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Conformal trace theorem for Julia sets of quadratic polynomials

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Abstract. If c is in the main cardioid of the Mandelbrot set, then the Julia set J of the map $\phi_c : z \mapsto z^2 + c$ is a Jordan curve of Hausdorff dimension $p \in [1, 2)$. We provide a full proof of a formula for the Hausdorff measure on J in terms of singular traces announced by the first named author in 1996.

1. *Introduction*

This paper illustrates the role of the quantized calculus in the concrete example of Julia sets. The calculus of infinitesimals fits perfectly in the operator formalism of quantum mechanics, where compact operators play the role of infinitesimals, with order governed by the rate of decay of the characteristic values, and where the logarithmic divergences familiar in physics give the substitute for integration of infinitesimals of order one, in the form of singular traces whose prototype is the Dixmier trace. It is the singularity of these traces, namely their vanishing on the ideal of ordinary trace class operators (more technically on $(\mathcal{L}_{1,\infty})_0$), which allows one to restore the simplifications familiar in the ordinary calculus of infinitesimals. Given a continuous map $Z : \mathbb{T} \to \mathbb{C}$, it admits a derivative in the sense of distributions, but taking the absolute value of this derivative and raising it to a real power do not make sense in distribution theory and this prevents one from making sense of the expression $\int f(Z)|dZ|^p$, which would be a natural candidate for the conformal measure in the context of Julia sets. Both the absolute value and the raising to real powers do make sense in the quantized calculus, where moreover the role of the derivative of Z is played by the quantized differential $[F, Z]$, where Z is viewed as the multiplication operator M_Z and F is the Hilbert transform. Thus, $|dZ|^p$ makes sense as an operator while as stated above the role of the integration is played by the

Dixmier trace. The main result of this paper is that the formula $\int f(Z)|dZ|^p$ now makes sense and recovers the Hausdorff measure on J in the context of Julia sets, as we describe more specifically now. This result was announced by the first named author in [**2**, p. 23] and later in [**4**] but the detailed proofs were not given.

We recall the definition of Julia sets of quadratic polynomials, as outlined in [**1**, Chs. 8 and 3]. Let $c \in \mathbb{C}$ and consider the family of quadratic complex polynomials $\phi_c(z) := z^2 + c$. For $k \geq 0$, let ϕ_c^k be the k-fold iteration of ϕ_c. The Mandelbrot set M is the set of all c such that $\{\phi_c^k(0)\}_{k=0}^{\infty}$ is bounded.

The Julia set J of ϕ_c is defined to be the boundary of the set of points $z \in \mathbb{C}$ such that $\phi_c^k(z)$ is bounded as $k \to \infty$. If $c = 0$, then J is simply the unit circle in $\mathbb{C}$, and it can be shown that when c is sufficiently small then J is a Jordan curve. More precisely, J is a Jordan curve if and only if

$$c \in \left\{ \frac{\mu}{2}\left(1 - \frac{\mu}{2}\right) : |\mu| < 1 \right\} \subset M. \tag{1.1}$$

See [**1**, Ch. 8, Theorem 1.3] for a proof of this claim. The set in (1.1) is called the main cardioid of the Mandelbrot set. It is well known that if $c \neq 0$ is in the main cardioid of the Mandelbrot set M, then the Hausdorff dimension p of J satisfies $1 < p < 2$ (see Corollary 5.3 below for a proof of this fact).

Let $F : L_2(\mathbb{T}) \to L_2(\mathbb{T})$ be the Hilbert transform, defined on exponential basis functions $e_n(z) := z^n$, $n \in \mathbb{Z}$, by $Fe_n = \mathrm{sgn}(n)e_n$. Given an essentially bounded function f on the unit circle $\mathbb{T}$, the symbol M_f denotes the operator on $L_2(\mathbb{T})$ of pointwise multiplication by f. Since J is a Jordan curve, due to the Riemann mapping theorem there is a conformal mapping Z from the exterior of the unit disc $\{z \in \mathbb{C} : |z| > 1\}$ to the unbounded component of $\mathbb{C} \setminus J$. By the Carathèodory theorem on continuous extensions of conformal maps, Z extends to a continuous bijection $Z : \mathbb{T} \to J$. We may therefore consider Z as a function on the circle, and correspondingly we write M_Z as the corresponding linear operator on $L_2(\mathbb{T})$. It is known that Z may be chosen such that for all $z \in \mathbb{T}$ we have $Z(z^2) = Z(z)^2 + c$ (see §5.3).

Our main result is the following theorem (all unexplained symbols and notions will be defined in §2).

THEOREM 1.1. *Let $c \neq 0$ be in the main cardioid of the Mandelbrot set. Let $p \in (1, 2)$ be the Hausdorff dimension of the Julia set J of ϕ_c. Let m_p be the p-dimensional Hausdorff measure on J. Then:*

(a) $[F, M_Z] \in \mathcal{L}_{p,\infty}$;

(b) *for every continuous Hermitian trace φ on $\mathcal{L}_{1,\infty}$, there exists a constant $K(\varphi, c)$ such that for every $f \in C(J)$, we have*

$$\varphi(M_{f\circ Z}|[F, M_Z]|^p) = K(\varphi, c) \int_J f \, dm_p;$$

(c) *if ω is a dilation-invariant extended limit on $L_\infty(0, \infty)$ such that $\omega \circ \log$ is also dilation invariant, then $K(\mathrm{tr}_\omega, \phi_c) > 0$. Here, tr_ω is a Dixmier trace corresponding to the extended limit ω.*

Theorem 1.1 should be compared with [**5**, Theorem 1.1], which concerns geometric measures on limit sets of finitely generated quasi-Fuchsian groups. The statement of the result is very similar; however, it should be noted that the methods of proof used in this text are completely different to those used in [**5**]. We follow a proof outlined by the first named author in [**2**, Ch. 4], which proceeds by identifying the functional $f \mapsto \varphi(M_{f\circ Z}|[F, M_Z]|^p)$ on the space $C(J)$ with the (essentially unique) p-conformal measure with respect to ϕ_c on J (as defined by Sullivan [**17**, Theorem 3]). Another theorem of Sullivan [**17**, Theorem 4] identifies this p-conformal measure with the p-dimensional Hausdorff measure.

This paper is broken up as follows:

(1) Section 2 provides necessary preliminaries on the theory of traces on operator ideals;
(2) Section 3 collects necessary results concerning commutators of multiplication operators and the Hilbert transform. Many of the proofs relevant to this section are provided in Appendix A;
(3) Section 4 proves that if $\mathcal{C}$ is any Jordan curve in the complex plane with finite upper s-dimensional Minkowski content, and ζ is a conformal equivalence between the exterior of the unit disc $\mathbb{D}$ and the exterior of $\mathcal{C}$, then $[F, M_\zeta]$ (where M_ζ is considered as an operator on $L_2(\mathbb{T})$) is in the weak Schatten ideal $\mathcal{L}_{s,\infty}$;
(4) Section 5 collects properties of Julia sets of quadratic polynomials, and demonstrates that the Julia set of a quadratic polynomial $\phi_c(z) = z^2 + c$, where $c \neq 0$ is in the main cardioid of the Mandelbrot set, is a Jordan curve with Hausdorff dimension $p \in (1, 2)$ and with finite upper and strictly positive lower p-Minkowski content. Combined with the results of §4, this immediately yields Theorem 1.1(a);
(5) Section 6 then completes the proof of Theorem 1.1(b), by showing that the functional $f \mapsto \varphi(M_{f\circ Z}|[F, M_Z]|^p)$ is p-conformal with respect to ϕ_c (in the sense of §5.2);
(6) Section 7 then provides a proof of Theorem 1.1(c) by referring to known results on the relationship between Dixmier traces and zeta-function residues.

2. *Singular traces and operator ideals*

This section introduces notation and terminology concerning singular traces (in particular Dixmier traces) and operator ideals. Let H be a complex separable Hilbert space with orthonormal basis $\{e_n\}_{n=0}^\infty$. Denote by $\mathcal{L}(H)$ the $*$-algebra of bounded linear operators on H and denote by $\mathcal{K}(H)$ the set of compact operators. Given an operator $T \in \mathcal{L}(H)$, the singular value function $s \mapsto \mu(s, T)$ is defined to be the distance of T to the set of all operators of rank at most s:

$$\mu(s, T) := \inf\{\|T - R\| : \operatorname{rank}(R) \leq s\}, \quad s \geq 0.$$

Given $p \in [1, \infty]$, the p-Schatten class $\mathcal{L}_p$ is defined to be the set of operators $T \in \mathcal{L}(H)$ such that $\{\mu(n, T)\}_{n=0}^\infty$ is in the sequence space ℓ_p. The weak Schatten class $\mathcal{L}_{p,\infty}$ is the set of operators $T \in \mathcal{L}(H)$ such that $\mu(n, T) = O(n^{-1/p})$. The Schatten p-class $\mathcal{L}_p$ (respectively the weak Schatten class $\mathcal{L}_{p,\infty}$) is equipped with the norm (respectively quasi-norm) given by $\|T\|_p := \|\{\mu(n, T)\}_{n=0}^\infty\|_{\ell_p}$ (respectively $\|T\|_{p,\infty} := \sup_{n\geq 0} n^{1/p}\mu(n, T)$).

A functional $\varphi : \mathcal{L}_{1,\infty} \to \mathbb{C}$ is called a continuous trace if it is continuous in the $\mathcal{L}_{1,\infty}$ quasi-norm and, for all $A \in \mathcal{L}_{1,\infty}$ and $B \in \mathcal{L}(H)$, we have $\varphi(BA) = \varphi(AB)$. A trace φ is Hermitian if $\varphi(A^*) = \overline{\varphi(A)}$ for all $A \in \mathcal{L}_{1,\infty}$.

There is a bijective correspondence between traces on $\mathcal{L}_{1,\infty}$ and certain functionals on ℓ_∞, which we will describe here for later use. A continuous linear functional $\theta \in \ell_\infty^*$ is called translation invariant if it is invariant under translations in the sense that

$$\theta(x_0, x_1, \ldots) = \theta(0, x_0, x_1, \ldots) \quad \text{for all } (x_0, x_1, \ldots) \in \ell_\infty.$$

Additionally, a functional θ is Hermitian if $\theta(x^*) = \overline{\theta(x)}$ for all $x \in \ell_\infty$.

The following result is a combination of [**15**, Theorems 4.1 and 4.9]. Note that in [**15**] the implicit assumption is made that all functionals are Hermitian.

THEOREM 2.1. *For every continuous Hermitian trace φ on $\mathcal{L}_{1,\infty}$, there exists a unique translation-invariant Hermitian functional $\theta \in \ell_\infty^*$ such that for all $A \geq 0$ in $\mathcal{L}_{1,\infty}$, we have*

$$\varphi(A) = \theta\bigg(\frac{1}{\log 2}\bigg\{\sum_{k=2^n-1}^{2^{n+1}-2} \mu(k, A)\bigg\}_{n\geq 0}\bigg). \tag{2.1}$$

Moreover, for every translation-invariant $\theta \in \ell_\infty^*$, the right-hand side of (2.1) defines a trace on $\mathcal{L}_{1,\infty}$.

COROLLARY 2.2. *Every continuous Hermitian trace φ on $\mathcal{L}_{1,\infty}$ can be written as a difference $\varphi = \varphi_+ - \varphi_-$, where φ_- and φ_+ are positive continuous traces.*

Proof. Due to Theorem 2.1, the result will follow from the assertion that for any translation-invariant Hermitian linear functional θ on ℓ_∞ that there are positive translation-invariant linear functionals θ_+, θ_- such that $\theta = \theta_+ - \theta_-$. This fact is established in [**15**, Lemma 4.8], thus completing the proof. □

In §7, we also refer to the specific subclass of traces on $\mathcal{L}_{1,\infty}$ of Dixmier traces. A linear positive linear functional ω on the von Neumann algebra $L_\infty(0, \infty)$ is called an extended limit if ω vanishes on all functions of bounded support and $\omega(1) = 1$. The dilation semigroup $\{\sigma_s\}_{s>0}$ on $L_\infty(0, \infty)$ is defined by

$$(\sigma_s f)(t) = f(t/s).$$

A dilation-invariant extended limit is defined to be an extended limit ω such that $\omega \circ \sigma_s = \omega$ for all $s > 0$.

Given a dilation-invariant extended limit ω, the Dixmier trace tr_ω is defined on $0 \leq A \in \mathcal{L}_{1,\infty}$ by

$$\mathrm{tr}_\omega(A) = \omega\bigg(t \mapsto \frac{1}{\log(1+t)} \int_0^t \mu(s, A)\, ds\bigg).$$

It is proved in [**11**, Theorem 6.3.6] that tr_ω extends by linearity to a continuous trace on $\mathcal{L}_{1,\infty}$.

3. *Commutators of multiplication operators and the Hilbert transform*

Denote by $\mathbb{D}$ the open unit disc in the complex plane. Given $f \in L_1(\mathbb{T})$†, let $\hat{f}(n)$ be the nth Fourier coefficient of f. It is well known that f has a holomorphic extension to the interior of the unit disc if and only if $\hat{f}(n) = 0$ for all $n < 0$. In this case we identify f with its holomorphic extension. The Hilbert transform $F : L_2(\mathbb{T}) \to L_2(\mathbb{T})$ is defined on functions $f \in L_2(\mathbb{T})$ by

$$(Ff)(z) = \sum_{n \in \mathbb{Z}} \operatorname{sgn}(n) \hat{f}(n) z^n, \quad z \in \mathbb{T}.$$

Given $f \in L_\infty(\mathbb{T})$, the symbol M_f stands for the operator on $L_2(\mathbb{T})$ given by pointwise multiplication by f. We are concerned with conditions on f which are necessary and sufficient for the commutator $[F, M_f]$ to be in the Schatten p-class $\mathcal{L}_p$. The following result is a restatement of a result due to Peller [**13**, Ch. 6] and a full proof is included in Appendix A. Recall that $dz\, d\overline{z}$ denotes the Lebesgue measure on $\mathbb{C}$.

THEOREM 3.1. *Let f be a function on $\mathbb{T}$ with holomorphic extension to $\mathbb{D}$ and let $p_0 > 1$. There exist constants k, $K > 0$ (depending on p_0) such that for all $p \in (p_0, 2)$, we have*

$$k\left(\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2}\, dz\, d\overline{z}\right)^{1/p} \le \|[F, f]\|_p \le K\left(\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2}\, dz\, d\overline{z}\right)^{1/p}.$$

We also utilize the following one-sided result, giving sufficient conditions for $[F, M_f]$ to be in the weak p-Schatten class and which is identical to [**5**, Lemma 3.5].

THEOREM 3.2. *Let $p > 1$ and let f be a function on $\mathbb{T}$ with holomorphic extension to $\mathbb{D}$. Define $h(z) := f'(z)(1 - |z|^2)$ and let ν be the measure on $\mathbb{D}$ given by $d\nu = dz\, d\overline{z}/(1 - |z|^2)^2$. Then there exists a constant $c_p > 0$ such that*

$$\|[F, M_f]\|_{p,\infty} \le c_p \|h\|_{L_{p,\infty}(\mathbb{D},\nu)}.$$

4. *Jordan curves with finite upper Minkowski content*

Let A be a Borel subset of $\mathbb{R}^d$ (we are mostly concerned with the case $d = 2$). The δ-neighbourhood of A is the set

$$S_\delta(A) = \{x \in \mathbb{R}^d : \operatorname{dist}(x, A) < \delta\}.$$

Let $|S_\delta(A)|$ denote the Lebesgue measure of $S_\delta(A)$ and let $0 \le s \le d$. The upper s-dimensional Minkowski content of A is defined by

$$M^s(A) := \limsup_{\delta \to 0} \delta^{s-d} |S_\delta(A)|.$$

By definition, $M^s(A)$ is finite if and only if $|S_\delta(A)| = O(\delta^{d-s})$ as $\delta \to 0$.

The lower s-dimensional Minkowski content is defined as

$$M_s(A) := \liminf_{\delta \to 0} \delta^{s-d} |S_\delta(A)|.$$

† Spaces $L_p(\mathbb{T})$ on the circle are always defined with respect to the Haar measure.

The above given definitions of upper and lower Minkowski content follow [**8**, Definition 3.2.37].

Let $\mathcal{C}$ be a Jordan curve in the plane, and let $\Omega \subset \mathbb{C}$ be the bounded component of $\mathbb{C} \setminus \mathcal{C}$, so that $\mathcal{C} = \partial\Omega$. By the Riemann mapping theorem, there is a conformal mapping $\xi : \mathbb{D} \to \Omega$ which by the Carathèodory theorem extends to a continuous function $\xi : \mathbb{T} \to \mathcal{C}$. This section is devoted to the proof of the fact that if $\mathcal{C}$ has finite upper s-dimensional Minkowski content, then the commutator $[F, M_\xi]$ (considered as an operator on $L_2(\mathbb{T})$) is in the weak Schatten s-class $\mathcal{L}_{s,\infty}$.

The following lemma appears as [**2**, equation (4.21)], and we supply a detailed proof for convenience.

LEMMA 4.1. *Let Ω be a domain in $\mathbb{C}$ whose boundary $\partial\Omega$ is a Jordan curve and let $\xi : \mathbb{D} \to \Omega$ be a conformal map. Then, for all $|z| < 1$,*

$$\tfrac{1}{4}(1 - |z|^2)|\xi'(z)| \leq \mathrm{dist}(\xi(z), \partial\Omega) \leq (1 - |z|^2)|\xi'(z)|.$$

Proof. Let h be any conformal mapping $h : \mathbb{D} \to \Omega$. Since h is conformal, it is bijective and we have $h'(0) \neq 0$. Hence, we may define a function k by

$$k(z) = \begin{cases} \dfrac{z}{h(z) - h(0)}, & z \neq 0, \\ \dfrac{1}{h'(0)}, & z = 0. \end{cases}$$

Since h is holomorphic, k is also holomorphic. Since $\partial\Omega$ is a Jordan curve, by the Carathèodory theorem [**9**, Theorem 3.1], h extends to a continuous function on the circle $\mathbb{T}$. Since $h(0)$ is in the interior of the curve ∂U, we have $\inf_{z\in\mathbb{T}} |h(z) - h(0)| > 0$, so it follows that k also extends continuously to $\mathbb{T}$. By the maximum modulus principle, since k is holomorphic in the open unit disc,

$$|k(0)| \leq \sup_{|z|=1} |k(z)|.$$

Equivalently,

$$\frac{1}{|h'(0)|} \leq \sup_{|z|=1} \frac{|z|}{|h(z) - h(0)|}.$$

Since $|z| = 1$, we then obtain

$$\inf_{|z|=1} |h(z) - h(0)| \leq |h'(0)|.$$

When $|z| = 1$, the point $h(z)$ lies in the boundary $\partial\Omega$, so immediately

$$\mathrm{dist}(h(0), \partial\Omega) \leq |h'(0)|. \tag{4.1}$$

We now refer to the Koebe 1/4 theorem [**14**, Theorem 14.14], which states that if h is a conformal mapping from $\mathbb{D}$ to a simply connected domain Ω, then Ω contains the disc centred at $h(0)$ with radius $|h'(0)|/4$. Equivalently, $\mathrm{dist}(h(0), \partial\Omega)$ is not less than $\frac{1}{4}|h'(0)|$, so

$$\tfrac{1}{4}|h'(0)| \leq \mathrm{dist}(h(0), \partial\Omega). \tag{4.2}$$

Combining (4.1) and (4.2),

$$\tfrac{1}{4}|h'(0)| \leq \mathrm{dist}(h(0), \partial\Omega) \leq |h'(0)|. \tag{4.3}$$

Let $|z| < 1$. Consider the function

$$h(w) := \xi\left(\frac{z-w}{1-\bar{z}w}\right), \quad |w| < 1.$$

Note that the map $w \mapsto (z-w)/(1-\bar{z}w)$ is a conformal automorphism of the unit disc, so the image of the unit disc under h is the same as the image under ξ. Thus, h is a conformal mapping from the unit disc to Ω. We can then simply compute

$$h(0) = \xi(z), \quad h'(0) = -\xi'(z)(1-|z|^2).$$

So, immediately from (4.3),

$$\tfrac{1}{4}(1-|z|^2)|\xi'(z)| \leq \mathrm{dist}(\xi(z), \partial\Omega) \leq (1-|z|^2)|\xi'(z)|. \qquad \square$$

The next result shows how we can use Lemma 4.1 to reduce the question of whether $(1-|z|^2)|\xi'(z)| \in L_{s,\infty}(\mathbb{D}, dz\, d\bar{z}/(1-|z|^2)^2)$ to a purely geometric question concerning $\mathcal{C}$.

PROPOSITION 4.2. *Let $\mathcal{C}$ be a Jordan curve in the plane with interior Ω and let $\xi : \mathbb{D} \to \Omega$ be a conformal map. Let h be the function on $\mathbb{D}$ given by $h(z) = |\xi'(z)|(1-|z|^2)$. Let D be the function on Ω defined by $D(z) = \mathrm{dist}(z, \partial\Omega) = \mathrm{dist}(z, \mathcal{C})$. Then, for all $s > 0$,*

$$h \in L_{s,\infty}\left(\mathbb{D}, \frac{dz\, d\bar{z}}{(1-|z|^2)^2}\right) \iff D \in L_{s,\infty}\left(\Omega, \frac{dw\, d\overline{w}}{\mathrm{dist}(w, \partial\Omega)^2}\right).$$

Proof. Restating Lemma 4.1, we have

$$\tfrac{1}{4}h(z) \leq D(\xi(z)) \leq h(z). \tag{4.4}$$

Rearranging the result of Lemma 4.1 yields

$$\frac{1}{(1-|z|^2)^2} \leq \frac{|\xi'(z)|^2}{\mathrm{dist}(\xi(z), \partial\Omega)^2} \leq \frac{16}{(1-|z|^2)^2}, \quad z \in \mathbb{D}. \tag{4.5}$$

The function ξ maps $\mathbb{D}$ conformally into Ω, so in particular it is injective. If $w = \xi(z)$, then $dw\, d\overline{w} = |\xi'(z)|^2\, dz\, d\bar{z}$, so from (4.5) for any Borel set $A \subseteq \mathbb{D}$, we have

$$\int_A \frac{dz\, d\bar{z}}{(1-|z|^2)^2} \leq \int_{\xi(A)} \frac{dw\, d\overline{w}}{\mathrm{dist}(w, \partial\Omega)^2} \leq 16 \int_A \frac{dz\, d\bar{z}}{(1-|z|^2)^2}.$$

Thus, the image of the measure $dz\, d\bar{z}/(1-|z|^2)^2$ under ξ is equivalent to the measure $dw\, d\overline{w}/\,\mathrm{dist}(w, \partial\Omega)^2$. Combining (4.4) and (4.5) yields the equivalence that $h \in L_{s,\infty}(\mathbb{D}, dz\, d\bar{z}/(1-|z|^2)^2)$ if and only if $D \in L_{s,\infty}(\Omega, dw\, d\overline{w}/\,\mathrm{dist}(w, \partial\Omega)^2)$. $\square$

The following is the key result which will yield $[F, M_\xi] \in \mathcal{L}_{s,\infty}$ if $\mathcal{C}$ has finite upper s-dimensional Minkowski content.

PROPOSITION 4.3. *If $\partial\Omega$ has finite upper s-dimensional Minkowski content, then*

$$z \mapsto \operatorname{dist}(z, \partial\Omega) \in L_{s,\infty}\left(\Omega, \frac{dz\, d\bar{z}}{\operatorname{dist}(z, \partial\Omega)^2}\right).$$

Proof. We partition the region Ω into countably many regions, $\{A_k\}_{k\geq 0}$, defined by

$$A_k := \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) \in [2^{1-k}, 2^{-k})\}$$

and define $A_{-1} := \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) > 2\}$. Then Ω is a disjoint union:

$$\Omega = \bigcup_{k=-1}^{\infty} A_k.$$

Let μ be the measure $d\mu = dz\, d\bar{z}/ \operatorname{dist}(z, \partial\Omega)^2$. Then, for all $n \geq 0$,

$$\mu(\{z \in \Omega : \operatorname{dist}(z, \partial\Omega) \geq 2^{-n}\}) = \sum_{k=-1}^{n} \mu(A_k).$$

Inside the region A_k, the function $z \mapsto 1/ \operatorname{dist}(z, \partial\Omega)^2$ is bounded from above by 2^{2k}. So, for $k \geq 0$,

$$\begin{aligned}\mu(A_k) &\leq 2^{2k}|A_k|\\ &= 2^{2k}(|S_{2^{1-k}}(\partial\Omega) \cap \Omega| - |S_{2^{-k}}(\partial\Omega) \cap \Omega|)\\ &\leq 2^{2k}|S_{2^{1-k}}(\partial\Omega)|.\end{aligned}$$

By the assumption that the s-dimensional Minkowski content is finite, there exists $C > 0$ such that for all k,

$$\begin{aligned}|S_{2^{1-k}}(\Omega)| &\leq C \cdot 2^{(1-k)(2-s)}\\ &= C \cdot 2^{2-s} \cdot 2^{-2k} \cdot 2^{ks}.\end{aligned}$$

Letting $K = C \cdot 2^{2-s}$, we obtain that for all $k \geq 0$ we have $\mu(A_k) \leq K2^{ks}$. So,

$$\begin{aligned}\mu(\{z \in \Omega : \operatorname{dist}(z, \partial\Omega) \geq 2^{-n}\}) &\leq \mu(A_{-1}) + K \sum_{k=0}^{n} 2^{ks}\\ &= O(2^{ns}).\end{aligned}$$

Thus, $\mu(\{z \in \Omega : \operatorname{dist}(z, \partial\Omega) \geq t\}) = O(t^{-s})$ as $t \to 0$. □

We obtain our main result concerning conformal maps from the unit disc to the interior of a Jordan curve.

THEOREM 4.4. *Let $\mathcal{C}$ be a Jordan curve in the plane with finite s-dimensional upper Minkowski content, and let ξ be a conformal map from the interior of the unit disc to the interior of $\mathcal{C}$. Then the extension of ξ to the boundary, considered as a function on the circle $\mathbb{T}$, satisfies*

$$[F, M_\xi] \in \mathcal{L}_{s,\infty}.$$

Proof. Let Ω denote the interior of $\mathcal{C}$ and let D be the function on Ω given by $D(w) := \operatorname{dist}(w, \mathcal{C})$. From Proposition 4.3, we have $D \in L_{s,\infty}(\Omega, dw\, d\overline{w}/ \operatorname{dist}(w, \mathcal{C})^2)$. Applying Proposition 4.2, it follows that the function $h(z) := (1 - |z|^2)|\xi'(z)|$ is in $L_{s,\infty}(\mathbb{D}, dz\, d\overline{z}/(1 - |z|^2)^2)$.

Due to Theorem 3.2, if $h \in L_{s,\infty}(\mathbb{D}, dz\, d\overline{z}/(1 - |z|^2)^2)$, then $[F, M_\xi] \in \mathcal{L}_{s,\infty}$. □

Theorem 4.4 concerns conformal equivalences between the open unit disc and the *interior* of a Jordan curve. In fact, similar results hold for equivalences between the *exterior* of the unit disc and the exterior of a Jordan curve.

THEOREM 4.5. *Let $\mathcal{C}$ be a Jordan curve in the plane with finite s-dimensional upper Minkowski content, and let ζ be a conformal map from the exterior of the unit disc, $\{z \in \mathbb{C} : |z| > 1\}$, to the exterior of $\mathcal{C}$. Then the extension of ζ to $\mathbb{T}$, considered as a function on the circle $\mathbb{T}$, satisfies*

$$[F, M_\zeta] \in \mathcal{L}_{s,\infty}.$$

Proof. Without loss of generality, we may assume that the point 0 is in the interior of $\mathcal{C}$, and also ζ may be chosen such that as $|z| \to \infty$ we have $|\zeta(z)| \to \infty$. Define the function η on $\mathbb{D} \setminus \{0\}$ by

$$\eta(z) := \zeta(z^{-1})^{-1}.$$

Since 0 is in the interior of $\mathcal{C}$, the range of $\zeta(z^{-1})$ is bounded away from zero, so η is bounded in any punctured neighbourhood of zero and so has holomorphic extension to $\mathbb{D}$, and by our assumption is extended to $\mathbb{D}$ by defining $\eta(0) = 0$. Since ζ is injective, η is also injective and hence is a conformal equivalence onto its image. Since $0 \notin \mathcal{C}$, the image $\mathcal{C}^{-1}$ is also a Jordan curve. Hence, η is a conformal equivalence between $\mathbb{D}$ and the interior of the Jordan curve $\mathcal{C}^{-1}$.

For all $\delta > 0$, by definition we have

$$S_\delta(\mathcal{C}^{-1}) = \bigcup_{z \in \mathcal{C}} B(z^{-1}, \delta).$$

Since the function $z \mapsto z^{-1}$ is Lipschitz when restricted to the complement of any ball containing 0, then for any $\varepsilon > 0$ there exists a constant $C > 0$ such that for all $|z| > \varepsilon$ and all $\delta < \varepsilon/2$, we have

$$B(z^{-1}, \delta) \subseteq B(z, C\delta)^{-1}.$$

Hence, for δ sufficiently small, the inclusion

$$S_\delta(\mathcal{C}^{-1}) \subseteq S_{C\delta}(\mathcal{C})^{-1}$$

holds.

The Jacobian of the function $z \mapsto z^{-1}$ is uniformly bounded on compact subsets of $\mathbb{C} \setminus 0$. Hence, there is a constant $K > 0$ (depending on $\mathcal{C}$) such that for δ sufficiently small,

$$|S_{C\delta}(\mathcal{C})^{-1}| \leq K|S_{C\delta}(\mathcal{C})| = O(\delta^{2-s}).$$

So, finally, $|S_\delta(\mathcal{C}^{-1})| = O(\delta^{2-s})$. Hence, $\mathcal{C}^{-1}$ has finite s-dimensional upper Minkowski content.

Let W be the unitary map on $L_2(\mathbb{T})$ which maps the basis function z^n to z^{-n} for all $n \in \mathbb{Z}$. Then $W^*FW = -F + R$, where R is a rank-one map, and $M_\zeta = (WM_\eta W^*)^{-1}$. Thus,

$$\begin{aligned}[F, M_\zeta] &= -(WM_\eta W^*)^{-1}[F, WM_\eta W^*](WM_\eta W^*)^{-1} \\ &= -(WM_\eta W^*)^{-1}W[-F+R, M_\eta]W^*(WM_\eta W^*)^{-1}.\end{aligned}$$

So, finally, $[F, M_\zeta] \in \mathcal{L}_{s,\infty}$. □

Theorem 4.5 will yield Theorem 1.1(a) once it is shown that the Julia sets of quadratic polynomials ϕ_c (when c is in the main cardioid of the Mandelbrot set) are Jordan curves with finite p-dimensional upper Minkowski content.

5. *Julia sets*

We now specialize to Jordan curves which arise as Julia sets of quadratic polynomials. We use the concepts of Hausdorff measure and Hausdorff dimension, conventionally defined as follows (see e.g. [**7**, §2.4]).

Let S be a Borel subset of $\mathbb{R}^d$ and let $A \subseteq S$ be Borel. Let $s, \delta > 0$ and define

$$\mathcal{H}^s_\delta(A) := \inf\left\{\sum_{j=1}^n r_j^s : \text{ there is a covering of } A \text{ with balls with radii } r_j < \delta\right\}.$$

The Hausdorff measure $m_s(A)$ is defined to be $\sup_{\delta>0} \mathcal{H}^s_\delta(A)$. The assignment $A \mapsto m_s(A)$ is then a Borel measure on S, and the Hausdorff dimension of S is defined to be the infimum of the set of all s such that $m_s(S)$ is zero.

Let $c \in \mathbb{C}$ and define ϕ_c to be the quadratic polynomial $\phi_c(z) = z^2 + c$. We denote the k-fold iteration of the function ϕ_c with itself by ϕ_c^k.

Definition 5.1. The Julia set J of ϕ_c is the boundary of the set of points $z \in \mathbb{C}$ such that $\phi_c^k(z)$ remains bounded as $k \to \infty$ (see [**7**, §14.1] and [**1**, Ch. 8]). The Mandelbrot set M is defined to be the set of c such that the sequence of iterates $\{\phi_c^n(0)\}_{n\geq 0}$ is bounded. Let p be the Hausdorff dimension of J.

It is well known (see [**7**, Theorem 14.16]) that J is connected if and only if $c \in M$. It is also well known that J is invariant under ϕ_c (see [**1**, Ch. 3, Theorem 1.3]).

The map ϕ_c has at most two fixed points, given as $\frac{1}{2}(1 \pm \sqrt{1-4c})$. A fixed point $z \in \mathbb{C}$ is called *attracting* if $|\phi_c'(z)| < 1$. Since $\phi_c'(z) = 2z$, one can then see that ϕ_c has an attracting fixed point if and only if $c \in \{(\mu/2)(1-\mu/2) : |\mu| < 1\}$. This set is called the main cardioid of M, and it is known that if c is in the main cardioid of M, then J is a Jordan curve (see [**1**, Ch. 8, Theorem 1.3]).

For a given c, the map ϕ_c is called *expanding* on J if for all $z \in J$, there exists $k > 0$ such that $|(\phi_c^k)'(z)| > 1$.

LEMMA 5.2. *If c is in the main cardioid of M, then ϕ_c is expanding on J.*

Proof. This is essentially a special case of the well-known necessary and sufficient criterion for a polynomial to be expanding on its Julia set, as found in [**6**, Theorem III.7.1], [**1**, Theorem V.2.2] and [**12**, Theorem 19.1]. The criterion states that for a polynomial to be expanding on its Julia set it is necessary and sufficient for the closure of the forward orbits of its critical points to be disjoint from its Julia set.

In the case of ϕ_c, there is a unique critical point at 0, so for ϕ_c to be expanding on J it is necessary and sufficient that the closure of $\{\phi_c^k(0)\}_{k=0}^{\infty}$ is disjoint from J. However, if c is in the main cardioid of the Mandelbrot set, ϕ_c has an attracting fixed point. However, it is known that the basin of attraction of an attracting periodic orbit contains a critical point (see [**1**, Theorem III.2.2]). Hence, $\{\phi_c^k(0)\}_{k=0}^{\infty}$ has a single limit point at the attracting fixed point of ϕ_c. The attracting fixed point is not in J. Hence, ϕ_c is expanding on J. □

From [**17**, Theorem 4], the Julia set of an expanding map has Hausdorff dimension strictly between 1 and 2. So, from Lemma 5.2, we can now state the following result.

COROLLARY 5.3. *If $c \neq 0$ is in the main cardioid of the Mandelbrot set, then the Hausdorff dimension p of J satisfies $1 < p < 2$.*

5.1. *Minkowski content of the Julia set.* The following proposition allows us to apply the results of §4 to the Julia set J. Let p denote the Hausdorff dimension of J.

PROPOSITION 5.4. *If c is in the main cardioid of M, then the Julia set J has finite upper p-dimensional Minkowski content and positive lower p-dimensional Minkowski content.*

Proof. Let $\delta > 0$. The set $S_\delta(J)$ can be written as a union of balls of radius δ,

$$S_\delta(J) = \bigcup_{z\in J} B(z, \delta).$$

By the Vitali covering lemma, there is a disjoint finite subset $\{B(z_j, \delta)\}_{j=1}^{K(\delta)}$ such that

$$\bigcup_{j=1}^{K(\delta)} B(z_j, \delta) \subseteq S_\delta(J) \subseteq \bigcup_{j=1}^{K(\delta)} B(z_j, 5\delta). \tag{5.1}$$

Since the finite set $\{B(z_j, \delta)\}_{j=1}^{K(\delta)}$ is disjoint, applying the Lebesgue measure to (5.1),

$$\sum_{j=1}^{K(\delta)} |B(z_j, \delta)| \le |S_\delta(J)| \le \sum_{j=1}^{K(\delta)} |B(z_j, 5\delta)|.$$

So,

$$K(\delta)\pi\delta^2 \le |S_\delta(J)| \le 25K(\delta)\pi\delta^2. \tag{5.2}$$

Let m_p denote the p-dimensional Hausdorff measure on J. Now, applying m_p to (5.1),

$$\sum_{j=1}^{K(\delta)} m_p(B(z_j, \delta) \cap J) \le m_p(J) \le \sum_{j=1}^{K(\delta)} m_p(B(z_j, 5\delta) \cap J).$$

Now we refer to [**18**, Theorem 2.3], where it is stated that there exist constants $\alpha, \beta > 0$ such that for all $r \in (0, 1)$ and $z \in J$, we have

$$\alpha r^p \le m_p(B(z, r) \cap J) \le \beta r^p.$$

So,

$$K(\delta)\alpha\delta^p \le m_p(J) \le K(\delta)\beta 5^p \delta^p. \tag{5.3}$$

Rearranging the inequalities (5.3), we obtain

$$\frac{m_p(J)}{5^p\beta}\delta^{-p} \le K(\delta) \le \frac{m_p(J)}{\alpha}\delta^{-p}. \tag{5.4}$$

Combining (5.2) and (5.4) yields

$$\frac{m_p(J)\pi}{5^p\beta}\delta^{2-p} \le |S_\delta(J)| \le \frac{25\pi m_p(J)}{\alpha}\delta^{2-p}.$$

Since $m_p(J)\pi/5^p\beta > 0$, the lower p-dimensional Minkowski content is positive and, since $25m_p(J)\pi/\alpha < \infty$, the upper p-dimensional Minkowski content is finite. □

Remark 5.5. Due to Proposition 5.4, the Julia set J of ϕ_c has finite upper and positive lower p-Minkowski content and hence has Minkowski dimension equal to p. So, for this particular class of Julia sets, the Hausdorff and Minkowski dimensions coincide. For general subsets of $\mathbb{R}^d$, one can only say that the Hausdorff dimension is a lower bound for the Minkowski dimension [**7**, equation (3.17)].

5.2. *Conformal measures on the Julia set.* Let $q \in (0, \infty)$ and let ν be a Borel measure on J. The measure ν is said to be q-conformal with respect to ϕ_c (in the sense of Sullivan [**17**, Theorem 3]) if for any measurable set $A \subseteq J$ such that $\phi_c|A$ is injective, we have

$$\nu(\phi_c(A)) = \int_A |\phi_c'(z)|^q \, d\nu(z).$$

Conditions which guarantee the uniqueness of a q-conformal measure with respect to ϕ_c have been previously studied; of particular interest is the case where $q = p$, the Hausdorff dimension of J. We refer to [**17**, Theorem 4], where it is proved that there is (up to a scaling factor) a unique p-conformal measure for ϕ_c when ϕ_c is an expanding map. Moreover, [**17**, Theorem 4] states that this essentially unique measure coincides with the Hausdorff measure on J.

5.3. *Conformal equivalence of the exterior of J with the exterior of the unit disc.* By the Riemann mapping theorem, we can choose a conformal map Z from the exterior of the unit disc to the exterior of J. By the Carathèodory theorem [**9**, Theorem 3.1], Z extends continuously to the boundary, $Z : \mathbb{T} \to J$. It is known as a special case of [**1**, Ch. 2, Theorem 4.1] and [**12**, Theorem 18.3] that Z can be chosen such that

$$Z(z^2) = \phi_c(Z(z)) = Z(z)^2 + c \quad \text{for all } |z| \ge 1. \tag{5.5}$$

The above equation is due to Böttcher, and implies that the map Z provides a conjugacy between the endomorphism $\phi_c : J \to J$ and the squaring map $z \mapsto z^2$ on the unit circle.

A combination of Corollary 5.4 and Theorem 4.5 immediately yields Theorem 1.1(a). That is, that $[F, M_Z] \in \mathcal{L}_{p,\infty}$.

The reason for considering Z as a mapping from the exterior of the unit disc to the exterior of J is precisely so that (5.5) holds.

Indeed, [**1**, Ch. 2, Theorem 4.1] shows that there is a conformal map Z such that $Z(z^2) = \phi_c(Z(z))$ defined for all z in a neighbourhood of a superattracting fixed point of the extension of ϕ_c to the Riemann sphere. The extension of the map $z \mapsto \phi_c(z)$ has precisely one superattracting fixed point on the Riemann sphere at ∞ (See for example the discussion of superattracting fixed points in [**12**, Ch. 4]).

For the remainder of this text, we assume that Z satisfies (5.5). This has a number of immediate consequences: for example it can be easily seen from (5.5) that $Z(-z) = -Z(z)$ for all $z \in \mathbb{T}$, so Z is an odd function and the Julia set J is invariant under the map $z \mapsto -z$.

6. *The conformal trace formula*

As in the previous section, we assume that $c \neq 0$ is in the main cardioid of the Mandelbrot set, so that the Julia set J of ϕ_c is a Jordan curve of Hausdorff dimension $1 < p < 2$, with finite upper and positive lower p-dimensional Minkowski content, and moreover that there is a unique (up to scaling) p-conformal measure on J with respect to ϕ_c. Everywhere in this section, Z is a fixed conformal map from the open unit disc $\mathbb{D}$ to the interior of J, identified with its continuous extension to $\mathbb{T}$, and satisfying (5.5).

Due to Theorem 4.5, we have $|[F, M_Z]|^p \in \mathcal{L}_{1,\infty}$, so the following functional is well defined and bounded on $C(J)$.

Definition 6.1. Let φ be a continuous trace on $\mathcal{L}_{1,\infty}$. Due to Theorem 4.5, we may consider the linear functional l_φ on $C(J)$ given by

$$l_\varphi(f) := \varphi(M_{f\circ Z}|[F, M_Z]|^p), \quad f \in C(J).$$

Remark 6.2. Suppose that φ in definition 6.1 is positive. By the Riesz theorem, there is a unique regular positive Borel measure ν_φ on J with the normalization $\nu_\varphi(J) = 1$ such that

$$\varphi(M_{f\circ Z}|[F, M_Z]|^p) = K(\varphi, c) \int_J f \, d\nu_\varphi,$$

where $K(\varphi, c)$ is a constant.

The first part of the following proposition appears as [**2**, Ch. 4, §3.β, Theorem 8(a)].

PROPOSITION 6.3. *Let $f \in C(\mathbb{T})$. Then:*
(1) $[M_f, [F, M_Z]] \in (\mathcal{L}_{p,\infty})_0$*; and*
(2) $[M_f, |[F, M_Z]|^p] \in (\mathcal{L}_{1,\infty})_0$.

Proof. For both part (1) and part (2), it suffices to prove the result for $f(z) = e_n(z) = z^n$, $n \in \mathbb{Z}$, due to linearity and continuity.

First we prove part (1). Since M_f commutes with M_Z,

$$M_{e_n}[F, M_Z]M_{e_n}^* = [M_{e_n}FM_{e_n}^*, M_Z].$$

However, it can be computed that

$$M_{e_n}FM_{e_n}^* e_k = \operatorname{sgn}(k-n)e_k \quad \text{for all } k \in \mathbb{Z}.$$

Hence, $M_{e_n}FM_{e_n}^* - F$ is a finite-rank operator and in particular is in $(\mathcal{L}_{p,\infty})_0$. Thus,

$$[M_{e_n}FM_{e_n}^*, M_Z] - [F, M_Z] = M_{e_n}[F, M_Z]M_{e_n}^* - [F, M_Z] \in (\mathcal{L}_{p,\infty})_0. \tag{6.1}$$

Multiplying (6.1) on the right by M_{e_n} yields $[M_{e_n}, [F, M_z]] \in (\mathcal{L}_{p,\infty})_0$, thus completing the proof of part (1).

Now we prove part (2). Applying [**2**, Ch. 4, §3.β, Proposition 10] to (6.1) yields

$$|M_{e_n}[F, M_Z]M_{e_n}^*|^p - |[F, M_Z]|^p \in (\mathcal{L}_{1,\infty})_0.$$

Since M_{e_n} is unitary, it follows that $|M_{e_n}[F, M_Z]M_{e_n}^*|^p = M_{e_n}|[F, M_Z]|^p M_{e_n}^*$, so

$$M_{e_n}|[F, M_Z]|^p M_{e_n}^* - |[F, M_Z]|^p \in (\mathcal{L}_{1,\infty})_0.$$

As in part (1), multiplying on the right by M_{e_n} yields $[M_{e_n}, |[F, M_Z]|^p] \in (\mathcal{L}_{1,\infty})_0$, thus completing the proof. □

We wish to show that ν_φ from Remark 6.2 is p-conformal with respect to ϕ_c, thus identifying it as the unique such measure on J (up to a constant). Let U be the linear map on $L_2(\mathbb{T})$ defined by $(Uh)(z) = h(z^2)$. Let e_n denote the nth exponential basis function on the circle, $e_n(z) := z^n$, $n \in \mathbb{Z}$. It is clear that U is a partial isometry on $L_2(\mathbb{T})$, since $Ue_n = e_{2n}$ and it is easily computed that $U^*e_{2n} = e_n$ and $U^*e_{2n+1} = 0$. One can then see that $U^*U = 1$ and $UU^* = P$, where P is the projection on $L_2(\mathbb{T})$ defined by $Pe_{2n} = e_{2n}$ and $Pe_{2n+1} = 0$ for all $n \in \mathbb{Z}$.

LEMMA 6.4. *Let $f \in C(\mathbb{T})$ and let φ be a continuous trace on $\mathcal{L}_{1,\infty}$. Then*

$$\varphi(M_f|[F, M_Z]|^p P) = \tfrac{1}{2}\varphi(M_f|[F, M_Z]|^p).$$

Proof. Let V be the unitary map $V = M_{e_1}$. Note that $1 - P = VPV^*$. So,

$$\varphi(M_f|[F, M_Z]|^p) = \varphi(M_f|[F, M_Z]|^p(P + VPV^*)).$$

Using the cyclicity of the trace φ, it follows that

$$\varphi(M_f|[F, M_Z]|^p) = \varphi(M_f|[F, M_Z]|^p P) + \varphi(V^*M_f|[F, M_Z]|^p VP). \tag{6.2}$$

However, V^* is just multiplication by e_{-1}, so it commutes with M_f. Now applying Lemma 6.3(2),

$$[V^*, |[F, M_Z]|^p] \in (\mathcal{L}_{1,\infty})_0. \tag{6.3}$$

So, commuting V^* and M_f, and recalling that $V^*V = 1$, we obtain

$$\begin{aligned}\varphi(V^*M_f|[F, M_Z]|^p VP) &= \varphi(M_f V^*|[F, M_Z]|^p VP)\\ &\overset{(6.3)}{=} \varphi(M_f|[F, M_Z]|^p V^*VP)\\ &= \varphi(M_f|[F, M_Z]|^p P).\end{aligned}$$

Thus, from (6.2), we obtain

$$\varphi(M_f|[F, M_Z]|^p) = 2\varphi(M_f|[F, M_Z]|^p P)$$

and dividing through by 2 yields exactly the desired result. □

The following theorem consists of special cases of parts (b) and (c) of [**2**, Ch. 4, §3.β, Theorem 8]; however, the proof of part (c) in that reference was not included and so for the convenience of the reader we supply a self-contained proof.

THEOREM 6.5. *Let f be a complex polynomial. Then:*
(1) $[F, f(M_Z)] - f'(M_Z)[F, M_Z] \in (\mathcal{L}_{p,\infty})_0$;
(2) $|[F, f(M_Z)]|^p - |f'(M_Z)|^p|[F, M_Z]|^p \in (\mathcal{L}_{1,\infty})_0$.

Proof. First we prove part (1). Due to linearity, it suffices to prove (1) for $f(z) = z^n$, $n \geq 0$. By the Leibniz rule,

$$[F, M_Z^n] = \sum_{k=1}^{n} M_Z^{n-k}[F, M_Z]M_Z^{k-1}.$$

From Lemma 6.3(1), $[[F, M_Z], M_Z^{k-1}] \in (\mathcal{L}_{p,\infty})_0$ for all $k \geq 1$, so

$$[F, M_Z^n] - nM_Z^{n-1}[F, M_Z] \in (\mathcal{L}_{p,\infty})_0.$$

Since $f'(z) = nz^{n-1}$, this completes the proof of part (1).

Now we prove part (2). Firstly, we apply [**2**, Ch. 4, §3.β, Proposition 10] to the difference $[F, f(M_Z)] - f'(M_Z)[F, M_Z]$, which gives us

$$|[F, f(M_Z)]|^p - |f'(M_Z)[F, M_Z]|^p \in (\mathcal{L}_{1,\infty})_0.$$

Note that†

$$|f'(M_Z)[F, M_Z]| = ||f'(M_Z)|[F, M_Z]|.$$

Hence,

$$|[F, f(M_Z)]|^p - ||f'(M_Z)|[F, M_Z]|^p \in (\mathcal{L}_{1,\infty})_0. \tag{6.4}$$

From Proposition 6.3(1), since the function $|f' \circ Z|$ is continuous on $\mathbb{T}$, we have

$$[|f'(M_Z)|, [F, M_Z]] \in (\mathcal{L}_{p,\infty})_0.$$

Again applying [**2**, Ch. 4, §3.β, Proposition 10], it follows that

$$||f'(M_Z)|[F, M_Z]|^p - |[F, M_Z]|f'(M_Z)||^p \in (\mathcal{L}_{1,\infty})_0. \tag{6.5}$$

Subtracting (6.5) from (6.4) yields

$$|[F, f(M_Z)]|^p - |[F, M_Z]|f'(M_Z)||^p \in (\mathcal{L}_{1,\infty})_0. \tag{6.6}$$

Hence, since $|[F, M_Z]|f'(M_Z)|| = ||[F, M_Z]||f'(M_Z)||$,

$$|[F, f(M_Z)]|^p - ||[F, M_Z]||f'(M_Z)||^p \in (\mathcal{L}_{1,\infty})_0.$$

Since the function $|f' \circ Z|^{1/2}$ is continuous on $\mathbb{T}$, from Proposition 6.3(1) (with M_f in that proposition given as $M_{|f' \circ Z|^{1/2}}$), the double commutator $[|f'(M_Z)|^{1/2}, [F, M_Z]]$ is in $(\mathcal{L}_{p,\infty})_0$. Taking the adjoint, we also have that $[|f'(M_Z)|^{1/2}, [F, M_Z]^*] \in (\mathcal{L}_{p,\infty})_0$. Thus, from [**5**, Lemma 6.2],

$$[|f'(M_Z)|^{1/2}, |[F, M_Z]|] \in (\mathcal{L}_{p,\infty})_0. \tag{6.7}$$

Multiplying (6.7) on the right by $|f'(M_Z)|^{1/2}$, it follows that

$$|f'(M_Z)|^{1/2}|[F, M_Z]||f'(M_Z)|^{1/2} - |[F, M_Z]||f'(M_Z)| \in (\mathcal{L}_{p,\infty})_0.$$

† For any operators A and B, we have $||A|B|^2 = B^*A^*AB = |AB|^2$, so $||A|B| = |AB|$.

Applying [**2**, Ch. 4, §3.β, Proposition 10], it follows that

$$(|f'(M_Z)|^{1/2}|[F, M_Z]||f'(M_Z)|^{1/2})^p - ||[F, M_Z]||f'(M_Z)||^p \in (\mathcal{L}_{1,\infty})_0. \tag{6.8}$$

Subtracting (6.8) from (6.6) yields

$$|[F, f(M_Z)]|^p - (|f'(M_Z)|^{1/2}|[F, M_Z]||f'(M_Z)|^{1/2})^p \in (\mathcal{L}_{1,\infty})_0. \tag{6.9}$$

From (6.7), $[|f'(M_Z)|^{1/2}, |[F, M_Z]|] \in (\mathcal{L}_{p,\infty})_0$, so we may apply [**5**, Lemma 5.3] to get

$$|[F, M_Z]|^p|f'(M_Z)|^p - (|f'(M_Z)|^{1/2}|[F, M_Z]||f'(M_Z)|^{1/2})^p \in (\mathcal{L}_{1,\infty})_0. \tag{6.10}$$

Subtracting (6.10) from (6.9) yields

$$|[F, f(M_Z)]|^p - |[F, M_Z]|^p|f'(M_Z)|^p \in (\mathcal{L}_{1,\infty})_0.$$

Taking the adjoint, we arrive at

$$|[F, f(M_Z)]|^p - |f'(M_Z)|^p|[F, M_Z]|^p \in (\mathcal{L}_{1,\infty})_0. \qquad \square$$

The following pair of lemmas (Lemmas 6.6 and 6.7) contain the details required to prove that ν_φ is p-conformal with respect to ϕ_c.

LEMMA 6.6. *Let φ be a positive continuous trace on $\mathcal{L}_{1,\infty}$ and let ν_φ be the corresponding measure from Remark 6.2. Then ν_φ satisfies the following transformation property: for all $g \in C(J)$, we have*

$$\int_J g\, d\nu_\varphi = \frac{1}{2}\int_J (g \circ \phi_c) \cdot |\phi_c'|^p \, d\nu_\varphi.$$

Proof. Let $q(z) = z^2$, so that (5.5) may be restated as

$$\phi_c \circ Z = Z \circ q. \tag{6.11}$$

Recall that U is defined as the linear operator on $L_2(\mathbb{T})$ given by $(Uh)(z) = h(z^2) = (h \circ q)(z)$. Since $U^*U = 1$, due to the cyclicity of φ, we have

$$\begin{aligned}\varphi(M_{g\circ Z}|[F, M_Z]|^p) &= \varphi(M_{g\circ Z}|[F, M_Z]|^pU^*U)\\ &= \varphi(UM_{g\circ Z}|[F, M_Z]|^pU^*).\end{aligned}$$

Since $UM_h = M_{h\circ q}U$ for all $h \in L_2(\mathbb{T})$, it follows that

$$\varphi(M_{g\circ Z}|[F, M_Z]|^p) = \varphi(M_{g\circ Z\circ q}U|[F, M_Z]|^pU^*).$$

From (6.11), it follows that

$$\varphi(M_{g\circ Z}|[F, M_Z]|^p) = \varphi(M_{g\circ\phi_c\circ Z}U|[F, M_Z]|^pU^*). \tag{6.12}$$

Since U commutes with F,

$$\begin{aligned}U[F, M_Z] &= [F, UM_Z]\\ &= [F, M_{Z\circ q}U]\\ &= [F, M_{Z\circ q}]U.\end{aligned}$$

The same argument yields

$$U[F, M_Z]^* = [F, M_{Z\circ q}]^* U.$$

So,

$$\begin{aligned} U|[F, M_Z]|^2 &= [F, M_{Z\circ q}]^* U[F, M_Z] \\ &= |[F, M_{Z\circ q}]|^2 U. \end{aligned}$$

By induction, for every $n \geq 1$,

$$U|[F, M_Z]|^{2n} = |[F, M_{Z\circ q}]|^{2n} U.$$

Hence, for any polynomial r, we have $Ur(|[F, M_Z]|^2) = r(|[F, M_{Z\circ q}]|^2)U$. Applying the continuous functional calculus with the function $r(t) = |t|^{p/2}$, it then follows that

$$U|[F, M_Z]|^p = |[F, M_{Z\circ q}]|^p U.$$

Returning to (6.12), and using $UU^* = P$, we now have

$$\varphi(M_{g\circ Z}|[F, M_Z]|^p) = \varphi(M_{g\circ\phi_c\circ Z}|[F, M_{Z\circ q}]|^p P). \tag{6.13}$$

Again applying (6.11), it follows that

$$\varphi(M_{g\circ Z}|[F, M_Z]|^p) = \varphi(M_{g\circ\phi_c\circ Z}|[F, M_{\phi_c\circ Z}]|^p P).$$

However, ϕ_c is a polynomial, so we can apply Theorem 6.5(2) to the right-hand side of the above to obtain

$$\begin{aligned} \varphi(M_{g\circ Z}|[F, M_Z]|^p) &= \varphi(M_{g\circ\phi_c\circ Z}|\phi_c'(M_Z)|^p|[F, M_Z]|^p P) \\ &= \varphi(M_{(g\circ\phi_c\circ Z)\cdot|\phi_c'\circ Z|^p}|[F, M_Z]|^p P). \end{aligned}$$

Now applying Lemma 6.4, it follows that

$$\varphi(M_{g\circ Z}|[F, M_Z]|^p) = \tfrac{1}{2}\varphi(M_{(g\circ\phi_c\circ Z)\cdot|\phi_c'\circ Z|^p}|[F, M_Z]|^p),$$

which is exactly the desired result. □

Recall from (5.5) that $Z(-z) = -Z(z)$ for all $z \in \mathbb{T}$ and that $J = -J$. The following lemma shows that the measure ν_φ from Remark 6.2 is invariant under the map $z \mapsto -z$.

LEMMA 6.7. *Let ν_φ be the measure from Remark 6.2 corresponding to a positive continuous trace φ. Then, for all $g \in C(J)$, we have*

$$\int_J g(z)\, d\nu_\varphi(z) = \int_J g(-z)\, d\nu_\varphi(z).$$

Proof. Since $Z(-z) = -Z(z)$ for all $z \in \mathbb{T}$, we have

$$(g \circ -Z)(z) = (g \circ Z)(-z).$$

Let V be the unitary map on $L_2(\mathbb{T})$ which maps $h(z)$ to $h(-z)$ for all $h \in L_2(\mathbb{T})$. Then $(V(g \circ Z))(z) = (g \circ Z)(-z) = (g \circ -Z)(z)$. From the definition of ν_φ, there is a constant $K = K(\varphi, c)$ such that

$$\begin{aligned} \int_J g(-z)\, d\nu_\varphi(z) &= K\varphi(M_{g\circ -Z}|[F, M_Z]|^p) \\ &= K\varphi(VM_{g\circ Z}V^*|[F, M_Z]|^p). \end{aligned}$$

Using the cyclicity of the trace φ and that V is unitary,

$$\varphi(VM_{g\circ Z}V^*|[F, M_Z]|^p) = \varphi(M_{g\circ Z}|V^*[F, M_Z]V|^p).$$

Since V commutes with F, and $V^*M_ZV = -M_Z$, it follows that

$$\begin{aligned}\int_J g(-z)\, d\nu_\varphi(z) &= K\varphi(M_{g\circ Z}|[F, -M_Z]|^p)\\ &= K\varphi(M_{g\circ Z}|[F, M_Z]|^p)\\ &= \int_J g(z)\, d\nu_\varphi(z).\end{aligned}$$ □

The following proposition is the main result of this section.

PROPOSITION 6.8. *The measure ν_φ from Remark 6.2 corresponding to a positive continuous trace φ is p-conformal with respect to the map ϕ_c.*

Proof. Recall that $\phi_c(z) = z^2 + c$, so it easily follows that for all $z, w \in \mathbb{T}$, we have $\phi_c(z) = \phi_c(w)$ if and only if either $w = z$ or $w = -z$. Hence, if A is a subset of $\mathbb{T}$, we have that $\phi_c|_A$ is injective if and only if $A \cap (-A) = \emptyset$.

Additionally, for any $z \in \mathbb{T}$, $\phi_c^{-1}(z)$ consists of exactly two points which differ by a sign. Thus,

$$\phi_c^{-1}(\phi_c(A)) = A \cup -A. \tag{6.14}$$

Let U be an open subset of J such that $U \cap (-U) = \emptyset$ and let g be a continuous function on J supported in $\phi_c(U)$. Since $U \cup (-U)$ is open, the complement $J \setminus (U \cup -U)$ is compact. Hence, $\phi_c(J \setminus U \cup (-U))$ is compact and so $\phi_c(U)$ is open. Since g is supported on $\phi_c(U)$, we trivially have

$$\int_{\phi_c(U)} g\, d\nu_\varphi = \int_J g\, d\nu_\varphi.$$

By Lemma 6.6, we have

$$\int_J g\, d\nu_\varphi = \frac{1}{2}\int_J (g\circ\phi_c)|\phi_c'|^p\, d\nu_\varphi.$$

Hence,

$$\int_{\phi_c(U)} g\, d\nu_\varphi = \frac{1}{2}\int_J (g\circ\phi_c)|\phi_c'|^p\, d\nu_\varphi. \tag{6.15}$$

Since g is supported on $\phi_c(U)$, the composition $g \circ \phi_c$ is supported on $\phi_c^{-1}(\phi_c(U))$, which from (6.14) is $U \cup (-U)$. By our assumption that $U \cap (-U) = \emptyset$,

$$\begin{aligned}\int_J (g\circ\phi_c)|\phi_c'|^p\, d\nu_\varphi &= \int_{U\cup(-U)} (g\circ\phi_c)|\phi_c'|^p\, d\nu_\varphi\\ &= \int_U (g\circ\phi_c)|\phi_c'|^p\, d\nu_\varphi + \int_{-U} (g\circ\phi_c)|\phi_c'|^p\, d\nu_\varphi.\end{aligned}$$

However, note that $g(\phi_c(z)) = g(\phi_c(-z))$ and $|\phi_c'(z)| = |\phi_c'(-z)|$ for all $z \in \mathbb{T}$. Hence, for all $z \in \mathbb{T}$, the identity $(g\circ\phi_c)(z)|\phi_c'|^p(z) = (g\circ\phi_c)(-z)|\phi_c'|^p(-z)$ holds. Due to

Lemma 6.7, we know that for all Borel sets A, we have $\nu_\varphi(A) = \nu_\varphi(-A)$. Thus, the integral over $-U$ is the same as the integral over U, so

$$\int_J (g \circ \phi_c)|\phi_c'|^p \, d\nu_\varphi = 2 \int_U (g \circ \phi_c)|\phi_c'|^p \, d\nu_\varphi. \tag{6.16}$$

Combining (6.15) and (6.16), we have established that the equality

$$\int_{\phi_c(U)} g \, d\nu_\varphi = \int_U (g \circ \phi_c)|\phi_c'|^p \, d\nu_\varphi$$

holds for all g supported in $\phi_c(U)$. Since $\phi_c|U$ is injective, as g varies over all continuous functions supported in $\phi_c(U)$, $(g \circ \phi_c)|U$ varies over all continuous functions supported in U. So,

$$\sup_{\operatorname{supp}(g)\subseteq\phi_c(U),\|g\|_\infty\le 1} \int_{\phi_c(U)} g \, d\nu_\varphi = \sup_{\operatorname{supp}(h)\subseteq U,\|h\|_\infty\le 1} \int_U h|\phi_c'|^p \, d\nu_\varphi.$$

Since ν_φ is positive, it follows from the Riesz theorem that we have an equality of measures,

$$\nu_\varphi(\phi_c(U)) = \int_U |\phi_c'|^p \, d\nu_\varphi,$$

for all open subsets U such that $U \cap (-U) = \emptyset$. Due to the regularity of the measure ν_φ, it follows that for all Borel subsets A with $A \cap (-A) = \emptyset$,

$$\nu_\varphi(\phi_c(A)) = \int_A |\phi_c'|^p \, d\nu_\varphi. \qquad \square$$

We may now finally complete the proof of Theorem 1.1(b).

COROLLARY 6.9. *Let φ be a continuous Hermitian (not necessarily positive) trace on $\mathcal{L}_{1,\infty}$. There is a constant $K(\varphi, c)$ such that for all $f \in C(J)$,*

$$\varphi(M_{f\circ Z}|[F, M_Z]|^p) = K(\varphi, c) \int_J f \, d\nu, \tag{6.17}$$

where ν is the (essentially unique) p-conformal measure on J with respect to ϕ_c.

Proof. If φ is positive, then this is simply a restatement of Proposition 6.8. For general traces φ, we may use Corollary 2.2 to write $\varphi = \varphi_+ - \varphi_-$ for positive traces φ_+ and φ_-. Then

$$\varphi(M_{f\circ Z}|[F, M_z]|^p) = K(\varphi_+, c) \int_J f \, d\nu_{\varphi_+} - K(\varphi_-, c) \int_J f \, d\nu_{\varphi_-}.$$

Then, applying Proposition 6.8 to the positive traces φ_+ and φ_- individually, we have that ν_{φ_+} and ν_{φ_-} are p-conformal. Hence, the measure $K(\varphi_+, c)\nu_{\varphi_+} - K(\varphi_-, c)\nu_{\varphi_-}$ is p-conformal, so there is a constant $K(\varphi, c)$ such that

$$K(\varphi_+, c)\nu_{\varphi_+} - K(\varphi_-, c)\nu_{\varphi_-} = K(\varphi, c)\nu,$$

where ν is the essentially unique p-conformal measure on J with respect to ϕ_c. $\square$

Remark 6.10. The p-conformal measure on J is identical to the p-dimensional Hausdorff measure on J by [**17**, Theorem 4], so Corollary 6.9 could also be stated with ν denoting the Hausdorff measure m_p.

7. *Non-triviality of the conformal trace formula*

The remaining task is to show that the formula (6.17) is non-trivial: that is, that there is φ such that $K(\varphi, c) > 0$. We show that indeed such a φ does exist and is given by a Dixmier trace tr_ω, where ω is a dilation-invariant extended limit such that $\omega \circ \log$ is also dilation invariant†. We achieve this using [**11**, Theorem 8.6.8], which states that if ω is a dilation-invariant extended limit on $L_\infty(0, \infty)$ such that $\omega \circ \log$ is still dilation invariant, then the Dixmier trace tr_ω is equal to the following ζ-function residue:

$$\mathrm{tr}_\omega(T) = (\omega \circ \log)\left(t \mapsto \frac{1}{t}\,\mathrm{tr}(T^{1+1/t})\right), \quad 0 \le T \in \mathcal{L}_{1,\infty}.$$

Hence, to show that $\mathrm{tr}_\omega(|[F, Z]|^p) > 0$, it suffices to show that

$$\liminf_{s \to 0} s \cdot \mathrm{tr}(|[F, Z]|^{p+s}) > 0.$$

The crucial result is the following, which is stated as [**2**, Ch. 4, §3.α, Proposition 7].

PROPOSITION 7.1. *Let $\mathcal{C}$ be a Jordan curve with interior Ω and let ξ be a conformal map $\xi : \mathbb{D} \to \Omega$. Since ξ extends continuously to $\mathbb{T}$, we may consider ξ as a function on $\mathbb{T}$. Let $p_0 > 1$. Then then there are positive constants C_{p_0} and c_{p_0} such that*

$$c_{p_0} \int_\Omega \mathrm{dist}(z, \partial\Omega)^{p-2}\, dz\, d\bar{z} \le \mathrm{tr}(|[F, M_\xi]|^p) \le C_{p_0} \int_\Omega \mathrm{dist}(z, \partial\Omega)^{p-2}\, dz\, d\bar{z}$$

for all $p > p_0$.

Proof. This result is an immediate consequence of Lemma 4.1 and Theorem 3.1. □

PROPOSITION 7.2. *Let $\mathcal{C}$ be a Jordan curve with finite upper p-dimensional Minkowski content and positive lower p-dimensional Minkowski content. Let Ω be the interior of $\mathcal{C}$, so that $\partial\Omega = \mathcal{C}$. Then*

$$\liminf_{s \to 0} s \cdot \int_\Omega \mathrm{dist}(z, \mathcal{C})^{p+s-2}\, dz\, d\bar{z} > 0.$$

Proof. By the assumption that $\mathcal{C}$ has positive lower p-Minkowski content and finite upper p-Minkowski content, there are constants $b, B > 0$ such that

$$b\delta^{2-p} \le |S_\delta(C) \cap \Omega| \le B\delta^{2-p} \quad \text{for all } \delta > 0.$$

Let $\lambda > 0$. Define $A_k \subseteq \Omega$, $k \ge 1$, by

$$A_k = \{z \in \Omega : \mathrm{dist}(z, \mathcal{C}) \in [\lambda^{-k}, \lambda^{1-k})\}.$$

So,

$$\begin{aligned} |A_k| &= |S_{\lambda^{1-k}}(\mathcal{C}) \cap \Omega| - |S_{\lambda^{-k}}(\mathcal{C}) \cap \Omega| \\ &\ge (b\lambda^{(1-k)(2-p)} - B\lambda^{-k(2-p)}) \\ &= (b\lambda^{2-p} - B)\lambda^{-k(2-p)}. \end{aligned}$$

† For an extended limit $\omega \in L_\infty(0, \infty)^*$, the notation $\omega \circ \log$ denotes the extended limit defined as $f \mapsto \omega(f \circ \max\{\log, 0\})$.

Now fix $\lambda > 1$ such that $b_0 := b\lambda^{2-p} - B > 0$. Then

$$\int_\Omega \operatorname{dist}(z, \mathcal{C})^{p+s-2}\, dz\, d\bar{z} \geq \sum_{k=0}^{\infty} b_0 \lambda^{-kp-ks+kp} \lambda^{-2k+kp}$$
$$= \sum_{k=0}^{\infty} b_0 \lambda^{-ks}$$
$$= \frac{b_0}{1-\lambda^{-s}}.$$

From the l'Hôpital rule, the limit as $s \to 0$ of $s/(1-\lambda^{-s})$ is $1/\log(\lambda)$. Hence, $\liminf_{s\to 0} b_0 s/(1-\lambda^{-s}) = b_0/\log(\lambda) > 0$. □

Due to Lemma 5.4, we can apply the above proposition to immediately obtain the following result.

COROLLARY 7.3. *Let ω be a dilation-invariant extended limit on $L_\infty(0, \infty)$ such that $\omega \circ \log$ is still dilation invariant. Then*

$$\operatorname{tr}_\omega(|[F, M_Z]|^p) > 0.$$

A. *Appendix. Function spaces and commutators with the Hilbert transform*

This section of the appendix is devoted to a self-contained proof of Theorem 3.1.

Definition A.1. Let μ be the measure on $\mathbb{D}$ defined by

$$d\mu(z) = \frac{dz\, d\bar{z}}{(1-|z|^2)^2}$$

(i.e. the Poincaré disc model volume form). For $p \in (0, \infty]$, the space $A_p^{1/p}$ is defined to be the set of functions f holomorphic in the unit disc satisfying $z \mapsto (1-|z|^2)^2 |f''(z)| \in L_p(\mathbb{D}, \mu)$ with the seminorm $\|f\|_{A_p^{1/p}} := \|(1-|z|^2)^2 |f''(z)|\|_{L_p(\mathbb{D},\mu)}$.

We also define the space $C_p^{1/p}$ of functions f holomorphic in the interior of the unit disc satisfying $(1-|z|^2)|f'(z)| \in L_p(\mathbb{D}, \mu)$ with corresponding seminorm $\|f\|_{C_p^{1/p}} = \|(1-|z|^2)|f'(z)|\|_{L_p(\mathbb{D},\mu)}$.

The following result shows that for $p \in (1, 2)$, the spaces $A_p^{1/p}$ and $C_p^{1/p}$ coincide, and in fact there is an equivalence of (semi)norms with constants that are uniform for $p \in (1+\varepsilon, 2)$ for all $\varepsilon > 0$.

THEOREM A.2. *Let $p_0 > 1$ and let f be a function holomorphic in the unit disc satisfying $f'(0) = 0$. There exist constants $k, K > 0$ (depending only on p_0) such that for all $p_0 < p < 2$, we have*

$$k\|f\|_{A_p^{1/p}} \leq \|f\|_{C_p^{1/p}} \leq K\|f\|_{A_p^{1/p}}.$$

Proof. Let $h(z) := f'(z)(1-|z|^2)$ and $g(z) := f''(z)(1-|z|^2)^2$, and fix $p_0 > 1$. The assertion of this theorem is equivalent to saying that there are positive constants $k, K > 0$ such that for all $p \in [p_0, 2]$,

$$k\|h\|_{L_p(\mathbb{D},\mu)} \leq \|g\|_{L_p(\mathbb{D},\mu)} \leq K\|h\|_{L_p(\mathbb{D},\mu)}. \tag{A.1}$$

By applying the method of complex interpolation as in [**5**, Lemma 3.4], it suffices to prove (A.1) for $p = p_0$ and $p = 2$. Firstly, for $p = 2$, we note that the spaces $A_2^{1/2}$ and $C_2^{1/2}$ are Hilbert spaces, and that the functions $e_n(z) = z^n$, $n \geq 0$, are orthogonal in both spaces with dense linear span. Hence, for $p = 2$, it suffices to prove (A.1) for $f(z) = z^n$, $n \geq 0$. When $f(z) = z^n$, we have $h(z) = nz^{n-1}(1 - |z|^2)$ and $g(z) = n(n-1)z^{n-2}(1 - |z|^2)^2$. Then

$$\begin{aligned}\|h\|^2_{L_2(\mathbb{D},\mu)} &= \int_{\mathbb{D}} n^2|z|^{2n-2}\,dz\,d\overline{z}\\ &= \frac{2\pi n^2}{2n}\\ &= \pi n\end{aligned}$$

and, furthermore,

$$\begin{aligned}\|g\|^2_{L_2(\mathbb{D},\mu)} &= \int_{\mathbb{D}} n^2(n-1)^2|z|^{2n-4}(1-|z|^2)^2\,dz\,d\overline{z}\\ &= 2\pi n^2(n-1)^2\int_0^1 r^{2n-3}(1-r^2)^2 dr\\ &= \pi n^2(n-1)^2\frac{2}{n(n^2-1)}\\ &\leq Kn\end{aligned}$$

for some $K > 0$, hence proving (A.1) for $p = 2$.

The case $p = p_0$ is more subtle. We refer to [**10**, Proposition 1.11], a special case of which states that $f \in C_{p_0}^{1/p_0}$ if and only if $f \in A_{p_0}^{1/p_0}$. We explain how it is possible to modify the proof of [**10**, Proposition 1.11] to obtain the left-hand-side inequality of (A.1) for $p = p_0$. The right-hand-side inequality of (A.1) will then follow from the open mapping theorem, since [**10**, Proposition 1.11] establishes that there is a bijective correspondence between $A_{p_0}^{1/p_0}$ and $C_{p_0}^{1/p_0}$.

We refer to the following formula, given in the proof of [**10**, Proposition 1.11]. If $\beta > \alpha > -1$, $\xi \in L_{p_0}(\mathbb{D}, (1 - |z|^2)^\alpha\, dz\, d\overline{z})$ is holomorphic and n is a positive integer, then there is a universal constant C such that

$$(1-|z|^2)^n\xi^{(n)}(z) = C(1-|z|^2)^n\int_{\mathbb{D}}\frac{(1-|w|^2)^\beta}{(1-z\overline{w})^{2+n+\beta}}\overline{w}^n\xi(w)\,dw\,d\overline{w}.$$

We apply this result with $\xi = f'$, $n = 1$ and $\alpha = p_0 - 2$, which is possible since $p_0 > 1$, so $\alpha > -1$. Fix any $\beta > \alpha$. Thus,

$$(1-|z|^2)f''(z) = C(1-|z|^2)\int_{\mathbb{D}}\frac{(1-|w|^2)^\beta}{(1-z\overline{w})^{3+\beta}}\overline{w}f'(w)\,dw\,d\overline{w}.$$

Restating this in terms of h, g and μ,

$$g(z) = C(1-|z|^2)^2\int_{\mathbb{D}}\frac{(1-|w|^2)^{\beta+1}}{(1-z\overline{w})^{\beta+3}}\overline{w}h(w)\,d\mu(w).$$

It is established in [**10**, Theorem 1.9] that the right-hand side (considered as a function of h) is an integral operator which maps $L_{p_0}(\mu)$ to $L_{p_0}(\mu)$. Hence, there is a constant k such that for all $p > p_0$,

$$k\|g\|_{L_{p_0}(\mathbb{D},\mu)} \leq \|h\|_{L_{p_0}(\mathbb{D},\mu)}. \qquad \square$$

The following two theorems express the equivalence of the $\mathcal{L}_p$ norm of the commutator $[F, M_f]$ with the $A_p^{1/p}$ norm of f. This result is originally due to Peller [**13**, Ch. 6]. We include a proof here since the results in [**13**, Ch. 3] are not stated with bounds on the norms.

THEOREM A.3. *Let f be a function holomorphic in the unit disc such that $f'(0) = 0$. There exists a universal constant $K > 0$ such that for all $p \in [1, 2]$,*

$$\|[F, M_f]\|_p \le K \|f\|_{A_p^{1/p}}. \tag{A.2}$$

Proof. The required inequality (A.2) for $p \in [1, 2]$ can be obtained from the $p = 1$ and $p = 2$ cases by the method of complex interpolation as discussed in [**5**, Lemma 3.4].

The case $p = 2$ is omitted, since it can be verified by computing $\|[F, M_f]\|_p$ and $\|f\|_{A_2^{1/2}}$ for the exponential basis functions $f(z) = z^n$, $n \ge 0$. For the case $p = 1$, we use [**19**, Lemma 2], which implies that if $f \in L_1(\mathbb{D}, dz\, d\bar{z})$ is holomorphic and $f(0) = f'(0) = \cdots = f^{(4)}(0) = 0$, then

$$f(z) = \frac{1}{2} \int_{\mathbb{D}} \frac{(1 - |w|^2)^2 f''(w)}{\overline{w}^2 (1 - z\overline{w})^2}\, dw\, d\overline{w}.$$

Hence,

$$[F, f(z)] = \frac{1}{2} \int_{\mathbb{D}} \frac{(1 - |w|^2)^2 f''(w)}{\overline{w}^2} \left[F, \frac{1}{(1 - z\overline{w})^2}\right] dw\, d\overline{w}. \tag{A.3}$$

However,

$$\left[F, \frac{1}{(1 - z\overline{w})^2}\right] = -(1 - z\overline{w})^{-2} [F, (1 - z\overline{w})^2] (1 - z\overline{w})^{-2}.$$

Since $(1 - z\overline{w})^2 = 1 - 2z\overline{w} + z^2\overline{w}^2$, the commutator $[F, (1 - z\overline{w})^2]$ is of finite rank, and in fact of rank at most 5. So, there is a constant C such that

$$\|[F, (1 - z\overline{w})^{-2}]\|_1 \le C \|(1 - z\overline{w})^{-2}\|_\infty.$$

If $w \ne 0$, then the function on $\mathbb{T}$ $z \mapsto |1 - z\overline{w}|^{-2}$ is maximized when z is as close to $1/\overline{w} = w/|w|^2$ as possible, which is at the point $w/|w|$. Hence, $|1 - z\overline{w}|^{-2} \le (1 - |w|)^{-2}$. Applying the $\mathcal{L}_1$-triangle inequality to (A.3), it follows that

$$\|[F, M_f]\|_1 \le \frac{C}{2} \int_{\mathbb{D}} \frac{|f''(w)|(1 - |w|^2)^2}{|w|^2 (1 - |w|)^2}\, dw\, d\overline{w}.$$

Since $1 \le (1 + |w|) \le 2$, we may simplify this to

$$\|[F, M_f]\|_1 \le C \int_{\mathbb{D}} \frac{|f''(w)|}{|w|^2}\, dw\, d\overline{w}.$$

We now split the integral over $\mathbb{D}$ into the two regions $R_1 := \{z \in \mathbb{D} : |w| \le 1/2\}$ and $R_2 := \{z \in \mathbb{D} : |w| > 1/2\}$. For $w \in R_2$, we have

$$\frac{|f''(w)|}{|w|^2} \le 4|f''(w)|,$$

so

$$\int_{R_2} \frac{|f''(w)|}{|w|^2}\, dw\, d\overline{w} \le 4 \int_{R_2} |f''(w)|\, dw\, d\overline{w}.$$

Since f is holomorphic and $f(0) = f'(0) = \cdots = f^{(4)}(0) = 0$, we have the power series expansion

$$f(w) = \sum_{n>4} \hat{f}(n) w^n.$$

Substituting the series expansion into the integral,

$$\begin{aligned}\int_{R_1} \frac{|f''(w)|}{|w|^2}\, dw\, d\overline{w} &\le \sum_{n>4} \int_{R_1} n(n-1)|\hat{f}(n)||w|^{n-4}\, dw\, d\overline{w} \\ &= \sum_{n>4} 2\pi n(n-1)|\hat{f}(n)| \int_0^{1/2} r^{n-3} dr \\ &= \sum_{n>4} 2\pi \frac{n(n-1)}{n-2} |\hat{f}(n)| 2^{2-n}.\end{aligned}$$

Hence,

$$\int_{R_1} \frac{|f''(w)|}{|w|^2}\, dw\, d\overline{w} \le \sum_{n>4} 2\pi \frac{n(n-1)}{n-2} |\hat{f}(n)| 2^{2-n}. \tag{A.4}$$

Let ξ be a function holomorphic in $\mathbb{D}$; then, by the Cauchy integral formula, for every $r > 0$,

$$\hat{\xi}(n) = \frac{1}{2\pi i} \int_0^1 \frac{\xi(re^{2\pi i t})}{r^{n+1} e^{2\pi i(n+1)t}} \cdot 2\pi i e^{2\pi i t} r\, dt.$$

Hence,

$$|\hat{\xi}(n)| \le \int_0^1 |\xi(re^{2\pi i t})| r^{-n}\, dt.$$

So, in fact,

$$r^{n+1}|\hat{\xi}(n)| \le \int_0^1 |\xi(re^{2\pi i t})| r\, dt.$$

So, integrating over $0 \le r \le 1$,

$$\frac{1}{n+2} |\hat{\xi}(n)| \le \|\xi\|_{L_1(\mathbb{D})}.$$

Applying this result to $\xi = f''$, we have $\hat{\xi}(n) = (n+2)(n+1)\hat{f}(n+2)$, so

$$(n+1)|\hat{f}(n+2)| \le \|f''\|_{L_1(\mathbb{D})}. \tag{A.5}$$

Hence, for all $n > 1$,

$$(n-1)|\hat{f}(n)| \le \|f''\|_{L_1(\mathbb{D})} \quad \text{for all } n \ge 0.$$

Applying this to (A.4), it follows that there exists $C > 0$ such that

$$\begin{aligned}\int_{R_1} \frac{|f''(w)|}{|w|^2}\, dw\, d\overline{w} &\le 2\pi \left(\sum_{n>4} \frac{n}{n-2} 2^{2-n} \right) \|f''\|_{L_1(\mathbb{D})} \\ &\le C \|f\|_{A_1^1}.\end{aligned}$$

This proves the desired result when $f(0) = \cdots = f^{(4)}(0) = 0$. For arbitrary functions f holomorphic in $\mathbb{D}$, consider the function g given by

$$g(z) = f(z) - \sum_{k=0}^{4} \hat{f}(k) z^k.$$

By the preceding argument, there is a constant $K > 0$ such that $\|[F, M_g]\|_1 \leq K\|g\|_{A_1^1}$. Thus, there is a constant $C > 0$ such that

$$\|[F, M_f]\|_1 \leq C(\|f\|_{A_1^1} + |\hat{f}(1)| + |\hat{f}(2)| + |\hat{f}(3)| + |\hat{f}(4)).$$

Applying (A.5) with $n = 0, 1, 2$, there is a constant $K > 0$ such that

$$\|[F, M_f]\|_1 \leq C\|f\|_{A_1^1}. \qquad \square$$

THEOREM A.4. *There is a universal constant $k > 0$ such that for all $p \in [1, 2]$,*

$$k\|f\|_{A_p^{1/p}} \leq \|[F, M_f]\|_p.$$

Proof. We first prove the result for $p = 1$ and $p = 2$, with the general case following from interpolation. This result essentially follows from [**13**, Theorem 6.1.1], where it is proved that there is a constant $k > 0$ such that

$$k\|f\|_{B_{1,1}^1} \leq \|[F, M_f]\|_1,$$

where $B_{1,1}^1$ is the Besov space on the circle.

However, following [**16**, Ch. 5, Proposition 7], $\|f\|_{A_1^1} \leq \alpha\|f\|_{B_{1,1}^1}$. This proves the result for $p = 1$, and for $p = 2$ it is enough to check the inequality on exponential basis functions $f(z) = z^n$, for which it is easily verified. $\square$

Combining Theorems A.2, A.3 and A.4, we obtain the following corollary.

COROLLARY A.5. *Let $p_0 > 1$. There exist constants b_{p_0}, $B_{p_0} > 0$ (depending on p_0) such that for all $p \in (p_0, 2)$, we have*

$$b_{p_0}\|f\|_{C_p^{1/p}} \leq \|[F, M_f]\|_p \leq B_{p_0}\|f\|_{C_p^{1/p}}.$$

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REFERENCES

[**1**] L. Carleson and T. W. Gamelin. *Complex Dynamics (Universitext: Tracts in Mathematics)*. Springer, New York, 1993.

[**2**] A. Connes. *Noncommutative Geometry*. Academic Press, San Diego, CA, 1994.

[**3**] A. Connes. Brisure de symétrie spontanée et géométrie du point de vue spectral. *Séminaire Bourbaki, Vol. 1995/96 (Astérisque, 241, Exp. No. 816, 5)*. Société de Mathématique de France, Paris, 1997, pp. 313–349.

[**4**] A. Connes. Noncommutative differential geometry and the structure of space time. *Deformation Theory and Symplectic Geometry (Ascona, 1996) (Mathematical Physics Studies, 20)*. Kluwer Academic, Dordrecht, 1997, pp. 1–33.

[**5**] A. Connes, F. Sukochev and D. Zanin. Trace theorem for quasi-Fuchsian groups. *Mat. Sb.* **208**(10) (2017), 59–90 (in Russian).

[**6**] A. Douady and J. H. Hubbard. *Étude dynamique des polynômes complexes. Partie I (Publications Mathématiques d'Orsay [Mathematical Publications of Orsay], 84)*. Université de Paris-Sud, Département de Mathématiques, Orsay, 1984.

[**7**] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*, 2nd edn. John Wiley, Hoboken, NJ, 2003.

[**8**] H. Federer. *Geometric Measure Theory (Grundlehren der mathematischen Wissenschaften, 153)*. Springer, New York, 1969.

[**9**] J. B. Garnett and D. E. Marshall. *Harmonic Measure (New Mathematical Monographs, 2)*. Cambridge University Press, Cambridge, 2005.

[**10**] H. Hedenmalm, B. Korenblum and K. Zhu. *Theory of Bergman Spaces (Graduate Texts in Mathematics, 199)*. Springer, New York, 2000.

[**11**] S. Lord, F. Sukochev and D. Zanin. *Singular Traces: Theory and Applications (De Gruyter Studies in Mathematics, 46)*. De Gruyter, Berlin, 2013.

[**12**] J. Milnor. *Dynamics in One Complex Variable (Annals of Mathematics Studies, 160)*, 3rd edn. Princeton University Press, Princeton, NJ, 2006.

[**13**] V. V. Peller. *Hankel Operators and Their Applications (Springer Monographs in Mathematics)*. Springer, New York, 2003.

[**14**] W. Rudin. *Real and Complex Analysis*, 3rd edn. McGraw-Hill, New York, 1987.

[**15**] E. Semenov, F. Sukochev, A. Usachev and D. Zanin. Banach limits and traces on $\mathcal{L}_{1,\infty}$. *Adv. Math.* **285** (2015), 568–628.

[**16**] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions (Princeton Mathematical Series, 30)*. Princeton University Press, Princeton, NJ, 1970.

[**17**] D. Sullivan. Conformal dynamical systems. *Geometric Dynamics (Rio de Janeiro, 1981) (Lecture Notes in Mathematics, 1007)*. Springer, Berlin, 1983, pp. 725–752.

[**18**] M. Urbański. Measures and dimensions in conformal dynamics. *Bull. Amer. Math. Soc. (N.S.)* **40**(3) (2003), 281–321.

[**19**] K. H. Zhu. Analytic Besov spaces. *J. Math. Anal. Appl.* **157**(2) (1991), 318–336.