

# Lessons from Quantum Field Theory

Hopf Algebras and Spacetime Geometries

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*We dedicate this paper to Moshé Flato.*

## Abstract

We discuss the prominence of Hopf algebras in recent progress in Quantum Field Theory. In particular, we will consider the Hopf algebra of renormalization, whose antipode turned out to be the key to a conceptual understanding of the subtraction procedure. We shall then describe several occurrences of this, or closely related Hopf algebras, in other mathematical domains, such as foliations, Runge-Kutta methods, iterated integrals and multiple zeta values. We emphasize the unifying role which the Butcher group, discovered in the study of numerical integration of ordinary differential equations, plays in QFT.

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## 1 Introduction

The *Leitmotiv* of this survey paper is our belief that in some way the true geometry of spacetime is actually dictated by quantum field theories as currently used by particle physicists in the calculation of radiative corrections.

There are two major ingredients in this use of the theory, the first is the renormalization technique, with all its combinatorial intricacies, which is perfectly justified by its concrete physical roots and the resulting predictive power.

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The second is the specific Lagrangian of the theory, the result of a long dialogue between theory and experiment, which, of course, is essential in producing meaningful physical results.

This Lagrangian is a unique mixture of several pieces, some very geometrical and governed by the external symmetry group of the equivalence principle, i.e. the diffeomorphism group **Diff**, the others governed by the internal group of gauge transformations **Gauge**. The overall symmetry group is the semi-direct product  $\mathbf{G} = \mathbf{Gauge} \rtimes \mathbf{Diff}$ , which is summarized by the following exact sequence of groups

$$1 \rightarrow \mathbf{Gauge} \rightarrow \mathbf{G} \rightarrow \mathbf{Diff} \rightarrow 1.$$

But there is more group structure than the external and internal symmetries of this Lagrangian of gravity coupled with matter. Our goal in this paper is to explain how the understanding of the group-theoretic principle underlying the working machine of renormalization [18] should allow one to improve the understanding of the geometrical nature of particle physics which was proposed in [10].

The main point in the latter proposal is that the natural symmetry group  $\mathbf{G}$  of the Lagrangian is isomorphic to the group of diffeomorphisms  $\mathbf{Diff}(X)$  of a space  $X$ , provided one stretches one's geometrical notions to allow slightly noncommutative spaces. Indeed, the automorphism group  $\mathbf{Aut}(\mathcal{A})$  of a noncommutative algebra  $\mathcal{A}$  which replaces the diffeomorphism group of any ordinary manifold has exactly this very feature of being composed of two pieces, one internal and one external, which, again, can be given equivalently as an exact sequence of groups

$$1 \rightarrow \mathbf{Int}(\mathcal{A}) \rightarrow \mathbf{Aut}(\mathcal{A}) \rightarrow \mathbf{Out}(\mathcal{A}) \rightarrow 1.$$

In the general framework of NCG the confluence of the two notions of metric and fundamental class for a manifold led very naturally to the equality

$$ds = 1/D, \tag{1}$$

which expresses the line element  $ds$  as the inverse of the Dirac operator  $D$ , hence under suitable boundary conditions as a propagator. The significance of  $D$  is two-fold. On the one hand, it defines the metric by the above equation, on the other hand its homotopy class represents the K-homology fundamental class of the space under consideration. While this new geometrical framework was immediately useful in various mathematical examples of noncommutative spaces, including the noncommutative torus [12, 11], it is obvious that it cannot be a satisfactory answer for spacetime precisely because QFT will unavoidably dress the free propagator, as Fig.(1) indicates. It nevertheless emphasizes that spacetime itself ought to be regarded as a derived concept, whose structure in we believe follows from the properties of QFT if one succeeds describing the latter in purely combinatorial terms.

Whereas it was simple in the undressed case to recover the standard ingredients of intrinsic geometry directly from the Dirac propagator and the algebra  $\mathcal{A}$ ,

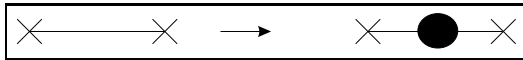


Figure 1: The dressed line element is obtained from the transition of the bare to the dressed propagator.

it is much more challenging in the dressed case, and we shall naturally propose the Schwinger-Dyson equation for the full fermion propagator as the proper starting point in the general case.

Let us now describe various appearances of Hopf algebras relevant to Quantum Field Theory. We will start with the Hopf algebra of renormalization, whose antipode turned out to be the key to a conceptual understanding of the subtraction procedure. We shall then describe several occurrences of this, or closely related Hopf algebras, in other mathematical domains, such as foliations [14], Runge-Kutta methods [6], iterated integrals [9, 20] and multiple zeta values [17].

We will finally address the Schwinger Dyson equation as a prototype of a generalized form of an ordinary differential equation (ODE).

## 2 The pertinence of Hopf algebras

### 2.1 Renormalization and the antipode in the algebra of rooted trees

Renormalization occurs in evaluating physical observable quantities which in simple terms can be written as formal functional integrals of the form

$$\int e^{-\mathcal{L}(\phi, \partial\phi)} P(\phi, \partial\phi)[d\phi], \quad \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I.$$

Computing such an integral in perturbative terms leads to a formal power series, each term  $G$  of which is obtained by integrating a polynomial under a Gaussian  $e^{-\mathcal{L}_0}$ . Such Feynman diagrams  $G$  are given by multiple integrals over spacetime or, upon Fourier transformation, over momentum space, and are typically divergent in either case. The renormalization technique consists in adding counterterms  $\mathcal{L}_G$  to the original Lagrangian  $\mathcal{L}$ , one for each diagram  $G$ , whose role is to cancel the undesired divergences. In good situations to which we are allowed to restrict ourselves if we take guidance from nature, this can be done by local counterterms, polynomials in fields and their derivatives.

The main calculational complication of this subtraction procedure occurs for diagrams which possess non-trivial subdivergences, i.e. subdiagrams which are already divergent. In that situation the procedure becomes very involved since it is no longer a simple subtraction, and this for two obvious reasons: i) the divergences are no longer given by local terms, and ii) the previous corrections (those for the subdivergences) have to be taken into account.

To have an example for the combinatorial burden imposed by these difficulties consider the problem below of the renormalization of a two-loop four-point function in massless scalar  $\phi^4$  theory in four dimensions, given by the following Feynman graph:

$$\Gamma^{[2]} = \text{Diagram of a two-loop four-point function with a divergent subgraph. It consists of a circle with two external lines on the left and two on the right. A vertical line connects the top of the circle to a vertex, which then branches into two lines that cross each other and rejoin the circle at the top.$$

It contains a divergent subgraph

$$\Gamma^{[1]} = \text{Diagram of a one-loop four-point function with a divergent subgraph. It consists of a circle with two external lines on the left and two on the right. Two lines cross each other inside the circle, forming a figure-eight shape.$$

We work in the Euclidean framework and introduce a cut-off  $\lambda$  which we assume to be always much bigger than the square of any external momentum  $p_i$ . Analytic expressions for these Feynman graphs are obtained by utilizing a map  $\Gamma_\lambda$  which assigns integrals to them according to the Feynman rules and employs the cut-off  $\lambda$  to the momentum integrations. Then  $\Gamma_\lambda[\Gamma^{[1,2]}]$  are given by

$$\Gamma_\lambda[\Gamma^{[1]}](p_i) = \int d^4k \frac{\Theta(\lambda^2 - k^2)}{k^2} \frac{1}{(k + p_1 + p_2)^2},$$

and

$$\Gamma_\lambda[\Gamma^{[2]}](p_i) = \int d^4l \frac{\Theta(\lambda^2 - l^2)}{l^2(l + p_1 + p_2)^2} \Gamma_\lambda[\Gamma^{[1]}](p_1, l, p_2, l).$$

It is easy to see that  $\Gamma_\lambda[\Gamma^{[1]}]$  decomposes into the form  $b \log \lambda$  (where  $b$  is a real number) plus terms which remain finite for  $\lambda \rightarrow \infty$ , and hence will produce a divergence which is a non-local function of external momenta

$$\sim \log \lambda \int d^4l \frac{\Theta(\lambda^2 - l^2)}{l^2(l + p_1 + p_2)^2} \sim \log \lambda \log(p_1 + p_2)^2.$$

Fortunately, the counterterm  $\mathcal{L}_{\Gamma^{[1]}} \sim \log \lambda$  generated to subtract the divergence in  $\Gamma_\lambda[\Gamma^{[1]}]$  will precisely cancel this non-local divergence in  $\Gamma^{[2]}$ .

That these two diseases actually cure each other in general is a very non-trivial fact that took decades to prove [1]. For a mathematician the intricacies of the proof and the lack of any obvious mathematical structure underlying it make it totally inaccessible, in spite of the existence of a satisfactory formal approach to the problem [15].

The situation is drastically changed by the understanding of the mechanism behind the actual computations of radiative corrections as the antipode in a very specific Hopf algebra, the Hopf algebra  $\mathcal{H}_R$  of decorated rooted trees discovered in [18, 13, 19].

This algebra is the algebra of coordinates in an affine nilpotent group  $G_R$ , and all the non-linear aspects of the subtraction procedure are subsumed by the action of the antipode, i.e. the operation  $g \rightarrow g^{-1}$  in the group  $G_R$ .

This is not only very satisfactory from the conceptual point of view, but it does also allow the automation of the subtraction procedure to arbitrary loop

order (see [5] for an example of Feynman graphs with up to twelve loops) and to describe the more elaborate notions of renormalization theory, -change of scales, renormalization group flow and operator product expansions-, from this group structure [5, 20, 21].

In [13] we characterized this Hopf algebra of renormalization,  $\mathcal{H}_R$ , abstractly as the universal solution for one-dimensional Hochschild cohomology. The Hopf algebra  $\mathcal{H}_R$  together with the collecting map (cf.[13])  $B_+ : \mathcal{H}_R \rightarrow \mathcal{H}_R$  is the universal solution of the following Hochschild cohomology equation. We let  $\mathcal{H}$  be a commutative (not necessarily cocommutative) Hopf algebra together with a linear map  $L : \mathcal{H} \rightarrow \mathcal{H}$ . The Hochschild equation is then  $bL = 0$ , i.e.

$$\Delta L(a) = L(a) \otimes 1 + (id \otimes L)\Delta(a).$$

We shall see later the intimate relation between this problem and the generalized form of ODE's.

Renormalization can now be summarized succinctly by saying that it maps an element  $g \in G_R$  to another element  $g_o^{-1}g \in G_R$ . Typically,  $g$  is associated with bare diagrams and  $g_o$  is a group element in accordance with a chosen renormalization scheme. In the ratio  $g_o^{-1}g$  the undesired divergences drop out. Hence,  $g_o^{-1}$  delivers the counterterm and  $g_o^{-1}g$  the renormalized Green function. It is precisely this group law which allowed the description of the change of scales and renormalization schemes in a comprehensive manner [20]. Indeed, if  $g_o^{-1}g$  is one renormalized Green function, and  $g_o^{-1}g' = (g_o^{-1}g)(g_o^{-1}g')$  another, then we immediately see the group law underlying the evolution of the renormalization group.

## 2.2 Foliations

Quite independently, the problem of computation of certain characteristic classes coming from differential operators on foliations led [14] to a very specific Hopf algebra  $\mathcal{H}(n)$  associated to a given integer  $n$ , which is the codimension of the foliation.

The construction of this Hopf algebra was actually dictated by the commutation relations of the algebra  $\mathcal{A}$  (corresponding to the noncommutative leaf space) with the operator  $D$  (cf.(1)) describing the transverse geometry in the sense of (1). The structure of this Hopf algebra revealed that it could be obtained by a general procedure from the pair of subgroups  $G_0, G_1$  of the group  $\mathbf{Diff}(\mathbf{R}^n)$  of diffeomorphisms of  $\mathbf{R}^n$ : One lets  $G_0$  be the subgroup of affine diffeomorphisms, while  $G_1$  is the subgroup of those diffeomorphisms which fix the origin and are tangent to the identity map at this point. This specific structure of a Hopf algebra not only led in this case to a solution of the computational problem, but also provided the proper generalization of Lie algebra cohomology to the general Hopf framework in the guise of cyclic cohomology (see the paper *Cyclic cohomology and Hopf algebras* by Connes and Moscovici in this volume).

In [13] we established a relation between the two Hopf algebras which led us to extend various results of [14] to the Lie algebra  $\mathcal{L}^1$  of rooted trees, in

particular we extended  $\mathcal{L}^1$  by an additional generator  $Z_{-1}$ , intimately related to natural growth of rooted trees.

Let us mention at this point that the extended Lie algebra  $\mathcal{L}$  has a radical as well as a simple quotient. The former is nilpotent and infinitely generated, the latter turns out to be isomorphic to the Lie algebra of  $\mathbf{Diff}(\mathbf{R})$ , thus exhibiting an intrinsic relation between the two groups.

Extension of this relation between  $\mathcal{H}(1)$  and  $\mathcal{H}_R$  leads to the Hopf algebra of decorated planar rooted trees, explored by R. Wulkenhaar [22], whose new feature is that the decoration of the root provides the information necessary to define the notion of parallel transport, while the decorations of other vertices are mere spacetime indices.

### 2.3 Numerical Integration Methods

Our description in [13] of the simplest realization of the Hopf algebra  $\mathcal{H}_R$  and its related group, together with the elaboration of this group structure in [5, 20], allowed C. Brouder [6] to recognize a far reaching relation with the combinatorics of numerical analysis as worked out by Butcher several decades ago.

Indeed, Butcher, in his seminal work on the algebraic aspects of Runge-Kutta methods (and other numerical integration methods) [7] discovered the very same Hopf algebra and group  $G_R$ .

Let us first remind the reader of the simplest numerical integration methods in the integration of a given ordinary differential equation

$$\dot{y} = f(y), \quad y(0) = y_0, \quad (2)$$

where  $t \rightarrow y(t)$  is a curve in Euclidean space  $E$ . The Runge method, designed to approximate the value  $y(h)$  of  $y$  at  $t = h$  is given by the formula,

$$\hat{y}(h) = y_0 + hf(y_0 + \frac{h}{2}f(y_0)). \quad (3)$$

The virtue of this simple iteration of  $f$  is that the Taylor expansion of  $\hat{y}$  for small  $h$  agrees up to order two with the Taylor expansion of the actual solution  $y(h)$ .

More generally a Runge-Kutta method is an iterative procedure, governed by two sets of data, traditionally denoted by  $a$  and  $b$ , where  $b_i$ ,  $i = 1, \dots, n$ , and  $a_{ij}$  are scalars while the implicit equations are

$$X_i(h) = y_0 + h \sum a_{ij} f(X_j(h)), \quad (4)$$

$$\hat{y} = y_0 + h \sum b_i f(X_i(s)). \quad (5)$$

Their solution is known to be a sum over rooted trees, involving the very combinatorial quantities ascribed to rooted trees by the study of QFT, namely tree factorials, CM weights and symmetry factors [7, 6, 5, 20]. It has been known since Cayley [8] that rooted trees are the correct way to label polynomials in

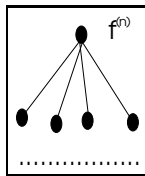


Figure 2: The tree structure of derivatives.

higher derivatives of a smooth map  $f : E \rightarrow E$ . For instance an  $n$ th derivative  $f^{(n)}$  corresponds to a vertex with  $n$  adjacent branches, as in Fig.(2).

What Butcher discovered is that Runge-Kutta methods naturally form a group, whose elements are actually scalar functions of rooted trees. He gave explicit formulas for the composition of two methods as well as for the inverse method. He also showed how the standard solution of a differential equation is obtained from a particular (continuous) method which he called the Picard method. There is a nuance between Runge-Kutta methods and the Butcher group of scalar functions of rooted trees, but Butcher proved that the Runge-Kutta methods are dense in the latter group.

In [20] scalar functions of rooted trees were used to parametrize and generalize iterated integrals, allowing a unified description of renormalization schemes. This immediately allowed C. Brouder to identify the above group  $G_R$  (in the simplest undecorated case) with the Butcher group.

The supplementary freedom in constructing group elements or operations on them provided by the Runge-Kutta description matches the freedom to choose a renormalization scheme or to describe the change of a chosen scheme. Hence the group product, as well as the counterterm, -the inverse group element, have immediate and equally elegant counterparts in the Runge-Kutta language worked out recently by Brouder [6].

Moreover, comparing the data  $a, b$  of a Runge-Kutta method with our characterization of the Hopf algebra  $\mathcal{H}_R$  by the Hochschild cohomology problem leads us to the following natural framework for the formulation of a universal differential equation (which also covers the case of the Schwinger-Dyson equation) given a non-linear map  $f : E \rightarrow E$ . The only nuance between our framework and the Butcher framework of an algebra  $B$  together with a linear map  $a : B \rightarrow B$  (cf.[7]) is that we now assume that the abelian algebra  $B$  is in fact a Hopf algebra while the map  $a$  satisfies our Hochschild equation

$$\Delta a(P) = a(P) \otimes 1 + (id \otimes a)\Delta(P).$$

The simplest example of such data is already given by the Picard method of [7] where the Hopf algebra structure (not provided in [7]) is given by

$$\begin{aligned} \Delta x &= x \otimes 1 + 1 \otimes x, \\ a(P)(x) &= \int_0^x P(u) du, \end{aligned}$$

cf.[13]. The difference between such data and a Runge-Kutta method is that we now have translation invariance available. Exactly as the Runge-Kutta method was producing an element in the Butcher group, the above data give a homomorphism from the underlying group (by the Milnor-Moore theorem) to the Butcher group, a situation which is dual to our theorem on the universality of  $\mathcal{H}_R$ .

We can now reformulate the ODE in general as an equation for  $y \in B \otimes E$ ,

$$y = 1 \otimes \eta_0 + (L \otimes id)f(y),$$

where  $\eta_0 \in E$  is the initial datum and  $f(y)$  is uniquely defined by  $f(y)(x) = f(y(x))$  for any  $x \in spec(B)$ .

This is very suggestive: somehow the usual solution curve for an ODE should not be considered as an ultimate solution and the universal problem should be thought of as a refinement of the idea of a scalar time parameter. We regard Butcher's work on the classification of numerical integration methods as an impressive example that concrete problem-oriented work can lead to far reaching conceptual results.

## 2.4 Iterated integrals and numbers from primitive diagrams

The circle of ideas described so far certainly allows us to come to terms with the combinatorics of the subtraction procedure so that we can now concentrate only on those diagrams without subdivergences, i.e. the decorations at vertices of rooted trees.

The proper definition for the integral in the new calculus used in Noncommutative Geometry is the Dixmier trace, i.e. the invariant coefficient of the logarithmic divergence of an operator trace. A superficial understanding of QFT would lead one to consider it as far too limited a tool to confront the divergences of QFT. This misgiving is based on the ignorance of the plain fact that the divergence of a subdivergence-free diagram is a mere overall logarithmic divergence, and that such diagrams, for which Fig.(3) gives archetypical examples, appear at all loop orders (the expert will notice that the decorations which are to be used are actually Feynman graphs with appropriate polynomial insertions which are the remainders when we shrink subgraphs to points).

In dimensional regularization the decorations appearing at the rooted trees deliver a first-order pole in  $D-4$ . In other words, the Dixmier trace, or rather its residue guise  $Res_-$ , is perfectly sufficient to disentangle and determine a general Feynman graph  $G$ , making full use of the decomposition dictated by the Hopf algebra structure, which gives the joblist [5] for the practitioner of QFT: the list of diagrams which correspond to analytic expressions suffering from merely an overall divergence.

Analyzing the corresponding residues of the decorations and their number-theoretic properties is a far reaching subject (discussed in part by M. Kontsevich in this issue) which by itself reveals some interesting Hopf algebra structures, based on the algebraic properties of iterated integrals.



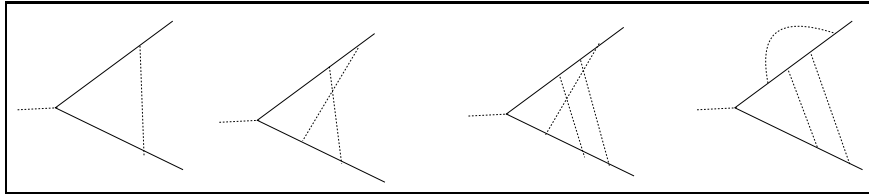


Figure 3: Diagrams free of subdivergences and hence delivering a first order pole in dimensional regularization. They are typically reflecting the skeleton expansion of the theory.

Iterated integrals came to prominence with the work of K.T. Chen [9]. Essentially, they are governed by two rules, Chen's lemma

$$F_{r,t}^I = F_{r,s}^I + F_{s,t}^I + \sum_{I=(I',I'')} F_{r,s}^{I'} F_{s,t}^{I''}$$

and the shuffle product

$$F_{r,s}^{I'} F_{r,s}^{I''} = \sum_{\sigma} F_{r,s}^{\sigma[I]},$$

where the sum is over all  $(p, q)$  shuffles of the symmetric group  $S_{p+q}$  acting on a string  $I$  of, say,  $n = p + q$  integers parametrizing the one-forms which give the iterated integral as their integral over the  $n$ -dimensional standard simplex.

If we define  $t^I$  to be a tree without sidebranches decorated with these one-forms in the appropriate order, then each iterated integral corresponds to a map from such a decorated tree to a real number, so that Chen's lemma expresses the familiar group law of the Butcher group in this special case. It is then the shuffle product which guarantees that iterating one-forms in accordance with a general rooted tree [20] will not produce anything new: any such integral is a linear combination of the standard iterated integrals.

Most interestingly, the calculus of full perturbative QFT can be understood as a calculus of generalized iterated integrals, where the boundaries  $r, s$  are to be replaced by elements of the Butcher group: scalar functions of (decorated) rooted trees [20]. The group law still holds naturally for such generalized integrals, but the shuffle product only holds for the leading term in the asymptotic expansion of bare diagrams [21].

We hope that an investigation of this situation has far reaching consequences for the understanding of the number-theoretic content of Feynman diagrams, whose richness is underwritten by a wealth of empirical data which provide a plethora of interesting numbers like multiple zeta values [23], their alternating cousins [4], and even sums involving non-trivial roots of unity in numerators [3].

It is gratifying then that the regularization of iterated integrals based on forms  $dx/x, dx/(1-x)$ , providing the iterated integrals representation of multiple zeta values, is in full accord with the renormalization picture developed here. The appearance of such numbers in the solution of the Knizhnik-Zamolodchikov equation, and the relation to the Drinfel'd associator upon renormalizing this

solution so that it extends to the unit interval, clearly motivates one to investigate the differential equations which encapsulate the iterative structure of Green functions in perturbative QFT, the bare and renormalized Schwinger-Dyson equations.

Before we close this survey with a couple of remarks concerning these equations, let us mention yet another application where the freedom of having tree-indexed scales, hence elements of the Butcher group, will prove essential.

The formulation of operator product expansion clearly takes place in a configuration space of the same nature as the Fulton-MacPherson compactification [16], known to be stratified by rooted trees. However, there is a very important nuance which can already be fully appreciated in the case of two points. In that case, the Fulton-MacPherson space is simply the blow-up of the diagonal in the space  $X \times X$ , whereas the geometric data which encodes most of the semiclassical deformation aspect in a diffeomorphism invariant manner are provided by a smooth groupoid, called the tangent groupoid in [11].

Essentially, the relation between the latter and the former is the same as the relation between a linear space and the corresponding projective space. That the transition to a linear space is a crucial improvement can be appreciated from the fact that it is only in the linear space that Fourier transform takes its full power.

It is thus of great interest to extend the construction of the tangent groupoid from the two-point case to the full set-up of the configurations of  $n$  points. This clearly involves the freedom of having scales available for any strata in the compactification, hence again elements of the Butcher group. This is in full accord with the use of tree-indexed parameters in the momentum space description of the operator product expansion undertaken in [20], which clearly shows the necessity of allowing for the full linear space to be able to describe the variety of renormalization prescriptions employed by the practitioners of QFT.

## 2.5 The Schwinger-Dyson Equations

Recalling that equation (1) demands an understanding of the full dressed line element (cf. Fig.(1)), we finally consider Schwinger-Dyson equations. The propagator, the vertex and the kernel functions provide a system of such Schwinger-Dyson equations, whose solution is at the heart of any Lagrangian QFT [2]. Though the fermion propagator, hence the line element, comes to prominence in these equations due to gauge covariance, a full solution is not yet known for any QFT of interest.

In the simplest form, the Schwinger-Dyson equation is typically an equation of the form

$$\Gamma(q) = \gamma + \int d^4k \Gamma(k) P(k)^2 K(k, q),$$

which is the structure of the Schwinger-Dyson equation for the QED vertex at zero momentum transfer. Here,  $P$  is a propagator, hence a geometric series in a self-energy (so that the resulting equations are highly non-linear) and  $K$  is a

QED four-point kernel function, actually one which is a typical generator (in its undressed form) of decorations upon closure of two of its legs.

Nevertheless it is not difficult to identify an operator  $L_K$  such that the solution is the formal series

$$\Gamma(q) = \gamma + \sum_i L_K^i(\gamma),$$

where the operator  $L_K$  amounts to the operator  $B_+$  combined with all decorations provided by the kernel  $K$ . If the above Schwinger-Dyson equation is for the bare vertex, the one for the renormalized vertex  $\Gamma$  is obtained by multiplying it with the counterterm  $Z$  which amounts, in full accordance with the  $g_o^{-1}g$  notation used before, to the equation

$$\Gamma(q) = Z\gamma + \int d^4k \Gamma(k) P(k)^2 K(k, q),$$

which neatly summarizes the Hopf algebra structure of perturbative QFT. From the above structure, we conclude that Runge-Kutta methods are fully available for this system.

Let us close this paper by noting an amusing coincidence: If we restrict the kernel  $K$  to the first two terms  $K^{[1]}, K^{[2]}$  in its skeleton expansion, restrict  $P$  to the free propagators and label each of the two terms  $K^{[1]}, K^{[2]}$  by noncommuting variables  $k_1$  and  $k_2$  say, then we are naturally led to the equation

$$\Gamma(q) = \gamma + k_1 \left[ \int d^4k \frac{1}{k^4} \Gamma(k) K^{[1]} \right] + k_2 \left[ \int d^4k \frac{1}{k^4} \Gamma(k) K^{[2]} \right],$$

which obviously involves a sum over all words in  $k_1, k_2$  in its solution and is fascinatingly close to the K-Z equation in two variables considered before.

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