q}), \mathbb{Z}) \to H^2(\mathrm{SL}(2,\mathbb{Q}), \mathbb{R})
$$

we regard it as a class $\mathbf{e} \in H^2(\mathrm{SL}(2,\mathbb{Q}), \mathbb{R})$.

Proposition 3. *The 2-cocycle* $\mathrm{Re}\, GV \in Z^2(\mathrm{SL}(2,\mathbb{Q}),\mathbb{R})$ *represents the class* $-2\mathbf{e} \in H^2(\mathrm{SL}(2,\mathbb{Q}),\mathbb{R})$*, while* $\mathrm{Im}\, GV$ *is a coboundary.*

Proof. In view of the definition (2.9) and using (1.4), one has for $\gamma_1, \gamma_2 \in \mathrm{GL}^+(2,\mathbb{Q})$,

$$\begin{aligned} 12\pi i\, GV(\gamma_1,\gamma_2) &= 12\pi i \int_{z_0}^{\gamma_2 z_0} \mu_{\gamma_1}(z)\, dz \;=\; \int_{z_0}^{\gamma_2 z_0} \frac{d}{dz} \log \frac{\Delta|\gamma_1}{\Delta}\, dz \\ &= \int_{z_0}^{\gamma_2 z_0} (d \log \Delta|\gamma_1 - \, d\log \Delta) \\ &= \log \Delta|\gamma_1(\gamma_2 z_0) - \log \Delta|\gamma_1(z_0) \\ &- \; (\log \Delta(\gamma_2 z_0) - \log \Delta(z_0)) \end{aligned} \tag{2.11}$$

where, since both Δ and $\Delta|\gamma_1$ don't have zeros in $\mathbb{H}$ one lets $\log \Delta$ and $\log \Delta|\gamma_1$ be holomorphic determinations of the logarithm whose choice is unimportant at this stage, since the additive constant which depends only on γ_1 cancels out. Let

$$j(\gamma, z) \;=\; cz + d\,, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}^+(2,\mathbb{Q})\,,$$

be the automorphy factor. Since it has no zero in $\mathbb{H}$ one can choose for each γ a holomorphic determination $\log j^2(\gamma, z)$ of its logarithm (for instance using the principal branch for $\mathbb{C} \setminus [0,\infty)$ of the logarithm when $c \neq 0$ and taking $\log d^2$ when $c = 0$). One then has,

$$\log \Delta|\gamma(z) \;=\; \log \Delta(\gamma z) \;-\; 6 \;\; \mathrm{Log}\, j^2(\gamma, z) \;+\; 2\pi i\, k(\gamma)\,, \quad \forall z \in \mathbb{H}$$

for some $k(\gamma) \in \mathbb{Z}$. Thus (2.11) can be continued as follows:

$$\begin{aligned} 12\pi i GV(\gamma_1,\gamma_2) &= \log \Delta(\gamma_1\gamma_2 z_0) - \log \Delta(\gamma_1 z_0) - \log \Delta(\gamma_2 z_0) \\ &+ \log \Delta(z_0) - 6 \left(\log j^2(\gamma_1, \gamma_2 z_0) - \log j^2(\gamma_1, z_0) \right) \end{aligned} \tag{2.12}$$

again after the cancelation of the additive constants. The equality

$$\log j^2(\gamma_1\gamma_2, z_0) = \log j^2(\gamma_1, \gamma_2 z_0) + \log j^2(\gamma_2, z_0) - 2\pi i\, c(\gamma_1,\gamma_2)\,, \tag{2.13}$$

determines a cocycle $c \in Z^2(\mathrm{PSL}(2,\mathbb{R}),\,\mathbb{Z})$ (which is precisely the cocycle discussed in [2, §B-2], and whose cohomology class is independent of the choices of the branches $\log j^2(\gamma, z)$ of the logarithm). Inserting (2.13) into (2.12) one obtains

$$\begin{aligned}12\pi i GV(\gamma_1,\gamma_2) &= \log\Delta(z_0)+\log\Delta(\gamma_1\gamma_2 z_0)-\log\Delta(\gamma_1 z_0)-\log\Delta(\gamma_2 z_0)\\ &- 6\left(\log j^2(\gamma_1\gamma_2,z_0)-\log j^2(\gamma_2,z_0)-\log j^2(\gamma_1,z_0)\right)\\ &- 12\pi i\, c(\gamma_1,\gamma_2)\,. \end{aligned} \tag{2.14}$$

This identity shows that the cocycles $\mathrm{Re}\, GV$ and $-c$ are cohomologous in $Z^2(\mathrm{PSL}(2,\mathbb{Q}),\mathbb{R})$, and also that $\mathrm{Im}\, GV$ is a coboundary.
On the other hand, the restriction of $c \in Z^2_{\mathrm{bor}}(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z})$ to $\mathbb{T} = SO(2)$,

$$\begin{aligned} c(\gamma(\theta_1),\gamma(\theta_2)) &= \frac{1}{2\pi i}\left(\mathrm{Log}\, e^{2i\theta_1}+\mathrm{Log}\, e^{2i\theta_2}-\mathrm{Log}\, e^{2i(\theta_1+\theta_2)}\right)\,,\\ \text{where}\quad \gamma(\theta) &= \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\,,\qquad \theta\in[0,2\pi)\,. \end{aligned}$$

evidently represents the class $2\mathbf{e} \in H^2_{\mathrm{bor}}(\mathbb{T},\mathbb{Z})$, which concludes the proof. □

The Euler class $\mathbf{e} \in H^2(\mathrm{SL}(2,\mathbb{Q}),\mathbb{Z})$ occurs naturally in the context of the Chern character in K-homology [3], as the Chern character of a natural Fredholm module given by the "dual Dirac" operator relative to a base point $z_0 \in \mathbb{H}$. When moving the base point to the cusp $i\infty \in P^1(\mathbb{Q})$, it coincides with the restriction of the 2-cocycle $e \in Z^2(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z})$ introduced by Petersson (cf. [21]) and investigated in detail by Asai (cf. [1], where it is denoted w). It is defined, for $g_1, g_2 \in \mathrm{SL}(2,\mathbb{R})$, by the formula

$$e(g_1,g_2) = \frac{1}{2\pi i}\left(\log j(g_2,z)+\log j(g_1,g_2 z)-\log j(g_1 g_2,z)\right)\,, \tag{2.15}$$

with the logarithm chosen so that $\mathrm{Im}\log \in [-\pi,\pi)$; the above definition is independent of $z \in \mathbb{H}$.
Asai [1, §1-4] has shown that it can be given a simple expression, analogous to Kubota's cocycles [17] for coverings over local fields, which is as follows:

$$e(g_1,g_2) = -(x(g_1)|x(g_2)) + (-x(g_1)x(g_2)|x(g_1g_2))\,, \tag{2.16}$$

where

$$x(g) \;=\; \begin{cases} c & \text{if} \quad c > 0\,, \\ & \\ d & \text{if} \quad c = 0\,, \end{cases} \qquad \forall\, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

and for any two numbers $x_1, x_2 \in \mathbb{R}$, the (Hilbert-like) symbol $(x_1|x_2)$ is defined as

$$(x_1|x_2) \;=\; \begin{cases} 1 & \text{iff} \quad x_1 < 0 \quad \text{and} \quad x_2 < 0\,, \\ & \\ 0 & \qquad\qquad \text{otherwise}\,. \end{cases}$$

Replacing τ_0 by $\widetilde{\tau}_0$ one obtains, as in (2.8) and (2.9), the cohomologous group 2-cocycle on $\mathrm{GL}^+(2,\mathbb{Q})$

$$\widetilde{GV}\,(\gamma_1,\gamma_2) \;=\; \widetilde{\tau}_0(U_{\gamma_0}\,\delta_1(U_{\gamma_1}),U_{\gamma_2}) \;=\; \widetilde{\tau}_0(\mu_{\gamma_1{}^{-1}}|\gamma_0^{-1}\,U_{\gamma_0\gamma_1},U_{\gamma_2})$$

$$=\; \int_{z_0}^{\gamma_2 z_0} \mu_{\gamma_1} dz \;+\; z_0\,\mathbf{a}_0(\mu_{\gamma_1} - \mu_{\gamma_1}|\gamma_2) + \int_{z_0}^{i\infty} (\widetilde{\mu_{\gamma_1}|\gamma_2} - \widetilde{\mu_{\gamma_1}})dz\,. \tag{2.17}$$

In [8, §4] we found an explicit rational formula for $\mathrm{Re}\,\widetilde{GV}$, in terms of Rademacher-Dedekind sums, which we proceed now to recall.

Since by its very definition $\mathrm{Re}\,\widetilde{GV}$ descends to a 2-cocycle on $\mathrm{PGL}^+(2,\mathbb{Q})$, it suffices to express it for pairs of matrices with integer entries

$$\gamma_1 \;=\; \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\,, \quad \gamma_2 \;=\; \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \;\in\; M_2^+(\mathbb{Z})\,. \tag{2.18}$$

When $\gamma_2 \in B^+(\mathbb{Z})$, that is $c_2 = 0$, then

$$\mathrm{Re}\,\widetilde{GV}(\gamma_1,\gamma_2) \;=\; \frac{b_2}{d_2} \sum_{\mathbf{x}\cdot\check{\gamma}_1=\mathbf{0},\mathbf{x}\neq\mathbf{0}} \mathbf{B}_2(x_1)\,, \tag{2.19}$$

where $\mathbf{B}_2(x) := (x-[x])^2 - (x-[x]) + \frac{1}{6}$, with $[x]$ = greatest integer $\leq x$. When $c_2 > 0$, then

$$\mathrm{Re}\,\widetilde{GV}(\gamma_1,\gamma_2) \;=\; \frac{a_2}{c_2} \sum_{\mathbf{x}\cdot\check{\gamma}_1=\mathbf{0},\mathbf{x}\neq\mathbf{0}} \mathbf{B}_2(x_1) \;+\; \frac{d_2}{c_2} \sum_{\mathbf{x}\cdot\check{\gamma}_2\check{\gamma}_1=\mathbf{0},\mathbf{x}\neq\mathbf{0}} \mathbf{B}_2(x_1)$$

$$- \quad 2 \sum_{\mathbf{x}\cdot\check{\gamma}_1=\mathbf{0},\mathbf{x}\neq\mathbf{0}} \sum_{j=0}^{c_2'-1} \mathbf{B}_1\left(\frac{x_1+j}{c_2'}\right) \mathbf{B}_1\left(\frac{a_2'(x_1+j)}{c_2'}+x_2\right) \tag{2.20}$$

where $\frac{a_2'}{c_2'} = \frac{a_2}{c_2}$, $(a_2', c_2') = 1$, and $\mathbf{B}_1(x) := x - [x] - \frac{1}{2}$, for any $x \in \mathbb{R}$.

We shall obtain below a simpler expression for $\mathrm{Re}\,\widetilde{GV} \in Z^2(\mathrm{SL}(2,\mathbb{Q}),\mathbb{Q})$, through a transgression formula which involves only the classical Dedekind sums, through the Rademacher function. Let us recall that, for a pair of integers m, n with $(m,n) = 1$ and $n \geq 1$, the Dedekind sum is given by the formula

$$s(\frac{m}{n}) = \sum_{j=1}^{n-1} \mathbf{B}_1\left(\frac{j}{n}\right) \mathbf{B}_1\left(\frac{m\,j}{n}\right) . \tag{2.21}$$

The Rademacher function $\Phi : \mathrm{SL}(2,\mathbb{Z}) \to \mathbb{Z}$ is uniquely characterized (cf. [23]) by the coboundary relation

$$\Phi(\sigma_1\sigma_2) = \Phi(\sigma_1) - \Phi(\sigma_2) - 3\,\mathrm{sign}(c_1c_2c_3)\,, \quad \sigma_1, \sigma_2 \in \mathrm{SL}(2,\mathbb{Q}) \tag{2.22}$$

where $\sigma_3 = \sigma_1\sigma_2$, $\sigma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, $i = 1, 2, 3$, and is explicitly given by the following formula (cf. [22]):

$$\Phi(\sigma) = \begin{cases} \frac{b}{d} & \text{if} \quad c = 0, \\ \frac{a+d}{c} \quad - \quad 12\,\mathrm{sign}(c) s(\frac{a}{|c|}) & \text{if} \quad c \neq 0 \end{cases} \tag{2.23}$$

for any $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$.

We extend it, in a slightly modified version, to a function $\widetilde{\Phi} : \mathrm{GL}^+(2,\mathbb{Q}) \to \mathbb{Q}$, as follows. First, for any $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, we define

$$\widetilde{\Phi}(\sigma) = \begin{cases} \frac{b}{12d} \quad + \quad \frac{1-\mathrm{sign}(d)}{4} & \text{if} \quad c = 0, \\ \frac{a+d}{12c} \quad - \quad \mathrm{sign}(c)\left(\frac{1}{4} + s(\frac{a}{|c|})\right) & \text{if} \quad c \neq 0, \end{cases} \tag{2.24}$$

while for any $\beta \in B(\mathbb{Q})$, where $B(\mathbb{Q}) = \left\{ \beta = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{GL}^+(2,\mathbb{Q}) \right\}$, we set

$$\widetilde{\Phi}(\beta) \,=\, \frac{b}{12d} \,+\, \frac{1-\mathrm{sign}(d)}{4}. \tag{2.25}$$

Now given $\gamma \in \mathrm{GL}^+(2,\mathbb{Q})$, after factoring it in the form

$$\gamma \,=\, \sigma \cdot \beta \qquad \text{with} \qquad \sigma \in \mathrm{SL}(2,\mathbb{Z}) \quad \text{and} \quad \beta \in B(\mathbb{Q})\,,$$

we define

$$\widetilde{\Phi}(\gamma) \,=\, \widetilde{\Phi}(\sigma) \,+\widetilde{\Phi}(\beta)\,; \tag{2.26}$$

one easily checks, by elementary calculations, that the definition is consistent.

The transgression formula within the Euler class can now be stated as follows.

Theorem 4. *The function $\widetilde{\Phi} : \mathrm{SL}(2,\mathbb{Q}) \to \mathbb{Q}$ is uniquely characterized by the identity*

$$\frac{1}{2}\,\mathrm{Re}\,\widetilde{GV}(\gamma_1,\gamma_2) \,+\, e(\gamma_1,\gamma_2) \,=\, \widetilde{\Phi}(\gamma_1\gamma_2) \,-\, \widetilde{\Phi}(\gamma_1) \,-\, \widetilde{\Phi}(\gamma_2)\,, \tag{2.27}$$

for any $\gamma_1, \gamma_2 \in \mathrm{SL}(2,\mathbb{Q})$.

Proof. By Proposition 3, $[\frac{1}{2}\,\mathrm{Re}\,\widetilde{GV} + e] = 0$ in $H^2(\mathrm{SL}(2,\mathbb{Q}),\mathbb{R})$. Since

$$H^1(\mathrm{SL}(2,\mathbb{Q}),\mathbb{R}) \,=\, 0\,,$$

there exists a unique function $\Psi : \mathrm{SL}(2,\mathbb{Q}) \to \mathbb{R}$ such that

$$\frac{1}{2}\,\mathrm{Re}\,\widetilde{GV}(\gamma_1,\gamma_2) + e(\gamma_1,\gamma_2) = \Psi(\gamma_1\gamma_2) - \Psi(\gamma_1) - \Psi(\gamma_2)\,. \tag{2.28}$$

Restricting to $\mathrm{SL}(2,\mathbb{Z})$, and taking into account that $\mu_\sigma = 0$ for all $\sigma \in \mathrm{SL}(2,\mathbb{Z})$, one obtains

$$e(\sigma_1,\sigma_2) = \Psi(\sigma_1\sigma_2) - \Psi(\sigma_1) - \Psi(\sigma_2)\,, \qquad \sigma_1,\sigma_2 \in \mathrm{SL}(2,\mathbb{Z}).$$

This is the splitting formula (2.22) which uniquely characterizes the restriction (2.24) of $\widetilde{\Phi}$ to $\mathrm{SL}(2,\mathbb{Z})$ (see [1, Thm. 3]), so that

$$\Psi(\sigma) = \widetilde{\Phi}(\sigma)\,, \qquad \forall\, \sigma \in \mathrm{SL}(2,\mathbb{Z})\,. \tag{2.29}$$

Furthermore, taking in (2.28) $\gamma_1 = \sigma \in \mathrm{SL}(2,\mathbb{Z})$ and $\gamma_2 = \beta \in B_1^+(\mathbb{Q}) = \left\{\beta = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B(\mathbb{Q})\,;\, a > 0\,,\, a\,d = 1\right\}$, one obtains

$$\Psi(\sigma\beta) \;=\; \widetilde{\Phi}(\sigma) + \Psi(\beta)\,, \quad \forall\, \sigma \in \mathrm{SL}(2,\mathbb{Z})\,,\, \beta \in B_1^+(\mathbb{Q})\,, \tag{2.30}$$

because $e(\sigma,\beta) = 0$, by [1, Lemma 3]. In particular, for $\sigma = -I$, one has

$$\Psi(-\beta) \;=\; \widetilde{\Phi}(-I) + \Psi(\beta) \;=\; \Psi(\beta) + \frac{1}{2}, \qquad \forall\, \beta \in B_1^+(\mathbb{Q})\,, \tag{2.31}$$

since $\widetilde{\Phi}(-I) = \frac{1}{2}$, cf. [1, Lemma 4].
It remains to prove that $\Psi(\beta) = \widetilde{\Phi}(\beta)$ for any $\beta \in B_1^+(\mathbb{Q})$. Specializing (2.28) to $B_1^+(\mathbb{Q})$, and recalling that e vanishes on $B_1^+(\mathbb{Q})$, one obtains

$$\Psi(\beta_1\beta_2) - \Psi(\beta_1) - \Psi(\beta_2) \;=\; \frac{1}{2}\,\mathrm{Re}\,\widetilde{GV}(\beta_1,\beta_2)$$

and by (2.17) and (1.6) this can be computed as the real part of

$$\begin{aligned}
& \int_{z_0}^{\beta_2 z_0} (\phi_0|\beta_1 - \phi_0)dz \;+\; z_0\,\mathbf{a}_0(\phi_0|\beta_1 - \phi_0 - \phi_0|\beta_1\beta_2 + \phi_0|\beta_2) \\
+ & \int_{z_0}^{i\infty} (\widetilde{\phi_0|\beta_1\beta_2} - \widetilde{\phi_0|\beta_2} - \widetilde{\phi_0|\beta_1} + \widetilde{\phi}_0)dz \\
= & \int_{z_0}^{\beta_2 z_0} (\widetilde{\phi_0|\beta_1} - \widetilde{\phi}_0)dz + (\beta_2 z_0 - z_0)\,\mathbf{a}_0(\phi_0|\beta_1 - \phi_0) \\
+ & \; z_0\,\mathbf{a}_0(\phi_0|\beta_1 - \phi_0 - \phi_0|\beta_1\beta_2 + \phi_0|\beta_2) \\
+ & \int_{z_0}^{i\infty} (\widetilde{\phi_0|\beta_1\beta_2} - \widetilde{\phi_0|\beta_2} - \widetilde{\phi_0|\beta_1} + \widetilde{\phi}_0)dz \\
= & \int_{z_0}^{\beta_2 z_0} (\widetilde{\phi_0|\beta_1} - \widetilde{\phi}_0)dz + \int_{z_0}^{i\infty} (\widetilde{\phi_0|\beta_1\beta_2} - \widetilde{\phi_0|\beta_2} - \widetilde{\phi_0|\beta_1} + \widetilde{\phi}_0)dz
\end{aligned}$$

$$+ \quad \beta_2 z_0\, \mathbf{a}_0(\phi_0|\beta_1 - \phi_0) - z_0\, \mathbf{a}_0((\phi_0|\beta_1 - \phi_0)|\beta_2)\,.$$

Since obviously

$$\mathbf{a}_0(f|\beta) \;=\; \frac{a}{d}\,\mathbf{a}_0(f)\,, \quad \text{for} \quad \beta = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B(\mathbb{Q})\,, \tag{2.32}$$

the last line contributes

$$\left(\frac{a_2\, z_0 + b_2}{d_2} \;-\; \frac{a_2}{d_2}\, z_0 \right) \mathbf{a}_0(\phi_0|\beta_1 - \phi_0) \;=\; \frac{b_2}{d_2}\,\mathbf{a}_0(\phi_0|\beta_1 - \phi_0)$$

which can be further computed as

$$= \frac{1}{12}\,\frac{b_2}{d_2}\,\left(\frac{a_1}{d_1} - 1\right)\,,$$

by using (2.19) or more directly the Fourier expansion of ϕ_0 (cf. e.g. [27, Prop. 2.4.2]). On the other hand, when $z_0 \to i\infty$ then $\beta_2\, z_0 \to i\infty$ too, so that both integrals converge to 0. We conclude that

$$\Psi(\beta_1\beta_2) - \Psi(\beta_1) - \Psi(\beta_2) \;=\; \frac{1}{2}\,\mathrm{Re}\,\widetilde{GV}(\beta_1,\beta_2) \;=\; \frac{1}{12}\,\frac{b_2}{d_2}\,\left(\frac{a_1}{d_1} - 1\right)\,. \tag{2.33}$$

Using (2.25) it is elementary to check that, for $\beta_1, \beta_2 \in B_1^+(\mathbb{Q})$,

$$\widetilde{\Phi}(\beta_1\beta_2) - \widetilde{\Phi}(\beta_1) - \widetilde{\Phi}(\beta_2) \;=\; \frac{1}{12}\,\frac{b_2}{d_2}\,\left(\frac{a_1}{d_1} - 1\right)\,.$$

Thus, $\Psi|B_1^+(\mathbb{Q})$ and $\widetilde{\Phi}|B_1^+(\mathbb{Q})$ can only differ by a character of $B_1^+(\mathbb{Q})$. To show that they coincide, it suffices to prove that Ψ vanishes on the torus

$$T = \left\{ \delta = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\; ; \; \delta \in B_1^+(\mathbb{Q}) \right\}\,.$$

For any $\delta \in T$, one has

$$\sigma_0\,\delta \;=\; \delta^{-1}\,\sigma_0\,, \quad \text{where} \quad \sigma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,,$$

hence

$$\Psi(\sigma_0\,\delta) \;=\; \Psi(\delta^{-1}\,\sigma_0)\,.$$

¿From (2.28) it follows that

$$\Psi(\sigma_0) + \Psi(\delta) + \frac{1}{2}\,\mathrm{Re}\,\widetilde{GV}(\sigma_0, \delta) \;=\; \Psi(\delta^{-1}) + \Psi(\sigma_0) + \frac{1}{2}\,\mathrm{Re}\,\widetilde{GV}(\delta^{-1}, \sigma_0)\,,$$

therefore, since $\mathrm{Re}\,\widetilde{GV}(\sigma_0, \delta) = 0$,

$$\Psi(\delta) \;-\; \Psi(\delta^{-1}) \;=\; \frac{1}{2}\;\mathrm{Re}\,\widetilde{GV}(\delta^{-1}, \sigma_0)\,.$$

On the other hand, by (2.33)

$$\Psi(\delta) \;+\; \Psi(\delta^{-1}) \;=\; 0\,.$$

Hence

$$\Psi(\delta) \;=\; \frac{1}{4}\;\mathrm{Re}\,\widetilde{GV}(\delta^{-1}, \sigma_0)\,,$$

and it remains to show that the right hand side vanishes. We shall apply formula (2.20), which allows us to replace δ^{-1}, after multiplication by a scalar, with a diagonal matrix with positive integer entries $\rho = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \in M_2^+(\mathbb{Z})$. In this case, it simply gives

$$\begin{aligned} \frac{1}{2}\;\mathrm{Re}\,\widetilde{GV}(\rho, \sigma_0) \;&=\; -\sum_{\mathbf{x}\cdot\check{\rho}=\mathbf{0}} \mathbf{B}_1(x_1)\,\mathbf{B}_1(x_2) \;+\; \mathbf{B}_1(0)^2 \\ &=\; -\sum_{j=0}^{m-1} \mathbf{B}_1(\frac{j}{m}) \sum_{k=0}^{n-1} \mathbf{B}_1(\frac{k}{n}) + \mathbf{B}_1(0)^2 \;=\; 0\,, \end{aligned}$$

because of the distribution property of the first Bernoulli function. □

Remark 5. Note that while the cocycle e admits a K-homological interpretation as a Chern character, the cohomologous cocycle $\frac{1}{2}\,\mathrm{Re}\,\widetilde{GV}$ obviously should also have such an interpretation. This in turns would allow to put the above transgression on the same K-homological footing as in [5].

3 Modular symbol cyclic cocycles of higher weight

We begin in this section to extend the above results to the case of higher weight modular symbols. This will involve introducing the cyclic cohomology with coefficients $HC^*(\mathcal{A}, W)$ and describing the higher weight analogues of the basic invariant cocycles. They will be used in the next section to define the characteristic maps corresponding to the degenerate actions of weight $m \geq 2$ of $\mathcal{H}_1$ on $\mathcal{A} = \mathcal{A}_{G^+(\mathbb{Q})}$.

3.1 Cyclic cohomology with coefficients

We need to introduce the *cyclic cohomology with coefficients* $HC^*(\mathcal{A}, W)$, where $\mathcal{A}$ is the crossed product algebra $\mathcal{A}_{G^+(\mathbb{Q})}$ and W is a $\mathrm{GL}^+(2, \mathbb{Q})$-module. It is a special case of the Hopf-cyclic cohomology with coefficients, cf. [12, §3], in which the 'gauge' Hopf algebra is the group ring

$$\mathcal{G} \,=\, \mathbb{C}[G^+(\mathbb{Q})], \qquad G^+(\mathbb{Q}) := \mathrm{GL}^+(2, \mathbb{Q}),$$

equipped with its usual Hopf algebra structure; W is viewed as a left $\mathcal{G}$-module and as a trivial $\mathcal{G}$-comodule, and $\mathcal{A}$ is regarded as a left $\mathcal{G}$-comodule algebra with respect to the intrinsic coaction

$$a = f\, U_\gamma \longmapsto a_{(-1)} \otimes a_{(0)} := U_\gamma \otimes f\, U_\gamma \in \mathcal{G} \otimes \mathcal{A}\,.$$

Thus, by definition, $HC^*(\mathcal{A}, W)$ is the cohomology associated to the cyclic module

$$C^*(\mathcal{A}, W) \,:=\, \mathrm{Hom}^{\mathcal{G}}(\mathcal{A}^{*+1}, W)\,, \tag{3.1}$$

whose cyclic structure is defined by the operators

$$\begin{aligned}
\partial_i \phi(a^0 \otimes \cdots \otimes a^n) &= \phi(a^0 \otimes \cdots a^i a^{i+1} \otimes \cdots \otimes a^n)\,,\ 0 \leq i < n\,,\\
\partial_n \phi(a^0 \otimes \cdots \otimes a^n) &= S(a^n_{(-1)})\, \phi(a^n_{(0)} a^0 \otimes \cdots \otimes a^{n-1})\,,\\
\sigma_i \phi(a^0 \otimes \cdots \otimes a^n) &= \phi(a^0 \otimes \cdots a^i \otimes a^{i+1} \otimes \cdots \otimes a^n)\,,\ 0 \leq i < n\,,\\
\tau_n \phi(a^0 \otimes \cdots \otimes a^n) &= S(a^n_{(-1)})\, \phi(a^n_{(0)} \otimes a^0 \otimes \cdots \otimes a^{n-1})\,.
\end{aligned}$$

For each $m \in \mathbb{N}$ we denote by W_m the simple $\mathrm{SL}(2,\mathbb{C})$-module of dimension $m+1$, realized as the the space

$$W_m = \{P(T_1,T_2) \in \mathbb{C}[T_1,T_2]\,;\, P \,\text{is homogeneous of degree}\, m\}$$

and we let $\mathrm{GL}^+(2,\mathbb{R})$ act on W_m by

$$(g\,\cdot P)\,(T_1,T_2) \,=\, \det(g)^{-\frac{m}{2}}\,P(aT_1+cT_2, bT_1+dT_2), \quad g \,=\, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,.$$

Note that as a $\mathrm{GL}^+(2,\mathbb{R})$-module, W_m is the complexification of

$$W_m(\mathbb{R}) = \{P(T_1,T_2) \in \mathbb{R}[T_1,T_2]\,;\, P \,\text{is homogeneous of degree}\, m\}\,,$$

and we denote by $\mathrm{Re} : W_m \to W_m(\mathbb{R})$ the projection obtained by taking the real parts of the coefficients. We also note that as a $\mathrm{GL}^+(2,\mathbb{Q})$-module $W_m(\mathbb{R})$ has an obvious rational structure

$$W_m(\mathbb{Q}) = \{P(T_1,T_2) \in \mathbb{Q}[T_1,T_2]\,;\, P \,\text{is homogeneous of degree}\, m\}\,.$$

We denote by $F_m : \mathbb{H} \to W_m$ the polynomial function

$$F_m(z) \,=\, (zT_1 \,+\, T_2)^m\,, \tag{3.2}$$

and note that it satisfies, for any $g \in GL^+(2,\mathbb{R})$, the covariance property

$$F_m|g\,(z) \equiv \det(g)^{-\frac{m}{2}}\,(cz+d)^m\,F_m(gz) = g \cdot F_m(z)\,. \tag{3.3}$$

3.2 Cyclic 1-cocycles with coefficients: base point in $\mathbb{H}$

With these ingredients at hand, and after making the additional choice of a 'base point' $z_0 \in \mathbb{H}$, we proceed to define the *invariant cocycles* which will support characteristic maps associated to Hopf actions in the degenerate case.

Regarding $\mathrm{Hom}_{\mathcal{G}}(\mathcal{A}\otimes\mathcal{A}, W_m)$ as a graded linear space with respect to the weight filtration inherited from $\mathcal{A}$, we define the weight 2 element $\tau_m \in \mathrm{Hom}_{\mathcal{G}}(\mathcal{A}\otimes\mathcal{A}, W_m)$ as follows. Let

$$a^0 = \sum_\alpha f^0_\alpha U_\alpha \in \mathcal{A}_{w(a^0)} \quad \text{and} \quad a^1 = \sum_\beta f^1_\beta U_\beta \in \mathcal{A}_{w(a^1)}\,,$$

be two homogeneous elements in $\mathcal{A}$; by definition,

$$\tau_m(a^0,a^1) = \begin{cases} \sum_\alpha \int_{z_0}^{\alpha z_0} F_m\, f^0_\alpha\, f^1_{\alpha^{-1}}|\alpha^{-1}\, dz & \text{if} \quad w(a^0)+w(a^1)=m+2, \\ \\ 0 & \text{otherwise}\,. \end{cases}$$

Lemma 6. *For each $m \geq 2$, $\tau_{m-2} \in C^1(\mathcal{A}, W_{m-2})$ is a cyclic cocycle.*

Proof. For notational convenience, we shall omit the subscript $m-2$ in the ensuing calculations, which are of course similar to those in the proof of Proposition 2.
Let $f^0U_{\gamma_0}, f^1U_{\gamma_1}, f^2U_{\gamma_2} \in \mathcal{A}$ be such that

$$w(f^0)+w(f^1)+w(f^2)=m \qquad \text{and} \qquad \gamma_0\gamma_1\gamma_2 = 1\,.$$

Then

$$\begin{aligned}
& b\tau(f^0U_{\gamma_0}, f^1U_{\gamma_1}, f^2U_{\gamma_2}) \;= \\
= \;& \tau(f^0f^1|{\gamma_0}^{-1}U_{\gamma_0\gamma_1}, f^2U_{\gamma_2}) - \tau(f^0U_{\gamma_0}, f^1f^2|{\gamma_1}^{-1}U_{\gamma_1\gamma_2}) \\
+ \;& {\gamma_2}^{-1}\tau(f^2f^0|{\gamma_2}^{-1}U_{\gamma_2\gamma_0}, f^1U_{\gamma_1}) \\
= \;& \int_{z_0}^{\gamma_0\gamma_1\, z_0} F f^0f^1|{\gamma_0}^{-1}f^2|{\gamma_1}^{-1}{\gamma_0}^{-1}dz - \int_{z_0}^{\gamma_0\, z_0} F f^0f^1|{\gamma_0}^{-1}f^2|{\gamma_1}^{-1}{\gamma_0}^{-1}dz \\
+ \;& {\gamma_2}^{-1}\int_{z_0}^{\gamma_2\gamma_0\, z_0} F f^2f^0|{\gamma_2}^{-1}f^1|{\gamma_0}^{-1}{\gamma_2}^{-1}dz \\
= \;& \int_{\gamma_0 z_0}^{\gamma_0\gamma_1 z_0} F f^0f^1|{\gamma_0}^{-1}f^2|\gamma_2 dz + \int_{z_0}^{\gamma_2\gamma_0\, z_0} F|{\gamma_2}^{-1}f^2f^0|{\gamma_2}^{-1}f^1|{\gamma_0}^{-1}{\gamma_2}^{-1}dz \\
= \;& \int_{\gamma_0\, z_0}^{{\gamma_2}^{-1}\, z_0} F f^0f^1|{\gamma_0}^{-1}f^2|\gamma_2 dz + \int_{{\gamma_2}^{-1}\, z_0}^{\gamma_0\, z_0} F f^2|\gamma_2 f^0f^1|{\gamma_0}^{-1}dz \quad = \quad 0\,,
\end{aligned}$$

and so $b\tau = 0$. It is also cyclic, because for $f^0U_{\gamma_0}, f^1U_{\gamma_1} \in \mathcal{A}$ such that

$$w(f^0)+w(f^1)=m \qquad \text{and} \qquad \gamma_0\gamma_1 = 1 \tag{3.4}$$

one has

$$
\begin{aligned}
& \lambda_1 \tau(f^0 U_{\gamma_0}, f^1 U_{\gamma_1}) = -{\gamma_1}^{-1} \tau(f^1 U_{\gamma_1}, f^0 U_{\gamma_0}) = \\
= \;& -{\gamma_1}^{-1} \int_{z_0}^{\gamma_1 z_0} F f^1 f^0 | {\gamma_1}^{-1} dz = -\int_{z_0}^{\gamma_1 z_0} F | {\gamma_1}^{-1} f^1 f^0 | {\gamma_1}^{-1} dz \\
= \;& \int_{z_0}^{\gamma_0 z_0} F f^1 | \gamma_1 f^0 dz \;=\; \tau(f^0 U_{\gamma_0}, f^1 U_{\gamma_1}) \,.
\end{aligned}
$$

□

3.3 Cyclic 1-cocycles with coefficients: base point at cusps

We now extend the construction of subsection 2.2, allowing the base point z_0 to belong to the 'arithmetic' boundary of the upper half plane $P^1(\mathbb{Q})$, to the general case of weight $m \geq 2$. To this end we shall just apply the same procedure to the cocycle $\tau = \tau_{m-2}$. For $f^0 U_{\gamma_0}, f^1 U_{\gamma_1} \in \mathcal{A}$ such that

$$
w(f^0) + w(f^1) = m \qquad \text{and} \qquad \gamma_0 \gamma_1 = 1 \tag{3.5}
$$

it is given by the integral

$$
\tau(f^0 U_{\gamma_0}, f^1 U_{\gamma_1}) \;=\; \int_{z_0}^{\gamma_0 z_0} F f^0 \, f^1 | \gamma_1 \, dz \,,
$$

where $F = F_{m-2}$. Taking its derivative with respect to $z_0 \in \mathbb{H}$ gives

$$
\begin{aligned}
\frac{d}{dz_0} \tau(f^0 U_{\gamma_0}, f^1 U_{\gamma_1}) \;&=\; F|\gamma_0(z_0) \, (f^0 | \gamma_0 \, f^1)(z_0) - F(z_0) \, (f^0 \, f^1 | \gamma_1)(z_0) \\
&=\; \gamma_0 \cdot F(z_0) \, (f^0 | \gamma_0 \, f^1)(z_0) - F(z_0) \, (f^0 \, f^1 | \gamma_1)(z_0) \\
&=\; -b \epsilon_F \, (f^0 U_{\gamma_0}, f^1 U_{\gamma_1}),
\end{aligned}
$$

where

$$
\epsilon_F \;=\; F(z_0) \, \epsilon \,.
$$

As above, we split it into two functionals

$$\epsilon_F \,=\, F(z_0)\,\mathbf{a}_0 \,+\, F(z_0)\,\widetilde{\epsilon}\,,$$

and regularize the cocycle τ by adding the coboundaries of suitable anti-derivatives of the two components:

$$\widetilde{\tau} \,=\, \tau \,+\, b\left(\check{F}(z_0)\,\mathbf{a}_0\right) \,-\, \int_{z_0}^{i\infty} b\left(F(z)\,\widetilde{\epsilon}\right)\,dz\,,$$

where

$$\check{F}_{m-2}(z_0) \,=\, \int_0^{z_0} F(z)dz \,=\, \sum_{k=0}^{m-2} \frac{(m-2)!}{(k+1)!(m-k-2)!}\, z_0^{k+1}\, T_1^k\, T_2^{m-k-2}\,. \tag{3.6}$$

Explicitly, for $f^0 U_{\gamma_0}, f^1 U_{\gamma_1} \in \mathcal{A}$ satisfying (3.5),

$$\begin{aligned}
&\widetilde{\tau}_{m-2}(f^0 U_{\gamma_0}, f^1 U_{\gamma_1}) = \\
= \;& \int_{z_0}^{\gamma_0\, z_0} F f^0\, f^1|\gamma_1\, dz \,+\, \check{F}(z_0)\,\mathbf{a}_0(f^0\, f^1|\gamma_1) - \gamma_0 \cdot \check{F}(z_0)\,\mathbf{a}_0(f^0|\gamma_0\, f^1) \\
- \;& \int_{z_0}^{i\infty} F\,\widetilde{\epsilon}(f^0\, f^1|\gamma_1)\, dz \,+\, \int_{z_0}^{i\infty} \gamma_0 \cdot F\,\widetilde{\epsilon}(f^0|\gamma_0\, f^1)\, dz\,.
\end{aligned} \tag{3.7}$$

Since by its very definition $\widetilde{\tau}_{m-2} \in C^1(\mathcal{A}, W_{m-2})$ differs from the cocycle $\tau_{m-2} \in ZC^1(\mathcal{A}, W_{m-2})$ by a coboundary, it is itself a cyclic cocycle.

4 Transgression for degenerate actions

The higher weight counterparts of the above results involve cyclic cocycles with coefficients that are invariant under 'degenerate' actions of $\mathcal{H}_1$, associated to modular forms of arbitrary weight. In the case of cusp forms, the corresponding cup products by the universal Godbillon-Vey cocycle $\delta_1 \in ZC^1(\mathcal{H}_1; \delta, 1)$ transgress to 1-dimensional cohomology classes of congruence subgroups, implementing the Eichler-Shimura isomorphism, while the degenerate actions corresponding to Eisenstein series give rise to generalized functions of Rademacher type (cf. [20]) as well as to the Eisenstein cocycle of Stevens [28].

4.1 Degenerate actions of $\mathcal{H}_1$ of higher weight

We recall a few basic facts about cocycle perturbations of Hopf actions. Given a Hopf algebra $\mathcal{H}$ and an algebra $\mathcal{A}$ endowed with a Hopf action of $\mathcal{H}$, a 1-cocycle $u \in Z^1(\mathcal{H}, \mathcal{A})$ is an invertible element of the convolution algebra $\mathcal{L}(\mathcal{H}, \mathcal{A})$ of linear maps from $\mathcal{H}$ to $\mathcal{A}$, satisfying

$$u(h\,h') = \sum u(h_{(1)})\, h_{(2)}(u(h'))\,, \quad \forall h \in \mathcal{H}\,. \tag{4.1}$$

The conjugate under $u \in Z^1(\mathcal{H}, \mathcal{A})$ of the original action of $\mathcal{H}$ on $\mathcal{A}$ is given by

$$\tilde{h}(a) := \sum u(h_{(1)})\, h_{(2)}(a)\, u^{-1}(h_{(3)}) \tag{4.2}$$

The standard action of $\mathcal{H}_1$ commutes with the natural coaction of $G^+(\mathbb{Q})$ on $\mathcal{A}_{G^+(\mathbb{Q})}$. It is natural to only consider cocycles u with the same property. The values of such a 1-cocycle u on generators must belong to the subalgebra $\mathcal{M} \subset \mathcal{A}_{G^+(\mathbb{Q})}$, and have to be of the following form:

$$u(X) = \theta \in \mathcal{M}_2,\ u(Y) = \lambda \in \mathbb{C},\ u(\delta_1) = \omega \in \mathcal{M}_2\,.$$

By [8, Prop. 11], for each such data $\theta \in \mathcal{M}_2,\ \lambda \in \mathbb{C},\ \omega \in \mathcal{M}_2$,

1^0 there exists a unique 1-cocycle $u \in \mathcal{L}(\mathcal{H}_1, \mathcal{A}_{G^+(\mathbb{Q})})$ such that

$$u(X) = \theta,\ u(Y) = \lambda,\ u(\delta_1) = \omega\,;$$

2^0 the conjugate under u of the action of $\mathcal{H}_1$ is given on generators as follows:

$$\begin{aligned} \tilde{Y} = Y\,, \quad \tilde{X}(a) &= X(a) + [(\theta - \lambda\omega), a] - \lambda\,\delta_1(a) + \omega Y(a)\,, \\ \tilde{\delta}_1(a) &= \delta_1(a) + [\omega, a]\,, \qquad a \in \mathcal{A}_{G^+(\mathbb{Q})}\,; \end{aligned} \tag{4.3}$$

3^0 The conjugate under u of δ_2' is given by the operator

$$\tilde{\delta}_2'(a) = [X(\omega) + \frac{\omega^2}{2} - \Omega_4,\ a]\,, \qquad a \in \mathcal{A}_{G^+(\mathbb{Q})}$$

and there is no choice of u for which $\tilde{\delta}_2' = 0$.

The actions described above were called *projective* because $\tilde{\delta}'_2$ acts by an inner transformation.

For our purposes, it is the difference $\widetilde{\delta}_1 - \delta_1$ which matters. In view of (4.3), we may as well start from the trivial action and may also assume $\theta = 0$, $\lambda = 0$.

We call the *trivial action of weight* m of $\mathcal{H}_1$ on $\mathcal{A}_{G^+(\mathbb{Q})}$ the action defined by

$$Y(a) \,=\, \frac{w(f)}{m}\, a, \qquad X(a) \,=\, 0\,, \qquad \delta_n(a) \,=\, 0, \quad n \geq 1 \tag{4.4}$$

To any modular form $\omega \in \mathcal{M}_m$ we shall now associate a 'degenerate' action of the Hopf algebra $\mathcal{H}_1$ on the crossed product algebra $\mathcal{A}_{G^+(\mathbb{Q})}$ as follows.

Proposition 7. *Let* $\omega \in \mathcal{M}$ *be a modular of weight* $w(\omega) \,=\, m$.

1^0. *There exists a unique* 1*-cocycle* $u = u_\omega \in Z^1(\mathcal{H}_1, \mathcal{A}_{G^+(\mathbb{Q})})$ *such that*

$$u(Y) \,=\, 0, \qquad u(X) \,=\, 0, \qquad u(\delta_1) \,=\, \omega\,.$$

2^0. *The conjugate under* u_ω *of the trivial action of weight* m *of* $\mathcal{H}_1$ *on* $\mathcal{A} = \mathcal{A}_{G^+(\mathbb{Q})}$ *is determined by*

$$\tilde{Y}(a) = \frac{w(a)}{m}\, a, \quad \tilde{X}(a) = \omega\, \tilde{Y}(a), \quad \tilde{\delta}_1(a) = [\omega, a], \quad \forall\, a \in \mathcal{A}\,. \tag{4.5}$$

Under this action, one has, for any $n \geq 1$ *,*

$$\begin{aligned} \tilde{\delta}_n(a) \;&=\; (n-1)!\, \omega^{n-1}\, [\omega, a], \qquad \forall\, a \in \mathcal{A}\,, && (4.6)\\ \textit{resp.} \quad \tilde{\delta}_n(f\, U^*_\gamma) \;&=\; X^{n-1}(\omega \,-\, \omega|\gamma)\, f\, U^*_\gamma, \quad \forall\, f\, U^*_\gamma \in \mathcal{A}\,. && (4.7) \end{aligned}$$

Proof. The first statement is a variant of [8, Proposition 11, 1^0]. Its proof gives the following explicit description of the 1-cocycle $u \in Z^1(\mathcal{H}_1, \mathcal{A}_{G^+(\mathbb{Q})})$. First, recall that any element of $\mathcal{H}_1$ can be uniquely written as a linear combination of monomials of the form $P(\delta_1, \delta_2, ..., \delta_\ell)\, X^n\, Y^m$. We then let $\omega^{(k)}$ be defined by induction by

$$\omega^{(1)} \,:=\, \omega, \qquad \omega^{(k+1)} \,:=\, \omega\, Y(\omega^{(k)}) \,=\, k!\, \omega^{k+1}, \quad \forall\, k \geq 1\,.$$

Then

$$u\left(P(\delta_1,\delta_2,...,\delta_\ell)\,X^n\,Y^m\right) \;=\; \begin{cases} P(\omega^{(1)},\omega^{(2)},...,\omega^{(\ell)}) & \text{if} \quad n=m=0,\\ 0 & \text{otherwise}\,,\end{cases}$$

while its inverse u^{-1} is given by

$$u^{-1}\left(P(\delta_1,\delta_2,...,\delta_\ell)\,X^n\,Y^m\right) \;=\; \begin{cases} P(-\omega,0,...,0) & \text{if} \quad n=m=0,\\ 0 & \text{otherwise}\,.\end{cases}$$

The formulae (4.5) are obtained from 1^0 above and the definitions (4.2), (4.4) (cf. also [8, Proposition 11, 2^0]). Finally (4.6), and hence (4.7), follows by straightforward computation. □

The actions as in 2^0 above are *degenerate*, in the sense that $\tilde{\delta}_1$ acts by an inner transformation.

Lemma 8. *With respect to the degenerate action of weight m, one has*

$$\tau(h_{(1)}(a^0),h_{(2)}(a^1)) \;=\; \delta(h)\,\tau(a^0,a^1), \qquad \forall\, h\in\mathcal{H}_1,\; a^0,a^1\in\mathcal{A}\,, \tag{4.8}$$

for $\tau=\tau_{m-2}$ *or* $\tau=\widetilde{\tau}_{m-2}$.

Proof. We shall check the $\mathcal{H}_1$-invariance property (4.8) for $\tau=\tau_{m-2}$. As in the proof of Proposition 2, it suffices to verify it on the generators.
Starting with $\widetilde{Y}$, for $f^0U_{\gamma_0}, f^1U_{\gamma_1}$ satisfying (3.4) one has

$$\begin{aligned} &\tau(\widetilde{Y}(f^0U_{\gamma_0}),f^1U_{\gamma_1})+\tau(f^0U_{\gamma_0},\widetilde{Y}(f^1U_{\gamma_1}))=\\ &=\int_{z_0}^{\gamma_0\, z_0} F\widetilde{Y}(f^0f^1|\gamma_0{}^{-1})dz=\frac{w(f^0)+w(f^1)}{m}\int_{z_0}^{\gamma_0\, z_0} F f^0f^1|\gamma_0{}^{-1}dz\\ &=\int_{z_0}^{\gamma_0\, z_0} F f^0f^1|\gamma_0{}^{-1}dz \quad=\quad \delta(Y)\,\tau(f^0U_{\gamma_0},f^1U_{\gamma_1})\,. \end{aligned}$$

Passing to $\widetilde{X}$, the identity (4.8) is nontrivial only if $f^0U_{\gamma_0}, f^1U_{\gamma_1}\in\mathcal{A}$ satisfy

$$w(f^0)+w(f^1)=0 \qquad \text{and} \qquad \gamma_0\gamma_1=1\,. \tag{4.9}$$

One then has

$$\tau(\widetilde{X}(f^0)U_{\gamma_0}, f^1U_{\gamma_1}) + \tau(f^0U_{\gamma_0}, \widetilde{X}(f^1)U_{\gamma_1}) + \tau(\delta_1(f^0U_{\gamma_0}), \widetilde{Y}(f^1)U_{\gamma_1}) =$$

$$\begin{aligned} = \quad & \int_{z_0}^{\gamma_0 z_0} F\,\omega\widetilde{Y}(f^0)f^1|\gamma_0{}^{-1}dz + \int_{z_0}^{\gamma_0 z_0} F\,f^0\omega|\gamma_0{}^{-1}\widetilde{Y}(f^1)|\gamma_0{}^{-1}dz \\ + \quad & \int_{z_0}^{\gamma_0 z_0} F\,(\omega - \omega|\gamma_0{}^{-1})f^0\widetilde{Y}(f^1)|\gamma_0{}^{-1}dz \;=\; \int_{z_0}^{\gamma_0 z_0} F\,\omega\widetilde{Y}(f^0\,f^1|\gamma_0{}^{-1})dz \\ = \quad & \frac{w(f^0)+w(f^1)}{m}\int_{z_0}^{\gamma_0 z_0} F f^0 f^1|\gamma_0{}^{-1}dz \quad = \quad 0\,, \end{aligned}$$

the vanishing being a consequence of (4.9).
Finally, in the case of $\widetilde{\delta}_1$ and with $f^0U_{\gamma_0}, f^1U_{\gamma_1}$ as in (4.9), if the action is degenerate, one has

$$\tau(\widetilde{\delta}_1(f^0U_{\gamma_0}), f^1U_{\gamma_1}) + \tau(f^0U_{\gamma_0}, \widetilde{\delta}_1(f^1U_{\gamma_1})) =$$

$$= \int_{z_0}^{\gamma_0 z_0} F(\omega - \omega|\gamma_1)f^0f^1|\gamma_1 dz + \int_{z_0}^{\gamma_0 z_0} F f^0(\omega - \omega|\gamma_0)|\gamma_1 f^1|\gamma_1 dz \;=\; 0\,.$$

□

Remark 9. The fact that the only modular forms of weight 0 are the constants was used just in the case of the standard action. Thus, for all the degenerate actions, the statement remains valid for the enlarged crossed product $\widetilde{\mathcal{A}} = \widetilde{\mathcal{M}} \ltimes \mathrm{GL}^+(2,\mathbb{Q})$, where $\widetilde{\mathcal{M}}$ is the algebra consisting of meromorphic modular forms of all levels whose poles are located on cusps.

4.2 Transgression in the higher weight case

We now consider the degenerate action associated to a modular form $\omega \in \mathcal{M}_m$ with $m \geq 2$ and define the cup product of δ_1 with the invariant 1-trace τ as in Lemma 8 by the formula

$$gv_\omega\,(a^0, a^1, a^2) = \tau(a^0\,\widetilde{\delta}_1(a^1),\, a^2) = \tau(a^0\,\omega a^1 - a^1\omega,\, a^2)\,, \quad a^0, a^1, a^2 \in \mathcal{A}$$

Lemma 10. *One has $gv_\omega \in ZC^2(\mathcal{A}, W_{m-2})$, that is*

$$b(gv_\omega) \,=\, 0 \quad \textit{and} \quad B(gv_\omega) \,=\, 0\,.$$

Proof. As in the proof of Lemma 1, using the primitivity of δ_1, one has

$$b(gv_\omega)\,(a^0,a^1,a^2,a^3) \,=\, \tau\,(a^0a^1\delta_1(a^2),a^3) \,-\, \tau\,(a^0\delta_1(a^1a^2),a^3)$$

$$+ \quad \tau\,(a^0\delta_1(a^1),a^2a^3) \,-\, S(a^3_{(-1)})\,\tau\,(a^3_{(0)}a^0\delta_1(a^1),a^2)$$

$$= \quad -\tau\,(a^0\delta_1(a^1)a^2,a^3) \,+\, \tau\,(a^0\delta_1(a^1),a^2a^3)$$

$$- \quad S(a^3_{(-1)})\,\tau\,(a^3_{(0)}a^0\delta_1(a^1),a^2) \,=\, -b\tau\,(a^0\delta_1(a^1),a^2,a^3) \,=\, 0\,.$$

Next, since $\tau(a,1) \,=\, 0$ for any $a \in \mathcal{A}$, one has

$$\begin{aligned} B(gv_\omega)\,(a^0,a^1) \;&=\; g_\omega v\,(1,a^0,a^1) \,-\, S(a^1_{(-1)})\,gv_\omega\,(1,a^1_{(0)},a^0) \\ &=\; \tau\,(\delta_1(a^0),a^1) \,-\, S(a^1_{(-1)})\,\tau\,(\delta_1(a^1_{(0)}),a^0)\,. \end{aligned}$$

We now use the fact that the action of δ_1 commutes with the coaction of $\mathrm{GL}^+(2,\mathbb{Q})$, to continue

$$\begin{aligned} B(gv_\omega)\,(a^0,a^1) \;&=\; \tau\,(\delta_1(a^0),a^1) \,-\, S(\delta_1(a^1)_{(-1)})\,\tau\,(\delta_1(a^1)_{(0)},a^0) \\ &=\; \tau\,(\delta_1(a^0),a^1) \,+\, \tau\,(a^0,\delta_1(a^1)) \,=\, 0\,, \end{aligned}$$

the vanishing being a consequence of the $\mathcal{H}_1$-invariance property of $\tau \in ZC^1(\mathcal{A}, W_{m-2})$. ☐

Because degenerate actions are perturbations of the trivial action of weight m, one expects the cup product

$$[gv_\omega] \,=\, [\delta_1]\#[\tau_{m-2}]$$

to vanish. This is indeed the case, the vanishing being a consequence of a general transgression formula for degenerate actions. To state it, we introduce the cochain $tgv_\omega \in C^1(\mathcal{A}, W_{m-2})$,

$$tgv_\omega(a^0,a^1) \,=\, -\tau(a^0,\omega\,a^1)\,, \qquad a^0,a^1 \in \mathcal{A} \tag{4.10}$$

Proposition 11. *For any $\omega \in \mathcal{M}_m$,*

$$gv_\omega \,=\, b\,(tgv_\omega) \quad \textit{and} \quad B\,(tgv_\omega) \,=\, 0\,.$$

Proof. One has

$$b(tgv_\omega)(a^0, a^1, a^2) = -\tau(a^0a^1,\, \omega a^2) + \tau(a^0,\, \omega a^1a^2) - a^2_{(-1)}\tau(a^2a^0,\, \omega a^1);$$

after replacing the first term in the right hand side using

$$\begin{aligned} 0 \;&=\; b\tau(a^0, a^1, \omega a^2) = \\ &=\; \tau(a^0a^1,\, \omega a^2) - \tau(a^0,\, a^1\omega a^2) + a^2_{(-1)}\tau(\omega a^2a^0,\, a^1), \end{aligned}$$

one obtains

$$\begin{aligned} & b(tgv_\omega)\,(a^0, a^1, a^2) \,=\, \tau(a^0,\, \omega a^1a^2) - \tau(a^0,\, a^1\omega a^2) + \\ +\;\; & a^2_{(-1)}\left(\tau(\omega a^2a^0,\, a^1) - \tau(a^2a^0,\, \omega a^1)\right) \\ =\;\; & \tau(a^0,\, \widetilde{\delta}_1(a^1)a^2) + a^2_{(-1)}\left(\tau(\omega a^2a^0,\, a^1) - \tau(a^2a^0,\, \omega a^1)\right). \end{aligned}$$

We now use the identity

$$0 = b\tau(a^0, \widetilde{\delta}_1(a^1), a^2) = \tau(a^0\widetilde{\delta}_1(a^1), a^2) - \tau(a^0, \widetilde{\delta}_1(a^1)a^2) + a^2_{(-1)}\tau(a^2a^0, \widetilde{\delta}_1(a^1))$$

to replace the term $\tau(a^0,\, \widetilde{\delta}_1(a^1)a^2)$, thus obtaining

$$\begin{aligned} & b(tgv_\omega)\,(a^0, a^1, a^2) = \tau(a^0\widetilde{\delta}_1(a^1), a^2) + \\ +\;\; & a^2_{(-1)}\Big(\tau(a^2a^0, \widetilde{\delta}_1(a^1)) + \tau(\omega a^2a^0,\, a^1) - \tau(a^2a^0,\, \omega a^1)\Big) \\ =\;\; & \tau(a^0\widetilde{\delta}_1(a^1), a^2) + a^2_{(-1)}\left(-\tau(a^2a^0, a^1\omega) + \tau(\omega a^2a^0,\, a^1)\right). \end{aligned} \tag{4.11}$$

Next, using

$$\begin{aligned} 0 = b\tau(c^0, c^1, \omega) \;&=\; \tau(c^0c^1, \omega) - \tau(c^0, c^1\omega) + \tau(\omega c^0, c^1) \\ &=\; -\tau(c^0, c^1\omega) + \tau(\omega c^0, c^1)\,, \end{aligned}$$

for $c^0 = a^2 a^0$ and $c^1 = a^1$, one sees that the term in paranthesis vanishes, and therefore (4.11) reduces to

$$b(tgv_\omega)\,(a^0, a^1, a^2) \,=\, \tau(a^0 \widetilde{\delta_1}(a^1), a^2) \,=\, gv_\omega\,(a^0, a^1, a^2)\,.$$

On the other hand, by the very definition of $\tau = \tau_{m-2}$, one has

$$B(tgv_\omega)\,(a^0) \,=\, -\tau(1, \omega a^0) \,-\, \tau(a^0, \omega) \,=\, 0\,.$$

□

Let $\omega \in \mathcal{M}_m(\Gamma)$, for some congruence subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$. The transgression proper occurs when gv_ω is restricted to the subalgebra

$$\mathcal{A}^\Gamma \,=\, \mathcal{M} \ltimes \Gamma\,,$$

on which $\widetilde{\delta_1}$ vanishes. Then tgv_ω becomes a cocycle in $ZC^2(\mathcal{A}^\Gamma, W_{m-2})$ and its restriction to $\mathcal{A}_0^\Gamma = \mathbb{C}[\Gamma]$

$$tgv_\omega(U_{\gamma_0}, U_{\gamma_1}) = -\tau(U_{\gamma_0}, \omega\, U_{\gamma_1}) = -\int_{z_0}^{\gamma_0 z_0} F\omega\, dz\,, \qquad \gamma_0\gamma_1 = 1,$$

gives a group cocycle in $Z^1(\Gamma, W_{m-2})$. After a change of sign, we denote this cocycle

$$TGV_\omega(\gamma) := \int_{z_0}^{\gamma z_0} F\omega\, dz\,, \qquad \forall\, \gamma \in \Gamma. \tag{4.12}$$

Taking its real part, one obtains the linear map

$$\omega \in \mathcal{M}_m(\Gamma) \longmapsto ES(\omega) := [\mathrm{Re}\, TGV_\omega] \in H^1(\Gamma, W_{m-2}(\mathbb{R}))\,, \tag{4.13}$$

whose restriction to the cuspidal subspace $\mathcal{M}_m^0(\Gamma)$ gives the Eichler-Shimura embedding of $\mathcal{M}_m^0(\Gamma)$ into $H^1(\Gamma, W_{m-2}(\mathbb{R}))$.

In the remainder of this section we shall look closer at the restriction of the assignment (4.13) to the Eisenstein subspace $\mathcal{E}_m(\Gamma) \subset \mathcal{M}_m(\Gamma)$. To this end, we proceed to recall the construction of the Hecke lattice for higher weight Eisenstein series.

For $\mathbf{a} = (a_1, a_2) \in (\mathbb{Q}/\mathbb{Z})^2$, the holomorphic Eisenstein series $G_{\mathbf{a}}^{(m)}$ of weight $m > 2$ is defined by the absolutely convergent series

$$G_{\mathbf{a}}^{(m)}(z) := \sum_{\mathbf{k} \in \mathbb{Q}^2 \setminus \mathbf{0},\, \mathbf{k} \equiv \mathbf{a}\, (\mathrm{mod}\, 1)} (k_1 z + k_2)^{-m}\,;$$

averaging over $\left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^2$, and using as weights the additive characters (see (1.7)) $\left\{\chi_{\mathbf{x}}\,;\,\mathbf{x}\in\left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^2\right\}$, gives rise to the series

$$\phi_{\mathbf{x}}^{(m)}(z) := \frac{(m-1)!}{(2\pi i N)^m} \sum_{\mathbf{a}\in\left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^2} \chi_{\mathbf{x}}(\mathbf{a})\cdot G_{\mathbf{a}}^{(m)}(z)\,. \tag{4.14}$$

We now apply Proposition 11 for the case of the degenerate action defined by an Eisenstein series $\phi_{\mathbf{x}}^{(m)}$, $\mathbf{x}\in\left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^2$, and with respect to the $\mathcal{H}_1$-invariant cocycle $\widetilde{\tau}=\widetilde{\tau}_{m-2}$ (see (3.7)). Restricting to $\mathbb{C}[\Gamma(N)]$ one obtains the transgressed group cocycle $\widetilde{TGV}_{\phi_{\mathbf{x}}}\in Z^1(\Gamma(N),W_{m-2})$. We then take its real part $\Phi_{\mathbf{x}}^{(m)} := \mathrm{Re}\,\widetilde{TGV}_{\phi_{\mathbf{x}}}\in Z^1(\Gamma(N),W_{m-2}(\mathbb{R}))$, which still satisfies the cocycle relation

$$\Phi_{\mathbf{x}}^{(m)}(\alpha\beta) \,=\, \Phi_{\mathbf{x}}^{(m)}(\alpha) \,+\, \alpha\cdot\Phi_{\mathbf{x}}^{(m)}(\beta), \qquad \alpha,\beta\in\Gamma(N)\,. \tag{4.15}$$

We shall show that $\Phi_{\mathbf{x}}^{(m)}$ coincides with the generalized Rademacher function of [20, §2], and in particular $\Phi_{\mathbf{x}}^{(m)}\in Z^1(\Gamma(N),W_{m-2}(\mathbb{Q}))$. As a matter of fact, we shall explicitly compute its extension to $\mathrm{GL}^+(2,\mathbb{Q})$,

$$\Phi_{\mathbf{x}}^{(m)} \,=\, \mathrm{Re}\,\Psi_{\mathbf{x}}^{(m)}\,.$$

In turn,

$$\Psi_{\mathbf{x}}^{(m)}(\gamma) \,=\, \widetilde{TGV}_{\phi_{\mathbf{x}}}\,, \qquad \gamma\in\mathrm{GL}^+(2,\mathbb{Q})\,,$$

is obtained by specializing the formula (3.7) to $f^0=\phi_{\mathbf{x}}^{(m)}$, $f^1=1$ and thus is given by the expression

$$\begin{aligned}\Psi_{\mathbf{x}}^{(m)}(\gamma) \;&=\; \int_{z_0}^{\gamma z_0} F(z)\,\phi_{\mathbf{x}}^{(m)}(z)dz + \check{F}(z_0)\,\mathbf{a}_0(\phi_{\mathbf{x}}^{(m)}) - \gamma\cdot\check{F}(z_0)\,\mathbf{a}_0(\phi_{\mathbf{x}}^{(m)}|\gamma)\\ &-\; \int_{z_0}^{i\infty} F(z)\,\widetilde{\epsilon}(\phi_{\mathbf{x}}^{(m)})(z)dz + \int_{z_0}^{i\infty} \gamma\cdot F(z)\,\widetilde{\epsilon}(\phi_{\mathbf{x}}^{(m)}|\gamma)(z)dz\,,\end{aligned} \tag{4.16}$$

which is independent of $z_0\in\mathbb{H}$. The coboundary relation in Proposition 11 is equivalent with the following cocycle property

$$\Psi_{\mathbf{x}}^{(m)}(\alpha\beta) \,=\, \Psi_{\mathbf{x}}^{(m)}(\alpha) \,+\, \alpha\cdot\Psi_{\mathbf{x}|\alpha}^{(m)}(\beta), \qquad \alpha,\beta\in\mathrm{GL}^+(2,\mathbb{Q})\,, \tag{4.17}$$

and the collection $\{\Phi_{\mathbf{x}}^{(m)}\,;\,\mathbf{x}\in\mathbb{Q}^2/\mathbb{Z}^2\}$ is equivalent to the distribution valued *Eisenstein cocycle* of Stevens [28].

To state the precise result, we need to recall one more definition, that of a *generalized higher Rademacher-Dedekind sum* (cf. [23, 13, 28, 20]); given $a,c\in\mathbb{Z}$ with $(a,c)=1$ and $c\geq 1$, $0<k<m\in\mathbb{N}$ and $\mathbf{x}\in\left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^2$, it is defined by the expression

$$S_{\mathbf{x}}^{(m-k,k)}(\frac{a}{c})=\sum_{r=0}^{c-1}\frac{\mathbf{B}_{m-k}\left(\frac{x_1+r}{c}\right)}{m-k}\,\frac{\mathbf{B}_k\left(x_2+a\frac{x_1+r}{c}\right)}{k}\,, \tag{4.18}$$

where $\mathbf{B}_j:\mathbb{R}\to\mathbb{R}$ is the j-th periodic Bernoulli function

$$\mathbf{B}_j(x)\;=\;B_j(x-[x])\,,\qquad x\in\mathbb{R}\,.$$

Theorem 12. *Let $m\geq 2$ and let $\mathbf{x}\in\mathbb{Q}^2/\mathbb{Z}^2$, with $\mathbf{x}\neq 0$ if $m=2$.*
1^0. *For $\beta=\begin{pmatrix}a&b\\0&d\end{pmatrix}\in B(\mathbb{Q})$,*

$$\Phi_{\mathbf{x}}^{(m)}(\beta)\;=\;-\frac{\mathbf{B}_m(x_1)}{m}\int_0^{\frac{b}{d}}(tT_1+T_2)^{m-2}\,dt\,.$$

2^0. *For $\sigma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{Z})$, with $c>0$,*

$$\Phi_{\mathbf{x}}^{(m)}(\sigma)=\begin{cases}-\quad\dfrac{\mathbf{B}_m(x_1)}{m}\displaystyle\int_0^{\frac{a}{c}}(tT_1+T_2)^{m-2}\,dt\\[2ex]-\quad\dfrac{\mathbf{B}_m(ax_1+cx_2)}{m}\displaystyle\int_{-\frac{d}{c}}^{0}(t(aT_1+cT_2)+bT_1+dT_2)^{m-2}\,dt\\[2ex]+\quad\displaystyle\sum_{k=0}^{m-2}(-1)^k\binom{m-2}{k}S_{\mathbf{x}}^{(m-k-1,k+1)}(\frac{a}{c})T_1^k(aT_1+cT_2)^{m-k-2}.\end{cases}$$

Proof. 1^0. Restricting $\Psi_{\mathbf{x}}^{(m)}$ to $B(\mathbb{Q})$, one has

$$\begin{aligned}
\Psi_{\mathbf{x}}^{(m)}(\beta) &= \int_{z_0}^{\beta z_0} F(z)\,\widetilde{\epsilon}(\phi_{\mathbf{x}}^{(m)})(z)dz + \mathbf{a}_0(\phi_{\mathbf{x}}^{(m)}) \int_{z_0}^{\beta z_0} F(z)dz \\
&+ \check{F}(z_0)\,\mathbf{a}_0(\phi_{\mathbf{x}}^{(m)}) - \beta\cdot\check{F}(z_0)\,\mathbf{a}_0(\phi_{\mathbf{x}}^{(m)}|\beta) \\
&- \int_{z_0}^{i\infty} F(z)\,\widetilde{\epsilon}(\phi_{\mathbf{x}}^{(m)})(z)\,dz + \int_{z_0}^{i\infty} \beta\cdot F(z)\,\widetilde{\epsilon}(\phi_{\mathbf{x}}^{(m)}|\beta)(z)dz \\
&= \int_{z_0}^{\beta z_0} F(z)\,\widetilde{\epsilon}(\phi_{\mathbf{x}}^{(m)})(z)dz + \mathbf{a}_0(\phi_{\mathbf{x}}^{(m)})\,(\check{F}(\beta z_0) - \check{F}(z_0)) \\
&+ \check{F}(z_0)\,\mathbf{a}_0(\phi_{\mathbf{x}}^{(m)}) - \beta\cdot\check{F}(z_0)\,\mathbf{a}_0(\phi_{\mathbf{x}}^{(m)}|\beta) \\
&- \int_{z_0}^{i\infty} F(z)\,\widetilde{\epsilon}(\phi_{\mathbf{x}}^{(m)})(z)dz + \int_{z_0}^{i\infty} \beta\cdot F(z)\,\widetilde{\epsilon}(\phi_{\mathbf{x}}^{(m)}|\beta)(z)dz \\
&= \int_{z_0}^{\beta z_0} F(z)\widetilde{\epsilon}(\phi_{\mathbf{x}}^{(m)})(z)dz + \mathbf{a}_0(\phi_{\mathbf{x}}^{(m)})\check{F}(\beta z_0) - \beta\cdot\check{F}(z_0)\mathbf{a}_0(\phi_{\mathbf{x}}^{(m)}|\beta) \\
&- \int_{z_0}^{i\infty} F(z)\,\widetilde{\epsilon}(\phi_{\mathbf{x}}^{(m)})(z)dz + \int_{z_0}^{i\infty} \beta\cdot F(z)\,\widetilde{\epsilon}(\phi_{\mathbf{x}}^{(m)}|\beta)(z)\,dz\,.
\end{aligned}$$

When $z_0 \to i\infty$ all three integrals vanish. For the remaining terms we note that, using the weight m analogue of (2.32), one has

$$\mathbf{a}_0(\phi_{\mathbf{x}}^{(m)})\check{F}(\beta z_0) - \beta\cdot\check{F}(z_0)\mathbf{a}_0(\phi_{\mathbf{x}}^{(m)}|\beta) = \mathbf{a}_0(\phi_{\mathbf{x}}^{(m)}) \left(\check{F}(\beta z_0) - \left(\frac{a}{d}\right)^{\frac{m}{2}} \beta\cdot\check{F}(z_0)\right).$$

Since

$$\frac{d}{dz_0}\left(\check{F}(\beta z_0) - \left(\frac{a}{d}\right)^{\frac{m}{2}} \beta\cdot\check{F}(z_0)\right) \;=\; 0\,,$$

as one can easily check employing (3.3), the expression is constant in z_0, hence equal to its value at $z_0 = 0$

$$\check{F}(\beta z_0) - \left(\frac{a}{d}\right)^{\frac{m}{2}} \beta\cdot\check{F}(z_0) \;=\; \check{F}(\frac{b}{d})\,.$$

From the known Fourier transform of $\phi_{\mathbf{x}}^{(m)}$ (cf. [20], formula (2.1)), one reads that

$$\mathbf{a}_0(\phi_{\mathbf{x}}^{(m)}) \;=\; -\frac{\mathbf{B}_m(x_1)}{m}\,. \tag{4.19}$$

It follows that

$$\Phi_{\mathbf{x}}^{(m)}(\beta) \;=\; -\frac{\mathbf{B}_m(x_1)}{m}\int_0^{\frac{b}{d}}(tT_1+T_2)^{m-2}\,dt \qquad (4.20)$$

which proves the first statement.

2^0. Let us now compute the value of $\Psi_{\mathbf{x}}^{(m)}$ at $\sigma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ by choosing in (4.16) $z_0 = i$, which is a fixed point of σ_0. One has

$$\begin{aligned} \Psi_{\mathbf{x}}^{(m)}(\sigma_0) \;&=\; \mathbf{a}_0(\phi_{\mathbf{x}}^{(m)})\,\check{F}(i) - \sigma_0\cdot\check{F}(i)\,\mathbf{a}_0(\phi_{\mathbf{x}}^{(m)}|\sigma_0) \\ &-\; \int_i^{i\infty} F(z)\,\widetilde{\epsilon}(\phi_{\mathbf{x}}^{(m)})(z)\,dz + \int_i^{i\infty}\sigma_0\cdot F(z)\,\widetilde{\epsilon}(\phi_{\mathbf{x}}^{(m)}|\sigma_0)(z)\,dz\,. \end{aligned}$$

This can be related to the Mellin transform of $F\phi_{\mathbf{x}}^{(m)}$,

$$D(F\phi_{\mathbf{x}}^{(m)},s) \;=\; \int_0^{i\infty} F(z)\,\widetilde{\epsilon}(\phi_{\mathbf{x}}^{(m)})(z)\,y^{s-1}\,dz\,, \quad z = x+iy\,.$$

More precisely, as in the proof of [27, Prop. 2.3.3], one shows that

$$\Psi_{\mathbf{x}}^{(m)}(\sigma_0) \;=\; -D(F\phi_{\mathbf{x}}^{(m)},1)\,,$$

which in turn can be computed, as in [27, Prop. 2.2.1], from the Fourier transform of f $\phi_{\mathbf{x}}^{(m)}$. Taking the real part one obtains, cf. [28, Thm. 6.9 (a)],

$$\Phi_{\mathbf{x}}^{(m)}(\sigma_0) = \sum_{k=0}^{m-2}(-1)^k\binom{m-2}{k}\frac{\mathbf{B}_{m-k-1}(x_1)}{m-k-1}\,\frac{\mathbf{B}_{k+1}(x_2)}{k+1}\,T_1^k\,T_2^{m-k-2} \qquad (4.21)$$

In view of the Bruhat decomposition $\mathrm{GL}^+(2,\mathbb{Q}) = B(\mathbb{Q})\cup B(\mathbb{Q})\sigma_0 B(\mathbb{Q})$, the expressions (4.20) and (4.21), together with the cocycle relation (4.17), uniquely determine $\Phi_{\mathbf{x}}^{(m)}$ and allow its explicit calculation (comp. [27, Ch. 2], [28, §5], where the weight 2 case is treated in detail). This calculation has in fact been done by Nakamura [20], for a cocycle defined in a similar manner but using an alternate construction of the Eichler-Shimura period integrals. Since both (4.20) and (4.21) agree with Nakamura's formula [20, (2.8)] for the restriction of $\Phi_{\mathbf{x}}^{(m)}$ to $\mathrm{SL}(2,\mathbb{Z})$, one can conclude that $\Phi_{\mathbf{x}}^{(m)}(\sigma)$ is given by the cited formula for any $\sigma\in\mathrm{SL}(2,\mathbb{Z})$. □

A similar construction can be performed starting with any holomorphic function $F : \mathbb{H} \to W$ that satisfies the covariance law (3.3),

$$F|g\,(z) \equiv \det(g)^{-\frac{m-2}{2}}(cz+d)^{m-2}\,F(gz) = g \cdot F(z)\,, \qquad g \in GL^+(2,\mathbb{R}).$$

Of particular interest, in view of its connection to special values at non-positive integers of partial zeta functions for real quadratic fields (see [28, §7]), is the following choice (cf. [28, (6.4) (b)]). For $m = 2n \in 2\mathbb{N}$, one takes

$$W_{n-1,n-1} \;=\; (W_{n-1} \otimes W_{n-1})^{\mathrm{sym}} \;\subset\; W_{n-1} \otimes W_{n-1}\,,$$

i.e. the subspace fixed by the involution $P_1 \otimes P_2 \mapsto P_2 \otimes P_1$; its elements may be identified with the homogeneous polynomials $P \in \mathbb{C}[T_1, T_2, T_3, T_4]$ of degree $2(n-1)$ such that

$$P(T_1, T_2, T_3, T_4) \;=\; P(T_3, T_4, T_1, T_2)\,.$$

$\mathrm{GL}^+(2,\mathbb{R})$ acts on $W_{n-1,n-1}$ by the tensor product of the natural representations on the two factors W_{n-1}, and the function $F = F_{n-1,n-1} : \mathbb{H} \to W_{n-1,n-1}$ is defined by the formula

$$F_{n-1,n-1}(z) \;=\; (zT_1 + T_2)^{n-1}\,(zT_3 + T_4)^{n-1}\,.$$

For this specific choice, taking into account [28, Theorem 6.9], the statement of Theorem 12 becomes modified as follows.

Theorem 13. *Let $m = 2n \geq 2$ and let $\mathbf{x} \in \mathbb{Q}^2/\mathbb{Z}^2$, with $\mathbf{x} \neq 0$ if $m = 2$.*
1^0. *For $\beta = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B(\mathbb{Q})$,*

$$\Phi_{\mathbf{x}}^{(m)}(\beta) \;=\; -\frac{\mathbf{B}_m(x_1)}{m} \int_0^{\frac{b}{d}} (tT_1 + T_2)^{n-1}\,(tT_3 + T_4)^{n-1}\,dt\,.$$

2^0. *For* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, *with* $c > 0$,

$$\Phi_{\mathbf{x}}^{(m)}(\sigma) = \begin{cases} - & \dfrac{\mathbf{B}_m(x_1)}{m} \displaystyle\int_0^{\frac{a}{c}} (tT_1 + T_2)^{n-1} (tT_3 + T_4)^{n-1}\, dt \\ - & \dfrac{\mathbf{B}_m(ax_1 + cx_2)}{m} \displaystyle\int_{-\frac{d}{c}}^{0} (t(aT_1 + cT_2) + bT_1 + dT_2)^{n-1} \\ & \qquad\qquad \cdot (t(aT_3 + cT_4) + bT_3 + dT_4)^{n-1}\ dt \\ + & \displaystyle\sum_{k=0}^{n-1}\sum_{\ell=0}^{n-1} (-1)^{k+\ell} \binom{n-1}{k}\binom{n-1}{\ell} S_{\mathbf{x}}^{(m-k-\ell-1,k+\ell+1)}(\frac{a}{c}) \\ & \qquad \cdot T_1^k\, (aT_1 + cT_2)^{n-k-1}\, T_3^{\ell}\, (aT_3 + cT_4)^{n-\ell-1}. \end{cases}$$

Remark 14. Reformulating a result of Siegel [25, 26], Stevens has shown [28, §7] that the Eisenstein cocycle Φ of Theorem 13 can be used to calculate the values at nonpositive integers of partial zeta functions over a real quadratic field. This is achieved by specializing Φ to a certain Eisenstein series E, then evaluating it at the element $\sigma \in \mathrm{SL}(2,\mathbb{Z})$ that represents the action of a certain unit, and finally computing the polynomial $\Phi_E(\sigma)$ on the basis elements and their conjugates. It is intriguing to observe that the above transgressive construction confers Φ the secondary status reminiscent of the Borel regulator invariants that enter in the expression of the special values at non-critical points of L-functions associated to number fields ([16, 29, 30]).

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