TYPE III AND SPECTRAL TRIPLES

ALAIN CONNES AND HENRI MOSCOVICI

Abstract. We explain how a simple twisting of the notion of spectral triple allows to incorporate type III examples, such as those arising from the transverse geometry of codimension one foliations. We show that the classical cyclic cohomology valued Chern character of finitely summable spectral triples extends to the twisted case and lands in ordinary (untwisted) cyclic cohomology. The index pairing with ordinary (untwisted) K-theory continues to make sense and the index formula is given by the pairing of the corresponding Chern characters. This opens the road to extending the local index formula to the type III case.

1. Introduction

The basic paradigm of noncommutative geometry is that of spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ (*cf.* [5], [6]), where the algebra $\mathcal{A}$ encodes the space and the operator D encodes the metric. In the finite dimensional situation, *i.e.* when there is an $\alpha > 0$ such that the n-th characteristic value of the resolvent of D decays as $n^{-\alpha}$ for $n \to \infty$, the Dixmier trace [5] induces a nontrivial trace on the algebra $\mathcal{A}$. The existence of a trace is a characteristic of the type II situation in the Murray-von Neumann classification of rings of operators. Thus, in essence the theory is, in its finite dimensional form, restricted to the type II case.

We shall explain in this short note how a simple twisting of the notion of spectral triple allows to incorporate type III examples, such as those arising from the transverse geometry of codimension one foliations. Since the twisting of the commutators turns the functional induced by the Dixmier trace on the algebra $\mathcal{A}$ into a twisted trace, one would naturally expect that some kind of twisting should also occur at the homological level, and in particular involve the twisting of cyclic cohomology introduced by the authors in the context of Hopf cyclic cohomology [7]. The main point of this note, besides giving simple natural examples of the general notion and developing the elementary first steps of the theory, is to show that no twisting is in fact needed: the Chern character of finitely summable spectral triples extends to the twisted case and lands in ordinary (untwisted) cyclic cohomology. This opens the road to extending the local index formula, the hypoelliptic construction on the dual system, and the Thom isomorphism to twisted spectral triples of type III.

2. Two examples

2.1. **Dirac operator.** We start by recalling the classical comparison formula for the Dirac operators associated to conformally equivalent metrics, *cf.* [9], [2]. Given a compact spin manifold M^n, to each Riemannian metric g on M one can canonically associate a Dirac operator $\partial\!\!\!/ = \partial\!\!\!/^g$ acting on the Hilbert space $\mathfrak{H} = \mathfrak{H}^g := L^2(M, S^g)$ of L^2-sections of the spin bundle $S = S^g$, and thus a corresponding spectral triple (cf. [5]) $(\mathcal{A}, \mathfrak{H}, \partial\!\!\!/)$ over the algebra $\mathcal{A} := C^\infty(M)$. Let $h \in C^\infty(M)$ be a self-adjoint element and replace g by the rescaled metric $g' = e^{-4h}\, g$. After identifying the corresponding spin bundles via the $Spin_n$-equivariant

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transformation $\beta^g_{g'}$ from g-spinorial frames to g'-spinorial frames defined in [2], the gauge transformed operator ${}^g\partial\!\!\!/^{g'} := \beta^{g'}_g \circ \partial\!\!\!/^{g'} \circ \beta^g_{g'}$ has the expression, *cf.* [2, (26)],

$$ {}^g\partial\!\!\!/^{g'} = e^{(n+1)h} \circ \partial\!\!\!/^g \circ e^{-(n-1)h} . $$

In order to account for the change of the Riemannian volume form, $\mathrm{vol}_{g'} = e^{-2nh}\,\mathrm{vol}_g$, at the level of L^2-sections one needs to further rescale the canonical identification by setting

$$ \tilde{\beta}^g_{g'} := e^{nh} \circ \beta^g_{g'} = \beta^g_{g'} \circ e^{nh} : \mathfrak{H}^g \to \mathfrak{H}^{g'} . $$

This shows that the gauge transformed spectral triple is simply obtained by replacing $\partial\!\!\!/$ with

$$ \partial\!\!\!/' = e^h\, \partial\!\!\!/\, e^h . $$

2.2. **Perturbed spectral triple.** In the general case, when one starts from an arbitrary spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ (*cf.* [5], [6]) and a self-adjoint element $h = h^* \in \mathcal{A}$, it is natural to wonder what are the properties of the 'perturbed' triple

$$ (\mathcal{A}, \mathfrak{H}, D') , \quad D' = e^h\, D\, e^h. \tag{2.1} $$

The operator D' is still self-adjoint but the basic boundedness condition

$$ [D, a] \quad \text{is bounded} \quad \forall\, a \in \mathcal{A} , \tag{2.2} $$

will not necessarily hold, unless h is in the center of $\mathcal{A}$.
However, the following simple fact holds true.

Lemma 2.1. *For any self-adjoint element $h \in \mathcal{A}$, letting*

$$ \sigma(a) = e^{2h}\, a\, e^{-2h}, \quad a \in \mathcal{A}, \tag{2.3} $$

one has

$$ d'_\sigma a := D'\, a - \sigma(a)\, D' \quad \textit{is bounded} \quad \forall\, a \in \mathcal{A} . \tag{2.4} $$

Proof. The fact that the usual commutators $[D, b]$, $b \in \mathcal{A}$, are bounded implies the boundedness of

$$ d'_\sigma a = e^h\, D\, e^h\, a - e^{2h}\, a\, e^{-h}\, D\, e^h = e^h\, [D, b]\, e^h , \qquad b = e^h\, a\, e^{-h}. \tag{2.5} $$

□

2.3. **Transverse spectral triple.** As a next example we take a codimension 1 foliation and consider the corresponding noncommutative algebra $\mathcal{A}$ of 'transverse coordinates'. In fact, as a further simplification we restrict to a complete transversal and take for $\mathcal{A}$ the algebraic crossed product of the algebra $C^\infty(S^1)$ of smooth functions on S^1 by a group Γ of orientation preserving diffeomorphisms. Any element of $\mathcal{A} = C^\infty(S^1) \rtimes \Gamma$ is represented as a finite sum of the form

$$ a = \sum_\Gamma a_\phi\, U^*_\phi , $$

the product rule is determined by

$$ U^*_\phi\, f = (f \circ \phi)\, U^*_\phi , \quad U^*_\phi\, U^*_\psi = U^*_{\psi\,\phi} , \tag{2.6} $$

and the involution

$$ a = \sum_\Gamma a_\phi\, U^*_\phi \quad \longmapsto \quad a^* = \sum_\Gamma U_\phi\, \bar{a}_\phi . \tag{2.7} $$

One represents $\mathcal{A} = C^\infty(S^1) \rtimes \Gamma$ in the Hilbert space $\mathfrak{H} = L^2(S^1)$ by the $*$-representation

$$(\pi(g\, U_\phi^*)\, \xi)(x) = g(x)\, \phi'(x)^{\frac{1}{2}}\, \xi(\phi(x))\,, \quad \forall \xi \in \mathfrak{H}\,, \quad x \in \mathbb{R}/\mathbb{Z}. \tag{2.8}$$

In the role of D we take the operator $\partial\!\!\!/ = \frac{1}{i}\frac{d}{dx}$, while the automorphism $\sigma \in \mathrm{Aut}\mathcal{A}$ is defined on the monomials generating $\mathcal{A}$ by

$$\sigma(g\, U_\phi^*) = \frac{d\phi(x)}{dx}\, g\, U_\phi^*\,. \tag{2.9}$$

One then has the following boundedness property.

Lemma 2.2. *For any $a \in \mathcal{A}$, the twisted commutators*

$$\partial\!\!\!/ \circ \pi(a) \quad - \quad \pi(\sigma(a)) \circ \partial\!\!\!/\,, \tag{2.10}$$

$$\text{and} \qquad |\partial\!\!\!/| \circ \pi(a) \quad - \quad \pi(\sigma(a)) \circ |\partial\!\!\!/| \tag{2.11}$$

are bounded.

Proof. For any $a = g\, U_\phi^*$ one has:

$$\begin{aligned} \partial\!\!\!/(\pi(a)\xi)(x) &= \tfrac{1}{i}\tfrac{d}{dx}\left(g(x)\phi'(x)^{\frac{1}{2}}\right)\, \xi(\phi(x)) + g(x)\phi'(x)^{\frac{1}{2}}\, \partial\!\!\!/(\xi\circ\phi)(x) \\ &= \tfrac{1}{i}\tfrac{d}{dx}\left(g(x)\phi'(x)^{\frac{1}{2}}\right)\, \xi(\phi(x)) + g(x)\phi'(x)^{\frac{1}{2}}\, \left(\phi'(x)\, (\partial\!\!\!/(\xi))(\phi(x))\right)\,; \end{aligned}$$

equivalently,

$$\partial\!\!\!/((\pi(a))(\xi))(x) \,-\, \pi(\sigma(a))(\partial\!\!\!/(\xi))(x) \,=\, \frac{1}{i}\frac{d}{dx}\left(g(x)\phi'(x)^{\frac{1}{2}}\right)\, \phi'(x)^{-\frac{1}{2}}\,(\pi(U_\phi^*)\xi)(x), \tag{2.12}$$

which proves (2.10).

To prove (2.11) we shall switch from direct calculation to an equally elementary symbolic argument. Letting V_ϕ denote the translation operator by $\phi \in \Gamma$,

$$(V_\phi\, \xi)(x) \,=\, \xi(\phi^{-1}(x))\,, \quad \forall \xi \in \mathfrak{H}\,, \quad x \in \mathbb{R}/\mathbb{Z}\,,$$

one has for any $a = g\, U_\phi^* \in \mathcal{A}$,

$$\left(|\partial\!\!\!/| \circ \pi(a) \,-\, \pi(\sigma(a)) \circ |\partial\!\!\!/|\right) \circ V_\phi \,=\, |\partial\!\!\!/| \circ g\, \phi'^{\frac{1}{2}} \,-\, g\, \phi'^{\frac{1}{2}}\, \phi'\, V_\phi^{-1} \circ |\partial\!\!\!/| \circ V_\phi\,.$$

Now $V_\phi^{-1}\circ|\partial\!\!\!/|\circ V_\phi$ is a 1-st order (classsical) pseudodifferential operator whose principal symbol is $\dfrac{1}{\phi'}$ times the principal symbol of $|\partial\!\!\!/|$. It follows that the right hand is a pseudodifferential operator of order 0, hence bounded. □

Remark 2.3. The symbolic argument given above applies more generally to any pseudodifferential operator of arbitrary order $m \in \mathbb{R}$. Thus, if $P \in \Psi DO^m(S^1)$, then for all $a \in \mathcal{A}$

$$\left(P \circ \pi(a) \,-\, \pi(\sigma^m(a)) \circ P\right) \circ V_\phi \,\in\, \Psi DO^{m-1}(S^1)\,.$$

Remark 2.4. The canonical state φ on $\mathcal{A}$,

$$\varphi(f\, U_\phi^*) = 0 \;\; if \;\; \phi \neq 1\,, \quad \varphi(f) = \int f\, dx$$

is a σ^{-1}-*trace*, *i.e.* satisfies

$$\varphi(a\, b) \,=\, \varphi(b\, \sigma^{-1}(a))\,, \qquad \forall\, a, b \in \mathcal{A}\,, \tag{2.13}$$

and its modular automorphism group is precisely the one-parameter group of automorphisms

$$\sigma_t(g\,U_\phi^*) = \left(\frac{d\phi(x)}{dx}\right)^{it} g\,U_\phi^*\,, \qquad t\in\mathbb{R}\,, \tag{2.14}$$

whose value (after analytic continuation) at $t=-i$ coincides with σ.

3. σ-spectral triples and their basic properties

3.1. **Elementary properties.** The usual definition of a spectral triple extends to the context illustrated by the preceding examples as follows.

Definition 3.1. *With σ being an automorphism of $\mathcal{A}$, an* ungraded *σ-spectral triple $(\mathcal{A},\mathfrak{H},D)$ is given by an action of $\mathcal{A}$ in the Hilbert space $\mathfrak{H}$, while D is a self-adjoint operator with compact resolvent and such that*

$$D\,a \,-\, \sigma(a)\,D \quad \textit{is bounded} \quad \forall\, a\in\mathcal{A}\,. \tag{3.1}$$

A graded *σ-spectral triple is similarly defined, with the additional datum of a grading operator*

$$\gamma=\gamma^*\in\mathcal{L}(\mathfrak{H})\,,\quad \gamma^2=I$$

which commutes with the action of $\mathcal{A}$, while

$$D\,\gamma \,=\, -\,\gamma\,D\,. \tag{3.2}$$

A Lipschitz-regular *σ-spectral triple is one that satisfies the additional condition*

$$\forall\, a\in\mathcal{A}\,,\quad |D|\,a-\sigma(a)\,|D| \quad \textit{is bounded.} \tag{3.3}$$

In the case when $\mathcal{A}$ is an involutive algebra and the representation is involutive, to ensure the compatibility between the automorphism σ and the $*$-involution, we impose the additional *unitarity condition*:

$$\sigma(a^*) = \,(\sigma^{-1}(a))^*\,,\quad \forall\; a\in\mathcal{A}\,. \tag{3.4}$$

Lemma 2.2 shows that the transverse Dirac operator on a codimension 1 foliation gives rise to a Lipschitz-regular σ-spectral triple with σ given by the Jacobian of the holonomy. In particular, since this spectral triple is 1-summable, one gets examples of finitely summable σ-spectral triples for which the representation of $\mathcal{A}$ in $\mathfrak{H}$ generates a type III factor.

Let us spell out the extensions to the twisted case of some basic properties of spectral triples. For background on spectral triples, including notational conventions used below, we refer the reader to [5, IV.2] and [6, Appendix A], while for the notion of σ-trace see [7].
First of all, we note that any twisted spectral triple which is Lipschitz-regular can be canonically 'untwisted' by passage to its 'phase'. This is quite clear in the case of the second example (*cf.* §2.3), since the phase of $\partial\!\!\!/ = \frac{1}{i}\frac{d}{dx}$ is the Hilbert transform, and is actually easy to prove in full generality.

Proposition 3.2. *If the σ-spectral triple $(\mathcal{A},\mathfrak{H},D)$ is Lipschitz-regular and $F=D\,|D|^{-1}$, then $(\mathfrak{H},F)$ is a Fredholm module over $\mathcal{A}$. If moreover $(\mathcal{A},\mathfrak{H},D)$ is finitely summable, so is $(\mathfrak{H},F)$.*

Proof. Indeed, for any $a \in \mathcal{A}$,

$$D\,a \,-\, \sigma(a)\,D \,=\, |D|\,(F\,a \,-\, a\,F) \,+\, (|D|\,a - \sigma(a)\,|D|)\,F\,,$$

therefore

$$[F,a] \,=\, |D|^{-1}\left((D\,a \,-\, \sigma(a)\,D) \,-\, (|D|\,a - \sigma(a)\,|D|)\,F\right).$$

Thus, all these commutators are compact operators, and in fact they are quantized differentials of the same order as D^{-1}. □

Proposition 3.3. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a σ-spectral triple with $D^{-1} \in \mathcal{L}^{n,\infty}$.*

(1) *The linear functional*

$$a \in \mathcal{A} \mapsto \varphi(a) = ⨍ a\,D^{-n} \,:= \mathrm{Tr}_\omega(a\,D^{-n})$$

is a σ^{-n}-trace on $\mathcal{A}$: $\quad \varphi(a\,b) \,=\, \varphi(b\,\sigma^{-n}(a)), \quad \forall\, a, b \in \mathcal{A}$.

(2) *More generally, for any bounded operator $T \in \mathcal{L}(\mathfrak{H})$,*

$$Tr_\omega(T\,\sigma^{-n}(a)\,D^{-n}) \,=\, Tr_\omega(a\,T\,D^{-n})\,, \qquad \forall\, a \in \mathcal{A}\,. \tag{3.5}$$

(3) *When the σ-spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ is Lipschitz-regular, the same hold true when D^{-n} is replaced by $|D|^{-n}$.*

Proof. Let us show by induction that for any $1 \leq k \leq n$ one has

$$D^{-k}\,a - \,\sigma^{-k}(a)\,D^{-k} \in \mathcal{L}_0^{\frac{n}{k},\infty}\,, \quad \forall\, a \in \mathcal{A}. \tag{3.6}$$

Clearly,

$$D^{-1}\,a - \,\sigma^{-1}(a)\,D^{-1} = \,D^{-1}\,(a\,D - \,D\,\sigma^{-1}(a))\,D^{-1} \,\in\, \mathcal{L}_0^{n,\infty}.$$

To verify the inductive step we write

$$\begin{aligned} D^{-k}\,a - \,\sigma^{-k}(a)\,D^{-k} \;&=\; D^{-1}\big(D^{-(k-1)}\,a - \,\sigma^{-(k-1)}(a)\,D^{-(k-1)}\big) \\ &+\; \big(D^{-1}\,\sigma^{-(k-1)}(a) \,-\, \sigma^{-k}(a)\,D^{-1}\big)D^{-(k-1)} \end{aligned}$$

and observe that, by Hölder's inequality and the induction hypothesis, each of the two summands in the right hand side belongs to $\mathcal{L}_0^{\frac{n}{k},\infty}$.
Applying now (3.6) for $k = n$, one obtains

$$\varphi(T\,\sigma^{-n}(a)) = ⨍ T\,\sigma^{-n}(a)\,D^{-n} = ⨍ T\,D^{-n}\,a = \,\varphi(a\,T)\,, \quad \forall\, T \in \mathcal{L}(\mathfrak{H})\,.$$

To prove the third statement, one replaces D by $|D|$ throughout the above argument. □

We now consider the analogue of the bimodule of gauge potentials given in the usual case by the $\mathcal{A}$-bimodule $\Omega^1_D \subset \mathcal{L}(\mathfrak{H})$ of operators of the form

$$A = \Sigma a_i[D, b_i], \quad a_i, b_i \in \mathcal{A}\,. \tag{3.7}$$

Let $(\mathcal{A}, \mathfrak{H}, D)$ be a σ-spectral triple, then we let $\Omega^1_D \subset \mathcal{L}(\mathfrak{H})$ be the linear space of operators of the form

$$A = \Sigma a_i(D\,b_i - \sigma(b_i)\,D)\,, \quad a_i, b_i \in \mathcal{A}\,. \tag{3.8}$$

Proposition 3.4. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a σ-spectral triple, then Ω^1_D is an $\mathcal{A}$-bimodule for the action*

$$a \cdot \omega \cdot b = \sigma(a)\,\omega\, b\,, \quad \forall\, a, b \in \mathcal{A}\,, \quad \forall\, \omega \in \Omega^1_D \tag{3.9}$$

and the map

$$a \mapsto d_\sigma(a) = D\,a - \sigma(a)\,D \tag{3.10}$$

is a derivation of $\mathcal{A}$ in Ω^1_D.

Proof. One has

$$d_\sigma(ab) = D\,ab - \sigma(ab)\,D = (D\,a - \sigma(a)\,D)\,b + \sigma(a)\,(D\,b - \sigma(b)\,D)$$

which shows that

$$d_\sigma(ab) = d_\sigma(a) \cdot b + a \cdot d_\sigma(b)\,, \quad \forall\, a, b \in \mathcal{A} \tag{3.11}$$

Since σ is an automorphism of $\mathcal{A}$, the linear space Ω^1_D is the linear span of the $a \cdot d_\sigma(b)$ for $a, b \in \mathcal{A}$. By (3.11) this is stable under right multiplication by elements of $\mathcal{A}$. Thus Ω^1_D is an $\mathcal{A}$-bimodule. Finally (3.11) shows that d_σ is a derivation. □

3.2. **Chern character.** By Proposition 3.2, any σ-spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ of finite summability degree, *i.e.* such that with $D^{-1} \in \mathcal{L}^{n,\infty}$ for some $n \in \mathbb{N}$, which in addition is *Lipschitz-regular* has a well-defined Chern character in cyclic cohomology, namely the Chern character of its 'phase' Fredholm module $(\mathfrak{H}, F)$ over $\mathcal{A}$,

$$\Phi_F(a^0, a^1, \ldots, a^n) := \mathrm{Tr}\,(\gamma\, F\,[F, a^0]\,[F, a^1] \cdots [F, a^n])\,, \qquad \forall\, a^0, a^1, \ldots a^n \in \mathcal{A}, \tag{3.12}$$

with γ omitted in the ungraded case (*cf.* [3, Part I]).
On the other hand let us assume for a moment that $(\mathcal{A}, \mathfrak{H}, D')$, $D' = e^h\, D\, e^h$, is a graded twisted spectral triple as in §2.2, with the property that $D^{-1} \in \mathcal{L}^{n,\infty}$ for some even $n \in \mathbb{N}$. Applying (2.5) one sees that

$$D'^{-1}\, d'_\sigma a \,=\, e^{-h}\, D^{-1}\,[D, b]\, e^h\,, \quad \text{where} \quad b = e^h\, a\, e^{-h}.$$

Therefore, for any $a^0, a^1, \ldots, a^n \in \mathcal{A}$,

$$\mathrm{Tr}\,(\gamma\, D'^{-1} d'_\sigma a^0\, D'^{-1} d'_\sigma a^1 \cdots D'^{-1} d'_\sigma a^n) = \mathrm{Tr}\,(\gamma\, D^{-1}[D, b^0]\, D^{-1}[D, b^1] \cdots D^{-1}[D, b^n])\,, \tag{3.13}$$

with $b^i = e^h\, a^i\, e^{-h}$, $\forall\, i = 0, \ldots, n$. The right hand side of the above identity is a cyclic cocycle on $\mathcal{A}$ that represents, up to normalization, the Chern character of $(\mathcal{A}, \mathfrak{H}, D)$, *cf.* [3, Part I, §6]. It follows that the left hand side is also a cyclic cocycle, obtained via conjugation by an inner automorphism, and thus determining the same periodic cyclic cohomological class

$$Ch^*(\mathcal{A}, \mathfrak{H}, D') \,=\, Ch^*(\mathcal{A}, \mathfrak{H}, D) \in HP^*(\mathcal{A}).$$

This suggests that it should be possible to define a 'straight' Chern character for any finitely sumable twisted spectral triple, not just for those that are Lipschitz-regular. The proposition below confirms that this is indeed the case.

Proposition 3.5. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a graded σ-spectral triple such that $D^{-1} \in \mathcal{L}^{n,\infty}$ for some even $n \in \mathbb{N}$. Then the following multilinear form*

$$\Phi_{D,\sigma}(a^0, a^1, \ldots, a^n) := \mathrm{Tr}\,(\gamma\, D^{-1} d_\sigma a^0\, D^{-1} d_\sigma a^1 \cdots D^{-1} d_\sigma a^n)\,, \qquad \forall\, a^0, a^1, \ldots a^n \in \mathcal{A} \tag{3.14}$$

is a cyclic cocycle in $Z^n_\lambda(\mathcal{A})$.

Proof. The proof of Proposition 1 in [3, Part I, §6] (*cf.* also *infra*) applies verbatim to the twisted case if one simply replaces the representation $a \mapsto D^{-1}\, a\, D$ by $a \mapsto D^{-1}\, \sigma(a)\, D$, $\forall\, a \in \mathcal{A}$. $\square$

3.3. **Pairing with K-theory.** Implicit in the above proof is the existence 'behind the scene' of a pair of graded Fredholm module over $\mathcal{A}$, $(\tilde{\mathfrak{H}}^{\pm}, F^{\pm})$, canonically associated to the σ-spectral triple $(\mathcal{A}, \mathfrak{H}, D)$. To wit, we decompose $(\mathcal{A}, \mathfrak{H}, D)$ according to the grading by γ into

$$\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_- \,, \qquad D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}, \qquad a = \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix}, \quad \forall a \in \mathcal{A},$$

then define

$$\tilde{\mathfrak{H}}^{\pm} := \mathfrak{H}_{\pm} \oplus \mathfrak{H}_{\pm} \,, \qquad \pi^{\pm}(a) := \begin{pmatrix} a_{\pm} & 0 \\ 0 & D_{\pm}^{-1} \sigma(a_{\mp}) D_{\pm} \end{pmatrix} \quad \text{on} \quad \mathrm{Dom} D_{\pm}, \qquad F^{\pm} := \begin{pmatrix} 0 & I_{\pm} \\ I_{\pm} & 0 \end{pmatrix},$$

and note that for all $a \in \mathcal{A}$, firstly,

$$D_{\pm}^{-1} \sigma(a_{\mp}) D_{\pm} \,=\, a_{\pm} \,-\, D_{\pm}^{-1} (D_{\pm}\, a_{\pm} \,-\, \sigma(a_{\mp})\, D_{\pm}) \quad \text{is bounded,} \tag{3.15}$$

and secondly,

$$[F^{\pm}, \pi^{\pm}(a)] \,=\, \begin{pmatrix} 0 & -D_{\pm}^{-1}\, (D_{\pm}\, a_{\pm} - \sigma(a_{\mp})\, D_{\pm}) \\ D_{\pm}^{-1}\, (D_{\pm}\, a_{\pm} - \sigma(a_{\mp})\, D_{\pm}) & 0 \end{pmatrix} \in \mathcal{L}^{n,\infty}. \tag{3.16}$$

Lemma 3.6. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a graded σ-spectral triple such that $D^{-1} \in \mathcal{L}^{n,\infty}$ for some even $n \in \mathbb{N}$, and let $e \in \mathcal{A}$ be an idempotent. Denote by $f_{\pm}$ the bounded closure of of $D_{\pm}^{-1}\sigma(e_{\mp}) D_{\pm}$. Then $f_{\pm}^2 = f_{\pm}$ and $f_{\pm} e_{\pm} : e_{\pm}\mathfrak{H}_{\pm} \longrightarrow f_{\pm}\mathfrak{H}_{\pm}$ are Fredholm operators.*

Proof. The first claim is obvious and the second follows from the fact that $f_{\pm} - e_{\pm}$ is compact. $\square$

The integer $\mathrm{Index}(f_{\pm} e_{\pm})$ depends only on the K-theory class of the idempotent, and thus one can define a pair of index maps $\mathrm{Index}_{D,\sigma}^{\pm} : K_0(\mathcal{A}) \to \mathbb{Z}$, by setting

$$\mathrm{Index}_{D,\sigma}^{\pm}[e] \,=\, \mathrm{Index}(f_{\pm} e_{\pm})\,, \qquad \forall\, e^2 = e \in M_N(\mathcal{A}); \tag{3.17}$$

taken together, they give rise to a *double index map*

$$\mathrm{Index}_{D,\sigma} = \left(\mathrm{Index}_{D,\sigma}^{+},\, \mathrm{Index}_{D,\sigma}^{-}\right) : K_0(\mathcal{A}) \to \mathbb{Z} \times \mathbb{Z}. \tag{3.18}$$

On the other hand, the cyclic cocycle (3.14) is itself made of two cocycles in $Z_{\lambda}^{n}(\mathcal{A})$,

$$\Phi_{D,\sigma}^{\pm}(a^0, a^1, \ldots, a^n) \,:=\, \mathrm{Tr}\left(D_{\pm}^{-1}(D_{\pm}\, a_{\pm}^0 - \sigma(a_{\mp}^0)\, D_{\pm}) \cdots D_{\pm}^{-1}(D_{\pm}\, a_{\pm}^n - \sigma(a_{\mp}^n)\, D_{\pm})\right). \tag{3.19}$$

Proposition 3.7. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a graded σ-spectral triple such that $D^{-1} \in \mathcal{L}^{n,\infty}$ for some even $n \in \mathbb{N}$. For any $e^2 = e \in M_N(\mathcal{A})$, one has*

$$\mathrm{Index}_{D,\sigma}^{\pm}[e] \,=\, \Phi_{D,\sigma}^{\pm}(e, \ldots, e)\,. \tag{3.20}$$

If in addition $e^* = \sigma(e)$, then

$$\mathrm{Index}_{D,\sigma}^{+}[e] \,=\, -\,\mathrm{Index}_{D,\sigma}^{-}[e]\,. \tag{3.21}$$

Proof. Indeed, $\mathrm{Index}^{\pm}_{D,\sigma}$ is precisely the index map associated to the Fredholm module $(\tilde{\mathfrak{H}}_{\pm}, F_{\pm})$, and therefore is given by the corresponding index formula, *cf.* [3, Part I, §3, Theorem 1].

The second claim follows from the fact that, if $e^* = \sigma(e)$, then

$$\big(D\, e - \sigma(e)\, D\big)^* \;=\; -\,(D\, e - \sigma(e)\, D)\,, \tag{3.22}$$

which in turn implies

$$\overline{\Phi^+_{D,\sigma}(e, \ldots, e)} \;=\; -\Phi^-_{D,\sigma}(e, \ldots, e)\,.$$

□

3.4. **One-parameter group of automorphisms.** We now make the the additional assumption that the involutive Banach algebra $\mathcal{A}$ is equipped with a strongly continuous 1-parameter group of isometric automorphisms $\{\sigma_t\}_{t\in\mathbb{R}}$ such that

(1PG) *σ coincides with the value at $t = -i$ of the analytic extension of $\{\sigma_t\}_{t\in\mathbb{R}}$.*

The existence of such an analytic extension defined on a *dense subalgebra* $\mathcal{O}$ of $\mathcal{A}$, which is moreover *stable under holomorphic functional calculus*, is ensured by a theorem of Bost [1, Thm. 1.1.1].

To begin with, let us note that in the presence of the above hypothesis, which by the way is automatically satisfied by the transverse spectral triple, *cf.* (2.14), the double index map reduces to a single index pairing map.

Lemma 3.8. *Assume that $\mathcal{A}$ satisfies* (1PG)*. Then the two signed index maps coincide, i.e.*

$$\mathrm{Index}^+_{D,\sigma} \;=\; -\,\mathrm{Index}^-_{D,\sigma} :\; K_0(\mathcal{A}) \to \mathbb{Z}\,.$$

Proof. Let $e^2 = e = e^* \in M_N(\mathcal{O})$ be a projection. Instead of (3.22) we now use the identity

$$\big(D\,\sigma^{-1}(e) \;-\; e\,D\big)^* \;=\; -\big(D\,e \;-\; \sigma(e)\,D\big)\,, \tag{3.23}$$

together with the fact that the idempotents e and $\sigma^{-1}(e)$ are homotopic via $t \in [0,1] \mapsto \sigma_{it}(e)$, to obtain

$$\mathrm{Index}^+_{D,\sigma}[e] = \mathrm{Index}^+_{D,\sigma}[\sigma^{-1}(e)] = \overline{\Phi^+_{D,\sigma}(\sigma^{-1}(e), \ldots, \sigma^{-1}(e))} = -\Phi^-_{D,\sigma}(e, \ldots, e) = -\mathrm{Index}^-_{D,\sigma}[e]\,.$$

□

As a matter of fact, a slight elaboration of the above argument gives a more comprehensive result that elucidates the relationship between the two 'half-character' cocycles (3.19). Note however that unlike the classical case, where this relationship manifests itself already at the level of cocycles (*cf.* [3, Part I, §6]), in the twisted case it only occurs at the cohomological level.

Proposition 3.9. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a graded σ-spectral triple such that $D^{-1} \in \mathcal{L}^{n,\infty}$ for some even $n \in \mathbb{N}$, Under the assumption* (1PG)*, the two Chern characters $[\Phi^{\pm}_{D,\sigma}] \in HP^{\mathrm{ev}}(\mathcal{O})$ are related by the identity*

$$[\Phi^-_{D,\sigma}] \;=\; -[(\Phi^+_{D,\sigma})^*]\,, \tag{3.24}$$

where

$$(\Phi^+_{D,\sigma})^*(a_0, \ldots, a_n) \;:=\; \overline{\Phi^+_{D,\sigma}(a_n^*, \ldots, a_0^*)}\,, \qquad \forall\, a^0, a^1, \ldots, a^n \in \mathcal{O}\,.$$

Proof. For any $a \in \mathcal{O}$ and $t \in [0,1]$, let

$$\pi_t(a) := \big((\pi^+ \circ \sigma_{it})(a^*)\big)^* = \begin{pmatrix} \sigma_{-it}(a_+) & 0 \\ 0 & D_- \sigma_{i-it}(a_-) D_-^{-1} \end{pmatrix}, \tag{3.25}$$

and define the family of Fredholm modules $\{(\mathfrak{H}_t, F_t)\}_{t\in[0,1]}$, by taking

$$\mathfrak{H}_t = \mathfrak{H}_+ \oplus \mathfrak{H}_+ \quad \text{acted upon by } \mathcal{O} \text{ via } \pi_t\,, \text{ and } \quad F_t := \begin{pmatrix} 0 & I_+ \\ I_+ & 0 \end{pmatrix}.$$

As in (3.16) one has $[F_t, \pi_t(a)] \in \mathcal{L}^{n,\infty}$, because

$$D\,\sigma_{i-it}(a)\,D^{-1} - \sigma_{-it}(a) = \big(D\,\sigma_{i-it}(a) - \sigma(\sigma_{i-it}(a))\,D\big)\,D^{-1} \in \mathcal{L}^{n,\infty}.$$

Note also that

$$\pi_0(a) = \begin{pmatrix} a_+ & 0 \\ 0 & D_- \sigma^{-1}(a_-) D_-^{-1} \end{pmatrix} = \big(\pi^+(a^*)\big)^*, \tag{3.26}$$

and

$$\pi_1(a) = \begin{pmatrix} \sigma(a_+) & 0 \\ 0 & D_- a_- D_-^{-1} \end{pmatrix} = \begin{pmatrix} 0 & D_- \\ D_- & 0 \end{pmatrix} \pi^-(a) \begin{pmatrix} 0 & D_-^{-1} \\ D_-^{-1} & 0 \end{pmatrix}. \tag{3.27}$$

By Lemma 1 in [3, Part I, §5], the periodic cyclic cohomology class $Ch^*(\mathfrak{H}_t, F_t) \in HP^{\rm ev}(\mathcal{O})$, is independent of $t \in [0,1]$. We also recall that this class can be represented by the cyclic cocycle

$$\Phi_t(a^0, \ldots, a^n) := \frac{1}{2}\,\mathrm{Tr}\left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} F_t\,[F_t, \pi_t(a^0)] \cdots [F_t, \pi_t(a^n)]\right), \quad a^0, a^1, \ldots, a^n \in \mathcal{O}.$$

In particular, using (3.26) and (3.23), one has

$$\begin{aligned} \Phi_0(a^0, \ldots, a^n) &= \mathrm{Tr}\big((a_+^0 - D_-\sigma^{-1}(a_-^0)D_-^{-1}) \cdots (a_+^n - D_-\sigma^{-1}(a_-^n)D_-^{-1})\big) \\ &= (\Phi_{D,\sigma}^+)^*(a^0, \ldots, a^n)\,, \end{aligned}$$

while by (3.27)

$$\begin{aligned} \Phi_1(a^0, \ldots, a^n) &= \mathrm{Tr}\big((\sigma(a_+^0) - D_- a_-^0 D_-^{-1}) \cdots (\sigma(a_+^n) - D_- a_-^n D_-^{-1})\big) \\ &= -\Phi_{D,\sigma}^-(a^0, \ldots, a^n). \end{aligned}$$

□

3.5. **Local Hochschild cocycles.** As a first step in the direction of extending the local index formula of [6] to twisted spectral triple, we shall construct an analogue of the local Hochschild cocycle that gives the Hochschild class of the Chern character in the untwisted case, *cf.* [5, IV.2.γ]. We begin by revisiting the latter and then proceed in a heuristic manner. Given a graded spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ such that $D^{-1} \in \mathcal{L}^{n,\infty}$ for some $n \in 2\mathbb{N}$, the Hochschild class of its Chern character is represented in local form by the following cocycle:

$$\Psi_D(a^0, a^1, \ldots, a^n) := ⨍ \gamma\, a^0\, [D, a^1] \cdots [D, a^n]\, D^{-n}, \qquad \forall\, a^0, a^1, \ldots, a^n \in \mathcal{A}. \tag{3.28}$$

Noting that

$$[D, a]\, D^{-k} = D^{-k+1}\big(D^k\, a\, D^{-k} - D^{k-1}\, a\, D^{-k+1}\big), \qquad \forall\, a \in \mathcal{A}, \tag{3.29}$$

we can successively move D^{-n} to the left with loss of a power at each step, and thus rewrite the cocycle (3.28) in the form

(3.30) $$\Psi_D(a^0, a^1, \dots, a^n) \,=\, \fint \gamma\, a^0\, (D\, a^1\, D^{-1} \,-\, a^1) \,\cdots\, (D^n\, a^n\, D^{-n} \,-\, D^{n-1}\, a^n\, D^{-n+1}).$$

In the twisted case, inspired by the formulas (3.26), (3.27), we make the formal substitution

(3.31) $$D^k\, a\, D^{-k} \quad\longmapsto\quad D^k\, \sigma^{-k}(a)\, D^{-k}\,, \qquad \forall\, a \in \mathcal{A}\,,$$

and obtain the following candidate for a Hochschild character cocycle:

$$\Psi_{D,\sigma}(a^0, a^1, \dots, a^n) \,=\,$$
$$\fint \gamma\, a^0\, \big(D\, \sigma^{-1}(a^1)\, D^{-1} \,-\, a^1\big) \,\cdots\, \big(D^n\, \sigma^{-n}(a^n)\, D^{-n} \,-\, D^{n-1}\, \sigma^{-n+1}(a^n)\, D^{-n+1}\big).$$

The counterpart of (3.29) being

(3.32) $$d_\sigma(\sigma^{-k}(a))\, D^{-k} \,=\, D^{-k+1}\, \big(D^k\, \sigma^{-k}(a)\, D^{-k} \,-\, D^{k-1}\, \sigma^{-k+1}(a)\, D^{-k+1}\big)\,, \qquad \forall\, a \in \mathcal{A}\,,$$

one can reverse the process of distributing D^{-n} among the factors, which leads to the expression stated below.

Proposition 3.10. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a graded σ-spectral triple such that $D^{-1} \in \mathcal{L}^{n,\infty}$ for some even $n \in \mathbb{N}$. Then the following multilinear form*

(3.33)
$$\Psi_{D,\sigma}(a^0, a^1, \dots, a^n) \,:=\, \fint \gamma\, a^0\, d_\sigma(\sigma^{-1}(a^1)) \,\cdots\, d_\sigma(\sigma^{-n}(a^n))\, D^{-n}\,, \qquad \forall\, a^0, a^1, \dots a^n \in \mathcal{A}$$

is a Hochschild cocycle in $Z^n(\mathcal{A}, \mathcal{A}^*)$.
If the σ-spectral triple is ungraded, of summability degree $n \in \mathbb{N}$ odd, and is Lipschitz-regular, the corresponding Hochschild cocycle has the expression

(3.34)
$$\Psi_{D,\sigma}(a^0, a^1, \dots, a^n) \,:=\, \fint a^0\, d_\sigma(\sigma^{-1}(a^1)) \,\cdots\, d_\sigma(\sigma^{-n}(a^n))\, |D|^{-n}\,, \qquad \forall\, a^0, a^1, \dots a^n \in \mathcal{A}.$$

Proof. We check that $\Psi_{D,\sigma}$ is a Hochschild cocycle by computing its Hochschild coboundary, using the derivation rule (3.11), as follows:

$$b\Psi(a^0, a^1, ..., a^{n+1}) \,=\, \sum_{i=0}^{n} (-1)^i\, \Psi(a^0, \dots, a^i a^{i+1}, ..., a^{n+1}) \,+\, (-1)^{n+1} \Psi(a^{n+1} a^0, a^1, ..., a^n)$$
$$= \fint \gamma\, a^0\, a^1\, d_\sigma(\sigma^{-1}(a^2)) \cdots\, d_\sigma(\sigma^{-n}(a^{n+1}))\, D^{-n} \,-\, \fint \gamma\, a^0 a^1\, d_\sigma(\sigma^{-1}(a^2)) \cdots\, d_\sigma(\sigma^{-n}(a^{n+1}))\, D^{-n}$$
$$- \fint \gamma\, a^0\, d_\sigma(\sigma^{-1}(a^1))\, \sigma^{-1}(a^2) \cdots\, d_\sigma(\sigma^{-n}(a^{n+1}))\, D^{-n} \,+\, \dots$$
$$\dots + \,(-1)^n \fint \gamma\, a^0\, d_\sigma(\sigma^{-1}(a^1)) \cdots\, \sigma^{-n+1}(a^{n-1})\, \sigma^{-n+1}(a^n)\, \sigma^{-n}(a^{n+1})\, D^{-n}$$
$$+\, (-1)^n \fint \gamma\, a^0\, d_\sigma(\sigma^{-1}(a^1)) \cdots\, \sigma^{-n+1}(a^{n-1})\, d_\sigma(\sigma^{-n}(a^n))\, \sigma^{-n}(a^{n+1})\, D^{-n}$$
$$+\, (-1)^{n+1} \fint \gamma\, a^{n+1}\, a^0\, d_\sigma(\sigma^{-1}(a^1)) \cdots\, d_\sigma(\sigma^{-n}(a^n))\, D^{-n}\,.$$

The resulting expression vanishes because of successive cancelations, with the last two terms canceling each other in view of (3.5).

In the ungraded Lipschitz-regular case, the very same calculation holds true provided γ is replaced by the phase operator $F = D\,|D|^{-1}$. □

The cyclic group generated by $\sigma \in \mathrm{Aut}(\mathcal{A})$ acts in a natural way on the set of such Hochschild cocycles. For each integer $m \in \mathbb{Z}$, the corresponding 'gauge transformed' cocycle via the action of $\sigma^m \in \mathrm{Aut}(\mathcal{A})$, has the expression: $\forall\, a^0, a^1, \ldots a^n \in \mathcal{A}$,

$$\Psi^{(m)}_{D,\sigma}(a^0, a^1, \ldots, a^n) \,:=\, ⨍ \gamma\, \sigma^m(a^0)\, d_\sigma(\sigma^{m-1}(a^1)) \,\cdots\, d_\sigma(\sigma^{m-n}(a^n))\, D^{-n}\,. \tag{3.35}$$

4. Future developments

We conclude by listing, roughly in their increasing order of complexity, a few themes for future research in this direction.

4.1. **Symbolic calculus and local index formula.** The symbolic calculus developed for spectral triples (*cf.* [6, Appendix B]) needs to be adapted to allow, in particular, establishing that if $(\mathcal{A}, \mathfrak{H}, D)$ is a σ-spectral triple with $D^{-1} \in \mathcal{L}^{n,\infty}$ that satisfies a stronger regularity assumption, then

$$|D|^{-t}\left(|D|^t\, a \,-\, \sigma^t(a)\,|D|^t\right) \in \mathcal{L}^{n,\infty}\,, \qquad \forall\, t \in \mathbb{R}\,. \tag{4.1}$$

By Remark 2.3, the transverse spectral triple example fulfills this property. Note also that the extra regularity assumption (4.1) immediately reconciles the two definitions given above to the Chern character, *viz.* (3.12) and (3.14). Indeed, one can then produce the following homotopy between the cyclic cocycles Φ_F and $\Phi_{D,\sigma}$:

$$\Phi_t(a^0, a^1, \ldots, a^n) \,:=\, \mathrm{Tr}\left(\gamma\, D_t^{-1}(D_t\, a^0 - \sigma^{1-t}(a^0)\, D_t) \,\cdots\, D_t^{-1}(D_t\, a^n - \sigma^{1-t}(a^n)\, D_t)\right),$$

where $D_t = D\,|D|^{-t}$ and $t \in [0,1]$.

The full expression of the local formula for the Chern character of a finitely summable σ-spectral triple, based on σ-twisted commutators and extending the noncommutative local index formula in [6, Part II], remains to be worked out.

4.2. **Relation to type II and Thom isomorphism.** One should expect that the constructions in the foliation context and in the context of modular forms of hypoelliptic spectral triples on frame bundles extend to the general context of twisted spectral triples satisfying (1PG). The noncommutative space associated to the total space of the frame bundle corresponds to the cross product algebra by the one-parameter group $\{\sigma_t\}_{t\in\mathbb{R}}$. The K-homology classes on the base and on the total space as well as their local index cyclic cocycles should be related by a Thom isomorphism.

4.3. **Higher dimensions.** The most challenging task ahead consists in extending the above considerations to the case of foliations of higher codimension, which has been put in the framework of a higher form of Tomita's theory in [4], Section 3. The above notion of twisting only allows to handle the determinant part of the cocycle given by the Jacobian. One expects the general case to involve dual actions of Lie groups such as $\mathrm{GL}(n)$ and more generally of quantum groups.

4.4. **Relation with quantum groups.** The domain of quantum groups is a natural arena where twisting frequently occurs (see [8]) and where the above extension of the notion of spectral triple could be useful. One would expect that the higher dimensional generalizations alluded to above would also cover the "braided" situation that arises from quantum groups.

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A. Connes: Collège de France, 3, rue d'Ulm, Paris, F-75005 France, I.H.E.S. and Vanderbilt University

E-mail address: alain@connes.org

H. Moscovici: Department of mathematics, The Ohio State University, Columbus, OH 43210, USA

E-mail address: henri@math.ohio-state.edu