

Cyclic homology, Serre’s local factors and λ -operations

by

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Abstract

We show that for a smooth, projective variety X defined over a number field K , cyclic homology with coefficients in the ring $\mathbb{A}_\infty = \prod_{v|\infty} K_v$, provides the right theory to obtain, using λ -operations, Serre’s archimedean local factors of the complex L-function of X as regularized determinants.

Key Words: Cyclic homology, Deligne cohomology, Gamma factors, complex L-function, motives.

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1. Introduction

Cyclic cohomology was introduced and widely publicized by the first author of this paper in 1981 (*see* [3]) as an essential tool in noncommutative differential geometry (*see* [5, 6]). In the context of algebraic geometry the dual theory, cyclic homology, which had been anticipated in the commutative case in [26], was subsequently developed in a purely algebraic framework by J.L. Loday–D. Quillen (*see* [22, 23]), B. Feigin–B. Tsygan (*see* [31, 12]), M. Karoubi (*see* [19]), C. Hood–J. Jones (*see* [18]) and in the more recent work of C. Weibel (*see* [33, 34]) to which we refer as a basic reference throughout this paper.

In this article we show that for a smooth, projective variety X defined over a number field K , cyclic homology with coefficients in the ring $\mathbb{A}_\infty = \prod_{v|\infty} K_v$, provides the right theory to obtain, using λ -operations, Serre’s archimedean local factors of the complex L-function of X as regularized determinants. In [10], C. Deninger constructed a cohomological theory that achieves a similar goal by introducing an archimedean (yet simplified) analogue of J.M. Fontaine’s p -ring \mathbf{B}_{dR} . It is evident that due to the infinite number of poles of the Gamma functions which enter in Serre’s formula of archimedean local factors, one is required to look for an infinite dimensional cohomological theory (*see* [7]). Moreover, since the multiplicity of these poles is provided by the dimension (in a precise range) of the real Deligne

cohomology of the complex *resp.*, real variety $X_\nu = X \times_K K_\nu$ over K_ν , one regards this latter as a cohomological theory intimately related to the sought for “archimedean cohomology”. As pointed out in [10] there is however a mismatch happening at a *real* place ν and for odd Hodge weights w , between Deninger's proposed definition of archimedean cohomology and the real Deligne cohomology of $X_\nu = X/\mathbb{R}$. This mismatch is explained in [11] as deriving from a canonical perfect pairing between the spaces in question.

In this paper we show that for $\nu|\infty$ the cyclic homology of X_ν provides a conceptual general construction of the sought for archimedean cohomology of X in terms of what we define (*see* Definition 5.1) as *archimedean cyclic homology* $HC_*^{\text{ar}}(X_\nu)$. Cyclic homology is naturally infinite dimensional, it solves the above mismatch with the real Deligne cohomology and at the same time also unveils the subtle nature of the operator Θ , as generator of the λ -operations, whose regularized determinant yields the archimedean local factors. The endomorphism Θ has two constituents: the natural grading in cyclic homology and the action of the multiplicative semigroup \mathbb{N}^\times on cyclic homology of *commutative* algebras given by the λ -operations $\Lambda_k, k \in \mathbb{N}^\times$. More precisely, the action u^Θ of the multiplicative group \mathbb{R}_+^\times generated by Θ on cyclic homology, is uniquely determined by its restriction to the dense subgroup $\mathbb{Q}_+^\times \subset \mathbb{R}_+^\times$ where it is given by the formula

$$k^\Theta|_{HC_n(X_\nu)} = \Lambda_k k^{-n}, \quad \forall n \geq 0, k \in \mathbb{N}^\times \subset \mathbb{R}_+^\times. \quad (1)$$

Our main result is the following

Theorem 1.1 *Let X be a smooth, projective variety of dimension d over an algebraic number field K and let $\nu|\infty$ be an archimedean place of K . Then, the action of the operator Θ on the archimedean cyclic homology of X_ν satisfies the following formula*

$$\prod_{0 \leq w \leq 2d} L_\nu(H^w(X), s)^{(-1)^{w+1}} = \frac{\det_\infty(\frac{1}{2\pi}(s - \Theta))|_{HC_{\text{even}}^{\text{ar}}(X_\nu)}}{\det_\infty(\frac{1}{2\pi}(s - \Theta))|_{HC_{\text{odd}}^{\text{ar}}(X_\nu)}}, \quad s \in \mathbb{R}. \quad (2)$$

The left-hand side of (2) is the product of Serre's archimedean local factors of the complex L -function of X (*see* [29]). On the right-hand side, \det_∞ denotes the regularized determinant (*see e.g.* [27, 10]) with $HC_{\text{even}}^{\text{ar}}(X_\nu) = \bigoplus_{n=2k \geq 0} HC_n^{\text{ar}}(X_\nu)$,

$$HC_{\text{odd}}^{\text{ar}}(X_\nu) = \bigoplus_{n=2k+1 \geq 1} HC_n^{\text{ar}}(X_\nu).$$

By taking into account the fact that cyclic homology of a finite product of algebras (or disjoint union of schemes) is the direct sum of their cyclic homologies, (2)

determines the required formula for the product of all archimedean local factors in terms of cyclic homology with coefficients in the ring $\mathbb{A}_\infty = \prod_{\nu|\infty} K_\nu$.

Let us now describe in some details the archimedean cyclic homology groups $HC_*^{\text{ar}}(X_\nu)$ appearing on the right-side of (2). The nuance between archimedean cyclic homology $HC_*^{\text{ar}}(X_\nu)$ and ordinary cyclic homology $HC_*(X_\nu)$ as developed in the context of algebraic geometry, corresponds exactly to the difference between the real Deligne cohomology and reduced Deligne cohomology of X_ν , *i.e.* relative de Rham cohomology of X_ν (up to a shift of degrees). The cyclic homology groups $HC_*(X_\nu)$ have coefficients in \mathbb{C} for a complex place ν of K , *resp.*, in \mathbb{R} for a real place, and in this context they play the role of relative de Rham cohomology. The *real* Betti cohomology of a complex projective variety $X_{\mathbb{C}}$ is recovered in periodic cyclic homology theory by the inclusion (*see* Appendix B and §4.3, Proposition 4.4 for the second equality)

$$HP_*(C^\infty(X_{\text{sm}}, \mathbb{R})_{\text{top}}) \subset HP_*(C^\infty(X_{\text{sm}}, \mathbb{C})_{\text{top}}) \cong HP_*(X_{\mathbb{C}}), \quad (3)$$

where X_{sm} denotes the underlying smooth C^∞ manifold. The periodic cyclic homology groups are defined in topological terms as in [5] and reviewed in Appendix B. Then the *real* (λ)-twisted periodic cyclic homology $HP_*^{\text{real}}(X_{\mathbb{C}})$ is defined as the hyper-cohomology of a cochain complex (*see* Definition 4.6) described by the derived pullback of two natural maps of complexes in periodic cyclic theory. The resulting map $\tau : HP_*^{\text{real}}(X_{\mathbb{C}}) \rightarrow HP_*(X_{\mathbb{C}})$ is essentially the Tate twist $(2\pi i)^{\Theta_0}$ combined with the above inclusion (3). Here, Θ_0 denotes the generator of the λ -operations,

$$k^{\Theta_0}|_{HC_n(X_\nu)} = \Lambda_k, \quad \forall n \geq 0, k \in \mathbb{N}^\times \subset \mathbb{R}_+^\times \quad (4)$$

i.e. it differs from the above operator Θ by the grading in cyclic homology. The spectrum of Θ_0 consists of integers so that $(2\pi i)^{\Theta_0}$ makes sense. Let us first consider the case when the archimedean place ν of K is complex. Then, one obtains (*see* Proposition 5.3) the short exact sequence (which determines archimedean cyclic homology)

$$0 \rightarrow HP_{*+2}^{\text{real}}(X_\nu) \xrightarrow{S \circ \tau} HC_*(X_\nu) \rightarrow HC_*^{\text{ar}}(X_\nu) \rightarrow 0 \quad (5)$$

where the periodicity map S implements here the link between periodic cyclic homology and cyclic homology. If instead the archimedean place ν is real, one takes in the above construction the fixed points of the anti-linear conjugate Frobenius operator \bar{F}_∞ acting on $HC_*(X_\nu \otimes \mathbb{C})$ (*see* §5.3). While the exact sequence (5) suffices to define $HC_*^{\text{ar}}(X_\nu)$, we also provide in Definition 5.1 the general construction of a complex of cochains whose cohomology gives $HC_*^{\text{ar}}(X_\nu)$. This

complex is defined by the derived pullback of two maps connecting the negative cyclic complex (see [18]) resp., the real (λ)-twisted periodic cyclic complex to the periodic cyclic complex.

Besides its natural clarity at the conceptual level, this newly developed archimedean cyclic homology theory has also the following qualities:

- 1) It inherits by construction the rich structure of cyclic homology, such as the action of the periodicity operator S (that has to be Tate twisted in the complex case and squared in the real case).
- 2) It is directly connected to algebraic K -theory and the γ -filtration by the regulator maps, thus it acquires naturally a role in the theory of motives.
- 3) The fundamental formula derived from (2), that gives the product of all archimedean local factors in terms of cyclic homology with coefficients in the ring $\mathbb{A}_\infty = \prod_{v|\infty} K_v$, is clearly of adelic nature and evidently suggests the study of a generalization of these results, by implementing the full ring \mathbb{A}_K of adèles as coefficients. Cyclic homology with coefficients in the number field K should provide a natural lattice in the spirit of [2].
- 4) When translated in terms of the logarithmic derivatives of the two sides, formula (2) is very suggestive of the existence of a global Lefschetz formula in cyclic homology.
- 5) At the conceptual level, cyclic homology is best understood as a way to embed the category of non-commutative k -algebras in the abelian category of Λ -modules (see [4]), where Λ is a small category built from simplicial sets and cyclic groups. Any non-commutative algebra \mathcal{A} gives rise *canonically*, through its tensor powers $\mathcal{A}^{\otimes n}$, to a Λ -module \mathcal{A}^{\natural} . The functor $\mathcal{A} \mapsto \mathcal{A}^{\natural}$ from the category of algebras to the abelian category of Λ -modules retains all the information needed to compute cyclic homology groups. In fact, since these groups are computed as $\text{Tor}(k^{\natural}, \mathcal{A}^{\natural})$ the cyclic theory fits perfectly with the use of extensions in the theory of motives.
- 6) One knows (see [4]) that the classifying space of the small category Λ is $\mathbb{P}^\infty(\mathbb{C})$ and its cohomology accounts for the geometric meaning of the periodicity operator S . The immersion of the category of algebras in the abelian category of Λ -modules is refined *in the commutative case* by the presence of the λ -operations. As shown in §6.4 of [23], the presence of the λ -operations for a Λ -module $E : \Lambda^{\text{op}} \rightarrow (k\text{-Mod})$ is a consequence of the fact that E factors through the category $\mathfrak{F}\text{in}$ of finite sets. This clearly happens for $E = \mathcal{A}^{\natural}$, when the algebra \mathcal{A} is commutative. We expect that a deeper understanding in the framework of algebraic topology, of the relations between the cyclic category Λ and the category $\mathfrak{F}\text{in}$ of finite sets should shed light on the structure of the "absolute point" and explain the role of cyclic homology and the λ -operations in the world of motives.

The rest of the paper is organized as follows

In Section 2, we recall the definition of real Deligne cohomology and review briefly the result of [34] (*see* also [12]) which relate the Hodge filtration of the Betti cohomology of a smooth, projective variety over \mathbb{C} to the cyclic homology of the associated scheme. This provides a direct link between the reduced Deligne cohomology (in the sense of [23]) and the cyclic homology of the variety.

In Section 3, we first recall the well known formula describing the multiplicity of the poles of the archimedean local factors of X as the rank of some real Deligne cohomology groups. This result leads unambiguously to the definition of the archimedean cohomology of a smooth, projective algebraic variety X over a number field as an infinite direct sum of real Deligne cohomology groups. Then, we point out that by neglecting at first the nuance between real Deligne cohomology and reduced Deligne cohomology, this infinite direct sum is nothing but the cyclic homology group $\bigoplus_{n \geq 0} HC_n(X_{\mathbb{C}})$.

The remaining sections are then dedicated to describe in cyclic terms the difference between reduced and unreduced Deligne cohomology.

In Section 4 we use the computation of the cyclic homology of a smooth manifold (*see* [5], Theorem 46) and the stability property of periodic cyclic homology when passing from the algebraic cyclic homology of a smooth complex projective variety $X_{\mathbb{C}}$ to the associated C^{∞} -manifold X_{sm} to construct the real (λ) -twisted periodic cyclic homology $HP_*^{\text{real}}(X_{\mathbb{C}})$.

The scheme theoretic relation between X_{sm} and $X_{\mathbb{C}}$ is explored in depth in Appendix A where we extend the obvious map of locally ringed spaces $X_{\text{sm}} \rightarrow X_{\mathbb{C}}$, to a *morphism of schemes* $\pi_X : \text{Spec}(C^{\infty}(X_{\text{sm}}, \mathbb{C})) \rightarrow X_{\mathbb{C}}$ to which the general theory of [33] can be applied. The morphism π_X factors naturally through an affine complex scheme $W_{\mathbb{C}} = W \times \mathbb{C}$ where W is an affine smooth variety defined over \mathbb{R} which is an open subscheme of $X \times \bar{X}$.

In Section 5 we provide the definition and the first properties of archimedean cyclic homology theory. In §5.3 we provide the cyclic homology meaning of the anti-linear conjugate Frobenius operator \bar{F}_{∞} and develop the construction at the real places.

Finally, Section 6 contains a detailed proof of Theorem 1.1.

2. Reduced Deligne cohomology and cyclic homology

In this section we quickly review the result of [34] from which our paper develops. This basic statement relates the Hodge filtration on the Betti cohomology of a smooth, projective variety over \mathbb{C} to the cyclic homology of the associated scheme. Throughout this paper we follow the conventions of [32] to denote the shift of the indices in chain C_* *resp.*, cochain complexes C^* *i.e.*

$$C[p]_n := C_{n+p}, \quad C[p]^n := C^{n-p}.$$

Note that this convention is the opposite to the one used in [28].

2.1. Reduced Deligne cohomology

Let $X_{\mathbb{C}}$ be a smooth, projective variety over the complex numbers. One denotes by $\mathbb{R}(r)$ the subgroup $(2\pi i)^r \mathbb{R}$ of \mathbb{C} ($i = \sqrt{-1}$) and by $\mathbb{R}(r)_{\mathcal{D}}$ the complex of sheaves of holomorphic differential forms $\Omega^{\cdot}_{X(\mathbb{C})}$ on the complex manifold $X(\mathbb{C})$ associated to $X_{\mathbb{C}}$, whose cohomology defines the real Deligne cohomology of $X_{\mathbb{C}}$ (see [28]):

$$\mathbb{R}(r)_{\mathcal{D}} : \mathbb{R}(r) \xrightarrow{\epsilon} \Omega^0_{X(\mathbb{C})} \xrightarrow{d} \Omega^1_{X(\mathbb{C})} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{r-1}_{X(\mathbb{C})} \rightarrow 0.$$

Here $\mathbb{R}(r)$ is placed in degree 0 and $\Omega^p_{X(\mathbb{C})}$ in degree $p + 1$ ($\forall p \geq 0$) and the map ϵ is the inclusion of the twisted constants: $\mathbb{R}(r) \subset \mathbb{C} \subset \mathcal{O}_{X(\mathbb{C})} = \Omega^0_{X(\mathbb{C})}$. The real Deligne cohomology of $X_{\mathbb{C}}$ is defined as the hyper-cohomology of the above complex: $H^n_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(r)) := \mathbb{H}^n(X(\mathbb{C}), \mathbb{R}(r)_{\mathcal{D}})$. One has an evident short exact sequence of complexes

$$0 \rightarrow (\Omega^{\leq r}_{X(\mathbb{C})})[1] \rightarrow \mathbb{R}(r)_{\mathcal{D}} \rightarrow \mathbb{R}(r) \rightarrow 0. \quad (6)$$

Following [23] (§3.6.4), we call the *reduced Deligne complex* the kernel of the surjective map $\mathbb{R}(r)_{\mathcal{D}} \rightarrow \mathbb{R}(r)$ in (6), that is the truncated de Rham complex shifted by 1 to the right. The *reduced Deligne cohomology* is then defined as the hyper-cohomology of the reduced Deligne complex:

$$\tilde{H}^n_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(r)) := \mathbb{H}^n(X(\mathbb{C}), (\Omega^{\leq r}_{X(\mathbb{C})})[1]) = \mathbb{H}^n(\text{Cone}(F^r \Omega^{\cdot}_{X(\mathbb{C})} \xrightarrow{\iota} \Omega^{\cdot}_{X(\mathbb{C})})[1]). \quad (7)$$

The second equality in (7) is an immediate consequence of the definition of the Hodge sub-complex $F^r \Omega^{\cdot}_{X(\mathbb{C})} \xrightarrow{\iota} \Omega^{\cdot}_{X(\mathbb{C})}$. Up to a shift by 1, reduced Deligne cohomology is sometimes referred to as relative de Rham cohomology. The results of [30, 15] show that reduced Deligne cohomology is computed in the same way in terms of the de Rham complex of algebraic differential forms on the scheme $X_{\mathbb{C}}$ and thus makes sense for schemes over \mathbb{C} .

In Proposition 2.2 we shall review the key result of [34] (Theorem 3.3) which, for a smooth projective algebraic variety over \mathbb{C} , relates the components $HC_n^{(j)}(X_{\mathbb{C}})$ of the λ -decomposition of the cyclic homology $HC_n(X_{\mathbb{C}})$ with the reduced Deligne cohomology $\tilde{H}^{2j+1-n}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(j+1))$. We first need to briefly recall the basic properties of the λ -operations in cyclic homology.

2.2. λ -decomposition

We recall from [12, 23, 34] that one has a natural action of the multiplicative semigroup \mathbb{N}^{\times} of positive integers on the cyclic homology groups of a *commutative*

algebra (over a ground ring). This action is the counterpart in cyclic homology of Quillen's λ -operations in algebraic K -theory (see [23] 4.5.16)

Proposition 2.1 *Let A be a commutative algebra and $(C_k(A) = A^{\otimes(k+1)}, b, B)$ its mixed complex of chains. The λ -operations define (degree zero) endomorphisms Λ_* of $C_*(A)$ commuting with the grading and satisfying the following properties*

- $\Lambda_{nm} = \Lambda_n \Lambda_m, \quad \forall n, m \in \mathbb{N}^\times$
- $b\Lambda_m = \Lambda_m b, \quad \forall m \in \mathbb{N}^\times$
- $\Lambda_m B = mB\Lambda_m, \quad \forall m \in \mathbb{N}^\times$.

Proof: The statements are all proven in [23]: the first is equation (4.5.4.2), the second is Proposition 4.5.9 and the third is Theorem 4.6.6 of *op.cit.* . \square

This construction gives rise, for a commutative algebra A over a ground field of characteristic 0, to a *canonical* decomposition, called the λ -decomposition, of the cyclic homology of A as a direct sum

$$HC_n(A) = \bigoplus_{j \geq 0} HC_n^{(j)}(A)$$

which is uniquely determined as a diagonalization of the endomorphisms Λ_m i.e.

$$\Lambda_m(\alpha) = m^j \alpha, \quad \forall \alpha \in HC_n^{(j)}(A), m \in \mathbb{N}^\times.$$

2.3. Cyclic homology of smooth, projective varieties

In [34] (Lemma 3.0) it is proven that the λ -decomposition extends to the projective case, thus for a smooth, projective algebraic variety $X_{\mathbb{C}}$ over the complex numbers one has the finite decomposition

$$HC_n(X_{\mathbb{C}}) = \bigoplus_{j \geq 0} HC_n^{(j)}(X_{\mathbb{C}}).$$

We recall the following key result of [34]:

Proposition 2.2 *Let $X_{\mathbb{C}}$ be a smooth, projective algebraic variety over \mathbb{C} . Then one has canonical isomorphisms*

$$\begin{aligned} HC_n^{(j)}(X_{\mathbb{C}}) &\cong \tilde{H}_{\mathbb{D}}^{2j+1-n}(X_{\mathbb{C}}, \mathbb{R}(j+1)) \\ &\cong H_B^{2j-n}(X(\mathbb{C}), \mathbb{C})/F^{j+1} \quad \forall j \geq 0, \forall n \geq 0. \end{aligned}$$

In [34] (Theorem 3.3) it is proven that

$$HC_n^{(j)}(X_{\mathbb{C}}) = \mathbb{H}^{2j-n}(X_{\mathbb{C}}, \Omega_{X_{\mathbb{C}}}^{\leq j}) = \mathbb{H}^{2j-n}(X(\mathbb{C}), \Omega_{X(\mathbb{C})}^{\leq j}).$$

Moreover it follows from the degeneration of the ‘‘Hodge to de Rham’’ hyper-cohomology spectral sequence and the canonical identification of de Rham cohomology with the Betti cohomology that $\mathbb{H}^{2j-n}(X(\mathbb{C}), \Omega_{X(\mathbb{C})}^{\leq j}) \cong H_B^{2j-n}(X(\mathbb{C}), \mathbb{C})/F^{j+1}$.

3. Cyclic homology and archimedean cohomology

In this section we first recall the basic result which expresses the order of the poles of Serre's archimedean local factors $L_\nu(H^w(X), s)$ ($\nu|\infty$) of the complex L-function of a smooth, projective algebraic variety X over a number field in terms of the ranks of some real Deligne cohomology groups of $X_\nu := X \times_K K_\nu$ over K_ν . The outcome leads unambiguously to consider an infinite direct sum of real Deligne cohomology groups of X_ν as the most natural candidate for the archimedean cohomology of X . If one ignores at first the nuance between real Deligne cohomology and reduced Deligne cohomology, this infinite direct sum (when ν is a complex place of K) is nothing but the cyclic homology direct sum $\bigoplus_{n \geq 0} HC_n(X_{\mathbb{C}})$. In the last part of the section we shall describe the strategy on how to understand in full this difference.

3.1. Deligne cohomology and poles of archimedean factors

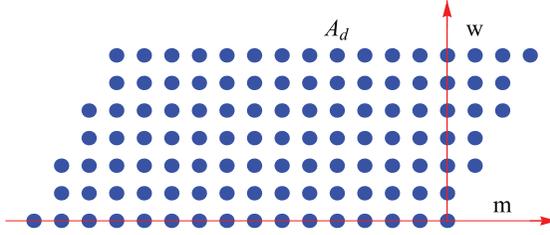
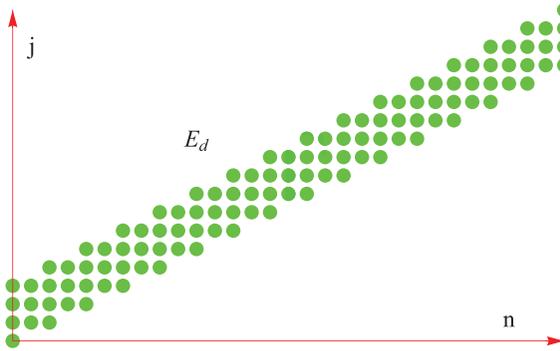
Let X be a smooth, projective variety over a number field K . We fix an archimedean place $\nu|\infty$ of K and a Hodge weight w of the singular cohomology of the complex manifold $X_\nu(\mathbb{C})$ and consider the local factor $L_\nu(H^w(X), s)$ (see [28]). In view of the definition of these factors given by Serre (see [29]) as a product of powers of shifted Γ -functions, these functions are completely specified by the multiplicities of their poles at some integer points on the real line. By a result of Beilinson ([1], see also [10, 28]) the order of these poles can be expressed in terms of the ranks of some real Deligne cohomology groups of X_ν , the poles only occur at integers $s = m \leq \frac{w}{2}$ and their multiplicity is provided by the well-known formula ([1], (3.1), (3.3), [28] §1,2, [10], Prop. 5.1)

$$\text{ord}_{s=m} L_\nu(H^w(X), s)^{-1} = \dim_{\mathbb{R}} H_D^{w+1}(X_\nu, \mathbb{R}(w+1-m)) \quad (8)$$

where for a complex place ν the real Deligne cohomology groups of X_ν are defined as in §2.1 ($X_\nu = X_{\mathbb{C}}$) and for a real place ν ($X_\nu = X_{/\mathbb{R}}$), these groups are defined as

$$H_D^q(X_\nu, \mathbb{R}(p)) := H_D^q(X_\nu \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{R}(p))^{\text{DR-conjugation}} \quad (9)$$

i.e. the subspace of $H_D^q(X_\nu \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{R}(p))$ of the elements fixed by the de Rham conjugation ([28] §2).

Figure 1: The subset A_d for $d = 3$.Figure 2: The subset E_d for $d = 3$.

3.2. Deligne cohomology and archimedean cohomology

Let us now consider the pairs (m, w) of integers which enter in formula (8), when $\dim X = d$ (so that w takes integer values between 0 and $2d$). They form the set of pairs of relative integers

$$A_d = \{(m, w) \mid 0 \leq w \leq 2d, m \leq w/2\}. \quad (10)$$

The infinite dimensional archimedean cohomology of X_v relevant to obtain the archimedean factors as regularized determinants is therefore dictated by (8) as the infinite direct sum

$$\bigoplus_{(m,w) \in A_d} H_{\mathcal{D}}^{w+1}(X_v, \mathbb{R}(w+1-m)). \quad (11)$$

We now rewrite this sum in an equivalent way by using the following

Lemma 3.1 *Let $d \geq 0$ be an integer. The map of sets which maps the pair of*

integers (n, j) to (m, w) under the relations

$$m = j - n, \quad w = 2j - n$$

is a bijection of $E_d = \{(n, j) \mid 0 \leq n \leq 2j \leq 2d + n\}$ with A_d .

Proof: The inverse map sends $(m, w) \in A_d$ to (n, j) where $n = -2m + w, j = -m + w$. One easily checks that the conditions are preserved, in fact one has

$$n \geq 0, 0 \leq 2j - n \leq 2d \implies j - n \leq (2j - n)/2$$

and

$$0 \leq w \leq 2d, m \leq w/2 \implies -2m + w \geq 0, 0 \leq 2(-m + w) - (-2m + w) \leq 2d.$$

□

Thus by Lemma 3.1, the sum in (11) can be equivalently rewritten as

$$\bigoplus_{(m,w) \in A_d} H_D^{w+1}(X_\nu, \mathbb{R}(w+1-m)) = \bigoplus_{(n,j) \in E_d} H_D^{2j+1-n}(X_\nu, \mathbb{R}(j+1)). \quad (12)$$

Moreover, by applying Proposition 2.2, one obtains the isomorphism

$$\bigoplus_{n \geq 0} HC_n(X_{\mathbb{C}}) \cong \bigoplus_{(n,j) \in E_d} \tilde{H}_D^{2j+1-n}(X_{\mathbb{C}}, \mathbb{R}(j+1)). \quad (13)$$

It follows that the difference between the archimedean cohomology of X_ν as in (12) and the cyclic homology direct sum as in (13) is expressed by the nuance between the real Deligne cohomology and the reduced Deligne cohomology of $X_{\mathbb{C}}$.

3.3. Passing from $\tilde{H}_D^*(X_\nu, \mathbb{R}(\cdot))$ to $H_D^*(X_\nu, \mathbb{R}(\cdot))$

To understand the difference between reduced Deligne cohomology $\tilde{H}_D^*(X_\nu, \mathbb{R}(\cdot))$ and real Deligne cohomology $H_D^*(X_\nu, \mathbb{R}(\cdot))$, we assume first that the place ν is complex, and introduce the long exact sequence associated to the short exact sequence (6). It is of the form

$$\begin{aligned} \dots \rightarrow H_B^w(X(\mathbb{C}), \mathbb{R}(r)) \rightarrow \tilde{H}_D^{w+1}(X_{\mathbb{C}}, \mathbb{R}(r)) \\ \rightarrow H_D^{w+1}(X_{\mathbb{C}}, \mathbb{R}(r)) \rightarrow H_B^{w+1}(X(\mathbb{C}), \mathbb{R}(r)) \rightarrow \dots \end{aligned}$$

In view of the isomorphism $\tilde{H}_D^{w+1}(X_{\mathbb{C}}, \mathbb{R}(r)) \cong H_{\text{dR}}^w(X(\mathbb{C}))/F^r$, this long exact sequence can be equivalently written as

$$\begin{aligned} \dots \rightarrow H_B^w(X(\mathbb{C}), \mathbb{R}(r)) \rightarrow H_{\text{dR}}^w(X(\mathbb{C}))/F^r \\ \rightarrow H_D^{w+1}(X_{\mathbb{C}}, \mathbb{R}(r)) \rightarrow H_B^{w+1}(X(\mathbb{C}), \mathbb{R}(r)) \rightarrow \dots \end{aligned}$$

By comparing with (12), we see that one has to estimate $H_D^{w+1}(X_{\mathbb{C}}, \mathbb{R}(r))$ for $(w, m) \in A_d$, with A_d as in (10) where $r = w + 1 - m$, i.e. for $(w, w + 1 - r) \in A_d$. One knows that for $w < 2r$ the natural map

$$H_B^w(X(\mathbb{C}), \mathbb{R}(r)) \rightarrow H_{\text{dR}}^w(X(\mathbb{C}))/F^r$$

is injective (in H_{dR}^w the intersection $F^r \cap \bar{F}^r = \{0\}$ when $2r > w$). For $w + 1 < 2r$ one gets a short exact sequence of the form

$$0 \rightarrow H_B^w(X(\mathbb{C}), \mathbb{R}(r)) \rightarrow \tilde{H}_D^{w+1}(X_{\mathbb{C}}, \mathbb{R}(r)) \rightarrow H_D^{w+1}(X_{\mathbb{C}}, \mathbb{R}(r)) \rightarrow 0. \quad (14)$$

This holds for the pair $(w, w + 1 - r) \in A_d$ since

$$(w, w + 1 - r) \in A_d \implies w + 1 - r \leq w/2 \implies w/2 + 1 \leq r \implies w + 1 < 2r.$$

Therefore, when the place ν is complex, the difference between the archimedean cohomology of X_{ν} as in (12) and the cyclic homology direct sum as in (13) will be taken care of by a suitable interpretation in cyclic terms of the real Betti cohomology $H_B^w(X(\mathbb{C}), \mathbb{R}(r))$ and of the map $H_B^w(X(\mathbb{C}), \mathbb{R}(r)) \rightarrow \tilde{H}_D^{w+1}(X_{\mathbb{C}}, \mathbb{R}(r))$.

When instead the place ν of K is real the corresponding real Deligne cohomology groups are defined in (9) as the fixed elements of the anti-linear de Rham conjugation \bar{F}_{∞} . In §5.3 we shall show that \bar{F}_{∞} admits a direct cyclic homology interpretation. Thus at a real place ν one has to further refine the above construction by taking the fixed points of the cyclic counterpart of \bar{F}_{∞} . This is how the remaining part of the paper develops.

4. Real (λ) -twisted cyclic homology of a smooth, projective variety

In this section we give the interpretation in cyclic terms of the real Betti cohomology and of the map $H_B^w(X(\mathbb{C}), \mathbb{R}(r)) \rightarrow \tilde{H}_D^{w+1}(X_{\mathbb{C}}, \mathbb{R}(r))$. We show that the computation of cyclic homology of smooth manifolds (see [5], Theorem 46) jointly with the stability property of the periodic cyclic homology when passing from the algebraic cyclic homology of a smooth complex projective variety $X_{\mathbb{C}}$ to the associated C^{∞} -manifold X_{sm} (using a natural morphism of schemes $\text{Spec}(C^{\infty}(X_{\text{sm}}, \mathbb{C})) \rightarrow X_{\mathbb{C}}$ whose definition is given in Appendix A and the Frechet topology of $C^{\infty}(X_{\text{sm}}, \mathbb{C})$ as explained in Appendix B), give rise (with the implementation of a natural involution on the Frechet algebra $C^{\infty}(X_{\text{sm}}, \mathbb{C})$ and of a Tate twist) to a *real subspace* $HP_*^{\text{real}}(X_{\mathbb{C}})$ of the periodic cyclic homology of $X_{\mathbb{C}}$. This new structure is more precisely described by means of a map of *real graded vector spaces*

$$HP_*^{\text{real}}(X_{\mathbb{C}}) \xrightarrow{\tau} HP_*(X_{\mathbb{C}})$$

which we deduce from a map of associated complexes.

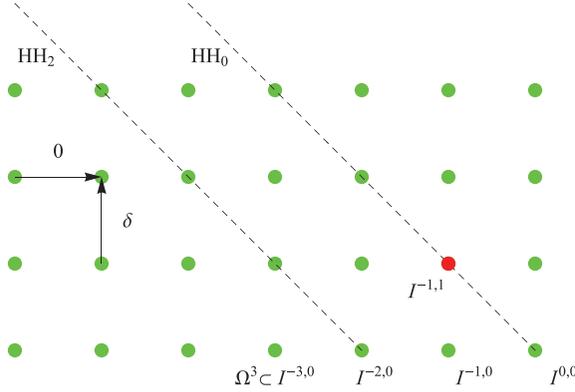


Figure 3: Bi-complex describing the Hochschild homology of $X_{\mathbb{C}}$, the vertical line $(I^{-m,*}, \delta)$ with abscissa $-m$ is an injective resolution of Ω^m .

4.1. Hochschild and cyclic homology of schemes

We first recall from [33] the definition of Hochschild and cyclic homology of a scheme X_k over a field k . We use the general conventions of [33] and re-index a chain complex C_* as a cochain complex by writing $C^{-n} := C_n, \forall n \in \mathbb{Z}$.

4.1.1 HOCHSCHILD HOMOLOGY

One lets C_*^h be the sheafification of the chain complex of presheaves $U \mapsto C_*^h(\mathcal{O}_{X_k}(U)) = \mathcal{O}_{X_k}(U)^{\otimes (*+1)}$, (the tensor products are over k), with the Hochschild boundary b as differential. Then, one re-indexes this complex as a negative (unbounded) cochain complex *i.e.* one lets $C_h^{-n} := C_n^h, \forall n$, and one finally defines the Hochschild homology $HH_n(X_k)$ of X_k as the Zariski hypercohomology $\mathbb{H}^{-n}(X_k, C_h^*)$

$$HH_n(X_k) := \mathbb{H}^{-n}(X_k, C_h^*) = H^{-n}(\Gamma(\text{Tot } I^{**})).$$

Here I^{**} is an injective Cartan-Eilenberg resolution of C_h^* and Tot is the total complex (using products rather than sums, *see* [32], 1.2.6 and [33], Appendix). We shall only work with algebras over $k = \mathbb{C}$ or \mathbb{R} and in that case the λ -decomposition (for commutative algebras) provides a natural decomposition of C_h as a direct sum of chain sub-complexes of the form

$$C_h = \bigoplus C_h^{(i)}$$

and one then sets $HH_n^{(i)}(X_k) := \mathbb{H}^{-n}(X_k, C_h^{(i)})$.

In the case of affine schemes $X_k = \text{Spec}(A)$ for a k -algebra A , one has the equality with Hochschild homology (*see* [35], 4.1, and [34] Proposition 1.3)

$$HH_n(X_k) = HH_n(A), \quad HH_n^{(i)}(X_k) = HH_n^{(i)}(A).$$

Let now $X_{\mathbb{C}}$ be the scheme over \mathbb{C} associated to a smooth, projective complex (classical) algebraic variety X_{alg} . In this case even though the scheme $X_{\mathbb{C}}$ has more points than the variety X_{alg} the categories of open sets are the same and so one can talk indifferently about sheaves over X_{alg} or over $X_{\mathbb{C}}$. By applying [34] (Corollary 1.4) one has

$$HH_n^{(j)}(X_{\mathbb{C}}) \cong H^{j-n}(X_{\mathbb{C}}, \Omega_{X_{\mathbb{C}}}^j), \quad \forall j, n.$$

Figure 3 shows the corresponding simplified version of the Cartan-Eilenberg injective resolution of the Hochschild complex. Using the Hochschild-Kostant-Rosenberg theorem one simplifies the chain complex \mathcal{C}_*^h by replacing the sheaf \mathcal{C}_j^h with the sheaf Ω^j of algebraic differential forms of degree j and the Hochschild boundary b by 0. The λ operations act as follows on Ω^j

$$\Lambda_k(\omega) = k^j \omega, \quad \forall k \in \mathbb{N}^\times, \omega \in \Omega^j. \quad (15)$$

One can then choose a Cartan-Eilenberg injective resolution where the horizontal boundaries are 0. The λ -decomposition is now read as the decomposition of the bi-complex in the sum of its vertical columns.

4.1.2 CYCLIC HOMOLOGY

We first recall the definition of the cyclic homology $HC_*(X_k)$ of a scheme X_k over a field k given in [34]. Sheafifying the usual (b, B) -bicomplex associated to a ring yields a bi-complex of sheaves (\mathcal{B}_{**}, b, B) on X_k and by definition ([34], (1.3))

$$HC_n(\mathcal{M}) := \mathbb{H}^{-n}(X_k, \text{Tot}\mathcal{B}_{**}(\mathcal{M}))$$

where the total chain complex $\text{Tot}\mathcal{B}_{**}$ is turned into a cochain complex by reindexing in negative degrees. More specifically, one obtains a (b, B) -bicomplex of the following form, where \mathcal{C}_*^h is as above the sheafification of the chain complex of presheaves $U \mapsto \mathcal{C}_*^h(\mathcal{O}_{X_k}(U)) = \mathcal{O}_{X_k}(U)^{\otimes(*+1)}$:

$$\mathcal{C}^{(\alpha, \beta)} := \mathcal{C}_{(\alpha-\beta)}^h, \text{ for } \alpha \leq 0, \alpha \geq \beta, \quad \mathcal{C}^{(\alpha, \beta)} := \{0\} \text{ otherwise,} \quad (16)$$

and where the horizontal coboundary is $d_1 = B$ and the vertical one is $d_2 = b$, as shown in Figure 4. The action of the λ -operations decomposes \mathcal{C}_a^h as a direct sum of the eigenspaces $\mathcal{C}_a^h = \bigoplus \mathcal{C}_a^{h, (j)}$ where j varies from 0 to a . Note that by construction

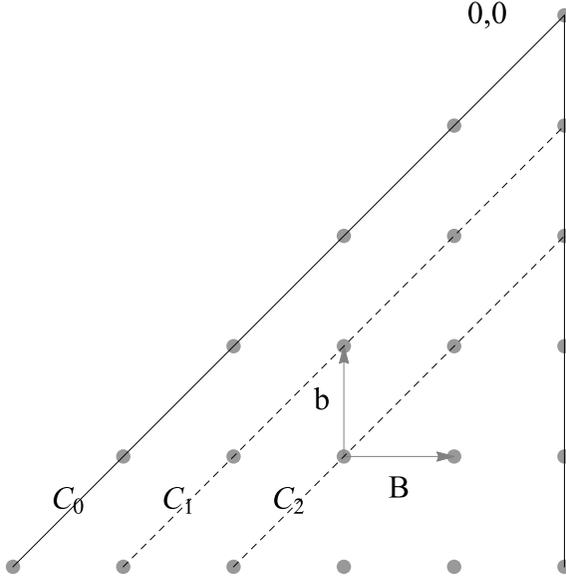


Figure 4: The bi-complex $(C^{*,*}, d_1, d_2)$ computing the cyclic homology.

the lower index a in C_a^h is always ≥ 0 . The above (b, B) bicomplex decomposes as the direct sum of the following subcomplexes $C^{*,*}(j)$

$$C^{(\alpha, \beta)}(j) := C_{(\alpha - \beta)}^{h, (\alpha + j)}, \text{ for } \alpha \leq 0, \alpha \geq \beta, \quad C^{(\alpha, \beta)}(j) := \{0\} \text{ otherwise,} \quad (17)$$

The basic compatibility properties of Proposition 2.1 show that the $C^{*,*}(j)$ are subcomplexes of the above (b, B) -bicomplex. In fact the following equality defines endomorphisms $\tilde{\Lambda}_k$ of the above (b, B) -bicomplex,

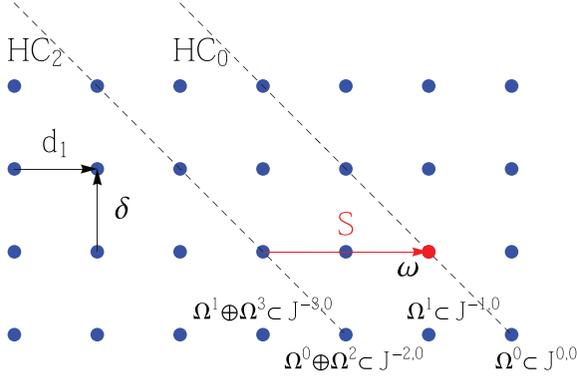
$$\tilde{\Lambda}_k \xi := k^{-\alpha} \Lambda_k \xi, \quad \forall \xi \in C^{(\alpha, \beta)}$$

and the subcomplexes $C^{*,*}(j)$ correspond to the decomposition into eigenspaces of the endomorphisms $\tilde{\Lambda}_k$, i.e. $\tilde{\Lambda}_k = k^j$ on $C^{*,*}(j)$. With these notations one has

$$HC_n(X_k) := \mathbb{H}^{-n}(X_k, \text{Tot} C^{*,*}), \quad HC_n^{(j)}(X_k) := \mathbb{H}^{-n}(X_k, \text{Tot} C^{*,*}(j)). \quad (18)$$

The periodicity map S is the endomorphism of the bicomplex $C^{*,*}$ given by translation by the vector $(1, 1)$. It fulfills the following relation with the endomorphisms $\tilde{\Lambda}_k$

$$S \tilde{\Lambda}_k = k \tilde{\Lambda}_k S, \quad \forall k \in \mathbb{N}^\times.$$


 Figure 6: Bicomplex describing the cyclic homology of $X_{\mathbb{C}}$, and map S .

of sheaves indexed in negative degrees $\mathcal{T}^n := \mathcal{T}_{-n}$ one obtains (using $j = -n - u$)

$$\mathcal{T}^n = \bigoplus_{0 \leq j \leq -n} \Omega_{X_{\mathbb{C}}}^{2j+n}, \quad (\mathcal{T}^*, d) = \bigoplus_{j \geq 0} (\Omega_{X_{\mathbb{C}}}^{2j+*}, d)_{(j+*) \leq 0} = \bigoplus_{j \geq 0} (\Omega_{X_{\mathbb{C}}}^{\leq j}, d)[-2j]$$

which is the product of the truncated de Rham complexes $(\Omega_{X_{\mathbb{C}}}^{\leq j}, d)$ shifted by $-2j$. Moreover this decomposition corresponds to the λ -decomposition of the mixed complex (\mathcal{M}, b, B) . To see this, note that the bicomplex $\mathcal{C}^{*,*}(j)$ of (17) gets replaced by a bicomplex which is zero outside the horizontal line $\beta = -j$ since by (15) and (17) one gets zero unless $\alpha + j = \alpha - \beta$. Thus the total complex gives exactly the shifted truncated de Rham complex $(\Omega_{X_{\mathbb{C}}}^{\leq j}, d)[-2j]$ as shown in Figure 5. Using a Cartan-Eilenberg injective resolution $(E^{*,*}; d, \delta)$ of the de Rham complex, one obtains a Cartan-Eilenberg injective resolution $(J^{*,*}; d_1, d_2)$ of the cochain complex \mathcal{T}^* of the following form ($r, s \in \mathbb{Z}$)

$$J^{r,s} := \bigoplus_{\substack{p \leq -r, \\ p \equiv r(2)}} E^{p,s} = \bigoplus_{p \equiv r(2)} E^{p,s} / F^{-r+1}. \quad (19)$$

This is the quotient of the strictly periodic bi-complex $(P^{r,s} = \bigoplus_{p \equiv r(2)} E^{p,s}; d, \delta)$ by the sub-complex

$$(N^{r,s} = \bigoplus_{\substack{p > -r, \\ p \equiv r(2)}} E^{p,s}; d, \delta) \quad (20)$$

corresponding to the Hodge filtration. The differentials in (19) are given by the truncation d_1 of the horizontal differential in the resolution $(E^{*,*}; d, \delta)$ of the de

Rham complex and by $d_2 = \delta$. The λ -decomposition corresponds to the following decomposition of $J^{*,*}$

$$J^{r,s} = \bigoplus_{j \geq 0} J^{r,s}(j), \quad J^{r,s}(j) = \begin{cases} E^{r+2j,s}, & \text{if } r+j \leq 0; \\ \{0\}, & \text{otherwise.} \end{cases}$$

Thus, since $r+j \leq 0 \iff r+2j \leq j$, $(J^{*,*}(j); d_1, d_2)$ provides a resolution of the truncated de Rham complex $\Omega_{X(\mathbb{C})}^{\leq j}[-2j]$. By computing the hypercohomology, one gets (see [34], Theorem 3.3)

$$HC_n^{(j)}(X_{\mathbb{C}}) = \mathbb{H}^{2j-n}(X_{\mathbb{C}}, \Omega_{X_{\mathbb{C}}}^{\leq j}), \quad HC_n(X_{\mathbb{C}}) = \prod_{j \in \mathbb{Z}} \mathbb{H}^{2j-n}(X_{\mathbb{C}}, \Omega_{X_{\mathbb{C}}}^{\leq j}).$$

The periodicity operator S , which in the bicomplex $C^{*,*}$ was the translation by the vector $(1, 1)$, acquires here a simple meaning as an endomorphism of the bi-complex $J^{*,*}$ given by translation by the vector $(2, 0)$, as Figure 6 shows.

4.2. Resolution using differential forms of type (p, q)

Let $X_{\mathbb{C}}$ be a smooth, projective variety over \mathbb{C} . In Appendix A (see A.2), we define a canonical morphism of schemes $\pi_X : \text{Spec}(C^\infty(X_{\text{sm}}, \mathbb{C})) \rightarrow X_{\mathbb{C}}$. In order to understand the effect of π_X on cyclic homology we use, as in [34], the results of [30, 15] to pass from the Zariski to the analytic topology and then use the Dolbeault resolution as a soft resolution of the de Rham complex. Thus the parallel discussion as given in §4.1.2 applies using instead of the injective sheaf $E^{p,q}$ on $X_{\mathbb{C}}$ the soft sheaf $A^{p,q}$ of smooth complex differential forms of type (p, q) on the manifold X_{sm} . Thus the hypercohomology \mathbb{H}^{-n} of the following bi-complexes ($r, s \in \mathbb{Z}$) computes the cyclic homologies $HC_n(X_{\mathbb{C}})$ resp., $HC_n^{(j)}(X_{\mathbb{C}})$

$$C^{r,s} := \bigoplus_{\substack{p \leq -r, \\ p \equiv r(2)}} A^{p,s}, \quad C^{r,s}(j) := \begin{cases} A^{r+2j,s}, & \text{if } r+j \leq 0; \\ \{0\}, & \text{otherwise.} \end{cases} \quad (21)$$

The differentials are defined as follows: $d_1 = \partial$ and $d_2 = \bar{\partial}$, where $d_1 = \partial$ is truncated according to the identification of $C^{*,*}$ with the quotient of the strictly periodic bicomplex by the sub-complex $N^{r,s}$ corresponding to the Hodge filtration as in (20). The periodic bi-complex and its λ -decomposition are given by

$$(\tilde{P}^{r,s} = \bigoplus_{p \equiv r(2)} A^{p,s}; \partial, \bar{\partial}), \quad \tilde{P}^{r,s}(j) = A^{r+2j,s}, \quad \forall r, s, j \in \mathbb{Z}. \quad (22)$$

The next result shows an equivalent analytic way to describe the cyclic homology of $X_{\mathbb{C}}$. As usual, $T_{\mathbb{C}}^*$ denotes the complexified cotangent bundle on the smooth manifold X_{sm} .

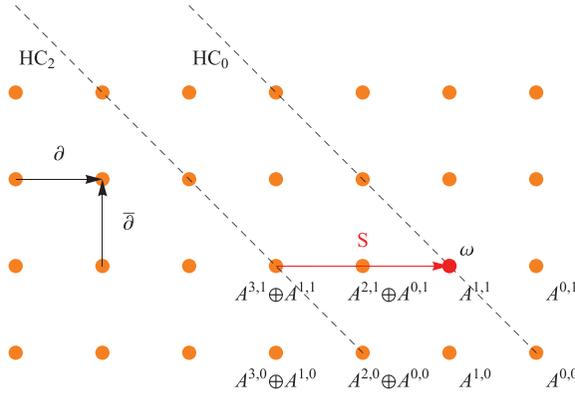


Figure 7: The bi-complex $(C^{*,*}, \partial, \bar{\partial})$ computing the cyclic homology of $X_{\mathbb{C}}$.

Lemma 4.1 *Let $X_{\mathbb{C}}$ be a smooth complex projective variety. Then in degree $\leq -\dim(X)$ the total complex $\text{Tot} C^{*,*}$ of (21) coincides with the complex of smooth differential forms of given parity on the associated smooth manifold X_{sm}*

$$\dots \xrightarrow{d} C^{\infty}(X_{\text{sm}}, \wedge^{\text{even}} T_{\mathbb{C}}^*) \xrightarrow{d} C^{\infty}(X_{\text{sm}}, \wedge^{\text{odd}} T_{\mathbb{C}}^*) \xrightarrow{d} \dots \quad (23)$$

where d is the usual differential. The total complex $\text{Tot} P^{*,*}$ of (22) coincides with (23), and the total complex $\text{Tot} P^{*,*}(j) = PC^*(j)$ is given by the sub-complex

$$\dots \xrightarrow{d} C^{\infty}(X_{\text{sm}}, \wedge^{k+2j} T_{\mathbb{C}}^*) \xrightarrow{d} C^{\infty}(X_{\text{sm}}, \wedge^{k+1+2j} T_{\mathbb{C}}^*) \xrightarrow{d} \dots \quad (24)$$

Proof: The component of degree k of the total complex $\text{Tot} C^{*,*}$ is described by

$$\bigoplus_{\substack{r+s=k \\ s \geq 0}} C^{r,s} = \bigoplus_{r+s=k} \bigoplus_{\substack{p \leq -r \\ p \equiv r(2)}} A^{p,s}.$$

If $k \leq -\dim(X)$ then $r \leq -\dim(X)$ and the condition $p \leq -r$ is automatic so that one derives

$$\bigoplus_{r+s=k} C^{r,s} = \bigoplus_{r+s=k} \bigoplus_{p \equiv r(2)} A^{p,s} = \bigoplus_{p+s \equiv k(2)} A^{p,s} = C^{\infty}(X_{\text{sm}}, \wedge^{k(\text{mod}2)} T_{\mathbb{C}}^*).$$

Moreover the co-boundary operator of $\text{Tot} C^{*,*}$ is $\partial + \bar{\partial} = d$ so the statement follows. The remaining statements follow from (22). \square

Next, we describe in more details the action of the periodicity map S of cyclic homology on the bi-complex $(C^{*,*}; \partial, \bar{\partial})$ as in (21).

Lemma 4.2 *The following map defines an endomorphism of degree 2 of the bi-complex $(C^{*,*}; \partial, \bar{\partial})$*

$$S\omega = \omega / F^{-r-1} \in C^{r+2,s}, \quad \forall \omega \in C^{r,s}. \quad (25)$$

Proof: It is enough to show that the sub-complex

$$(\tilde{N}^{r,s} = \bigoplus_{\substack{p > -r, \\ p \equiv r(2)}} A^{p,s}; d, \delta)$$

of the periodic bi-complex (22) is stable under the map S . A differential form ω belongs to $\tilde{N}^{r,s} \subset \tilde{P}^{r,s}$ if and only if its homogeneous components do so. Thus, for ω of type (p,s) this means that $p \equiv r \pmod{2}$ and that $p \geq -r + 1$. The element $S\omega$ is of type (p,s) and one needs to check that it belongs to $\tilde{N}^{r',s}$ with $r' = r + 2$. One has $p \equiv r' \pmod{2}$ and $p \geq -r' + 1$. \square

Correspondingly, the bi-complex that computes the Hochschild homology of $X_{\mathbb{C}}$ is described in Figure 8. It fills up a finite square in the plane and it is defined by

$$C_h^{r,s} := (A^{-r,s}; 0, \bar{\partial}).$$

This bi-complex maps by inclusion into the bi-complex $(C^{*,*}, \partial, \bar{\partial})$ and it coincides precisely with the kernel (bi-complex) of the map S . Thus, by using the surjectivity of S at the level of complexes, one derives an exact sequence of associated total complexes

$$0 \rightarrow C_h^* \rightarrow C^* \xrightarrow{S} C^{*+2} \rightarrow 0. \quad (26)$$

The corresponding long exact sequence of cohomology groups is then the *SBI* exact sequence

$$\dots \rightarrow HH_n(X_{\mathbb{C}}) \xrightarrow{I} HC_n(X_{\mathbb{C}}) \xrightarrow{S} HC_{n-2}(X_{\mathbb{C}}) \xrightarrow{B} HH_{n-1}(X_{\mathbb{C}}) \rightarrow \dots$$

Remark 4.3 Since the smooth manifold X_{sm} associated to $X_{\mathbb{C}} = X_{\text{alg}}$ (see [30]) can be endowed with the structure of a Kähler manifold, the map S is surjective at the level of cyclic homology (see [34] Proposition 4.1). We discuss (in the Kähler case) a specific example where the result might look surprising. We consider a cyclic homology class $[\omega] \in HC_0(X_{\mathbb{C}})$ coming from a form ω of type $(1,1)$ (see Figure 7). One has $\bar{\partial}\omega = 0$ although there is no condition on $\partial\omega$ since the horizontal co-boundary, is automatically zero due to the truncation. It seems surprising that $[\omega]$ can belong to the image of S since this means exactly that there exists a form ω'

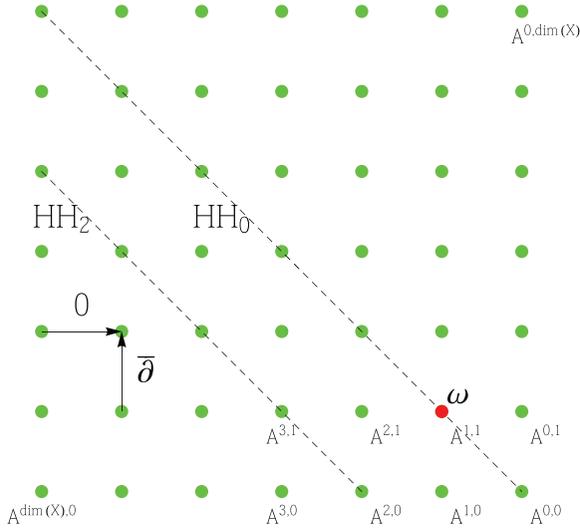


Figure 8: The bi-complex $C_h^{r,s} = (A^{-r,s}, 0, \bar{\partial})$ computing the Hochschild homology of $X_{\mathbb{C}}$.

also of type $(1, 1)$ and also representing the above class (*i.e.* $[\omega] = [\omega'] \in HC_0(X_{\mathbb{C}})$) and such that

$$\bar{\partial}\omega' = 0 = \partial\omega'.$$

The reason why this holds is that one can modify ω to $\omega' = \omega + \bar{\partial}\alpha$ with α of type $(1, 0)$ by using the Kähler metric so that ω' becomes harmonic for any choice of the three Laplacians

$$\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} = \partial\bar{\partial}^* + \bar{\partial}^*\partial = \frac{1}{2}(dd^* + d^*d).$$

Then it follows that ω' is automatically also $\bar{\partial}$ -closed.

4.3. The periodic cyclic homologies $HP_*(X_{\mathbb{C}})$ and $HP_*(C^\infty(X_{sm}, \mathbb{C})_{top})$

We shall now compare the periodic cyclic homology $HP_*(X_{\mathbb{C}})$ of a smooth complex projective variety $X_{\mathbb{C}}$ (viewed as a scheme over \mathbb{C}) with the periodic cyclic homology $HP_*(C^\infty(X_{sm}, \mathbb{C})_{top})$ of the underlying smooth manifold X_{sm} .

By [5] (*see* Theorem 46), the periodic cyclic homology $HP_*(A_{top})$ of the topological algebra associated to $A = C^\infty(X_{sm}, \mathbb{C})$ is expressed in terms of the de Rham cohomology of smooth differential forms, *i.e.* as the Betti cohomology with complex coefficients $H_B^*(X(\mathbb{C}), \mathbb{C})$. This requires taking into account the natural

Frechet topology on the algebra $A = C^\infty(X_{\text{sm}}, \mathbb{C})$: we refer to Appendix B for details.

Proposition 4.4 *Let $X_{\mathbb{C}}$ be a smooth, complex projective variety.*

(i) *The map $(\text{top}) \circ \pi_X^*$ of cyclic homology groups induced by the morphism of schemes (see Appendix A for the definition of π_X)*

$$\pi_X : \text{Spec}(A) \rightarrow X_{\mathbb{C}}, \quad A = C^\infty(X_{\text{sm}}, \mathbb{C})$$

and the morphism of functors $\text{top} : C(A) \rightarrow C(A_{\text{top}})$ (see Appendix B, (61)) is the composite of the following two isomorphisms

$$\begin{array}{ccc} HP_n^{(j)}(X_{\mathbb{C}}) & \xrightarrow{(\text{top}) \circ \pi_X^*} & HP_n^{(j)}(A_{\text{top}}) \\ & \searrow \sim \scriptstyle \zeta & \nearrow \sim \\ & H_B^{2j-n}(X(\mathbb{C}), \mathbb{C}) & \end{array}$$

(ii) *For any pair of integers n, j with $n \geq 2\dim(X)$ and $\frac{n}{2} \leq j \leq n$, the statement in (i) holds at the level of cyclic homology groups, i.e.*

$$HC_n^{(j)}(X_{\mathbb{C}}) \cong H_B^{2j-n}(X(\mathbb{C}), \mathbb{C}) \cong HC_n^{(j)}(A_{\text{top}})$$

where the upper index (j) refers to the corresponding component of the λ -decomposition.

Proof: (i) Take a covering \mathcal{U} of $X_{\mathbb{C}}$ made by affine Zariski open sets U_i . The morphism π_X is affine (as is any morphism of an affine scheme to a projective scheme) so the inverse images $V_i = \pi_X^{-1}(U_i)$ form an open affine covering \mathcal{V} of the scheme $\text{Spec}(A)$. The discussion of §4.1.2 applies using, instead of the injective resolution $(E^{*,*}, d, \delta)$ of the algebraic de Rham complex, the Čech bicomplex (see [17] Lemma III 4.2 and Theorem III 4.5)

$$(\check{C}^{p,q} = C^q(\mathcal{U}, \Omega^p), d, \delta)$$

where d is the de Rham coboundary and δ the Čech coboundary. In particular $HP_n^{(j)}(X_{\mathbb{C}})$ is the cohomology H^{-n} of the cochain complex

$$\left(\bigoplus_{r+s=*} \check{C}^{r+2j,s}, d + \delta \right).$$

Let then $\omega \in HP_n^{(j)}(X_{\mathbb{C}})$ be represented by a cocycle

$$\omega = \sum_{r+s=-n} \omega_{r,s}, \quad \omega_{r,s} \in C^s(\mathcal{U}, \Omega^{r+2j})$$

The image $\pi_X^*(\omega) \in HP_n^{(j)}(A)$ is then given by the corresponding Čech cocycle for the affine open covering \mathcal{V} of $\text{Spec}(A)$. Now the map π_X^* coincides on the affine Zariski open sets $U := U_{i_0, \dots, i_k} = \cap U_{i_j}$ with the inclusion $\Gamma(U, \Omega_{X_{\mathbb{C}}}^p) \xrightarrow{\iota} C^\infty(U, \wedge^{p,0} T_{\mathbb{C}}^*)$ of algebraic sections of $\Omega_{X_{\mathbb{C}}}^p$ into the space of smooth sections of the vector bundle $\wedge^{p,0} T_{\mathbb{C}}^*$ of (complex) differential forms of type $(p,0)$. This inclusion is the restriction of the corresponding inclusion for the analytic space X_{an} and the latter is a morphism for the following two resolutions of the constant sheaf \mathbb{C} for the usual topology. The first resolution is given by the sheaves of holomorphic differential forms

$$0 \rightarrow \mathbb{C} \rightarrow \Omega_{X(\mathbb{C})}^0 \xrightarrow{\partial} \Omega_{X(\mathbb{C})}^1 \rightarrow \dots$$

and the second resolution is the de Rham complex of sheaves of C^∞ differential forms

$$0 \rightarrow \mathbb{C} \rightarrow C^\infty(\bullet, \wedge^0 T_{\mathbb{C}}^*) \xrightarrow{d} C^\infty(\bullet, \wedge^1 T_{\mathbb{C}}^*) \rightarrow \dots$$

This shows that the Betti cohomology class of ω is the same as the Betti cohomology class of $\pi_X^*(\omega)$ which is represented by the Čech cocycle $\iota(\omega)$ in the Čech bicomplex $C^{a,b} = C^b(\mathcal{V}, \wedge^a T_{\mathbb{C}}^*)$ of smooth differential forms, associated to the covering \mathcal{V} . In this bicomplex the vertical lines are acyclic since one is in the affine case ([17] III Remark 3.5.1). In fact more precisely one can use the factorization provided in Theorem A.1 of π_X through a noetherian affine scheme W and apply Theorem III 4.5 and Theorem III 3.7 of [17] to prove the acyclicity at the level of W and thus conclude that $\iota(\omega)$ is cohomologous to a global section *i.e.* a closed differential form of degree $2j - n$. Passing from $HP_n^{(j)}(A)$ to $HP_n^{(j)}(A_{\text{top}})$ does not alter the above computation but ensures that the map $H_B^{2j-n}(X(\mathbb{C}), \mathbb{C}) \rightarrow HP_n^{(j)}(A_{\text{top}})$ is an isomorphism.

(ii) For $n \geq 2\dim(X)$, one has $HC_n(A_{\text{top}}) = \oplus_{j \geq n/2} H_B^{2j-n}(X(\mathbb{C}), \mathbb{C})$. The statement then follows from the λ -decomposition $HC_n^{(j)}(X_{\mathbb{C}})$ by applying Proposition 2.2 and Theorem 4.6.10 of [23] (adapted to the topological case) for $HC_n^{(j)}(A_{\text{top}})$. Note that as $j \geq n/2 \geq \dim(X)$, one has $F^{j+1} = \{0\}$ in Proposition 2.2, and as $n \geq 2\dim(X) = \dim(X_{\text{sm}})$, Theorem 4.6.10 of [23] remains valid for the boundary value $j = n$ since all forms of top degree are automatically closed. \square

Remark 4.5 (i) To understand the behavior of the map π_X^* we give a simple example. Let $X_{\mathbb{C}} = \mathbb{P}_{\mathbb{C}}^1$ be the complex projective line. It is obtained by gluing two affine lines $U_+ = \text{Spec}(\mathbb{C}[z])$ and $U_- = \text{Spec}(\mathbb{C}[1/z])$ on the intersection $U = \text{Spec}(\mathbb{C}[z, 1/z])$. Let $\mathcal{U} = \{U_{\pm}\}$ be the affine covering of $X_{\mathbb{C}}$, then the differential form $dz/z \in C^1(\mathcal{U}, \Omega_{X_{\mathbb{C}}}^1)$ determines a cocycle ω in the Čech bicomplex. Let $[\omega]_{\text{cyc}} \in HC_0(X_{\mathbb{C}})$ be the corresponding cyclic homology class as in Figure 6. One has $[\omega]_{\text{cyc}} \in HC_n^{(j)}(X_{\mathbb{C}})$ with $n = 0$, $2j - n = 2$ thus $[\omega]_{\text{cyc}} \in HC_0^{(1)}(X_{\mathbb{C}})$. Set

$A = C^\infty(X_{\text{sm}}, \mathbb{C})$ as above. As in any commutative algebra one has $HC_0(A) = HC_0^{(0)}(A)$ (see [23] Theorem 4.6.7), it follows that $\pi_X^*([\omega]_{\text{cyc}}) \in HC_0^{(1)}(A) = \{0\}$. Let now $[\omega]_{\text{per}} \in HC_2(X_{\mathbb{C}})$ be represented by the same cocycle ω in the Čech bicomplex, with $S[\omega]_{\text{per}} = [\omega]_{\text{cyc}}$ as in Figure 6. One has $[\omega]_{\text{per}} \in HC_2^{(2)}(X_{\mathbb{C}})$ and $\pi_X^*([\omega]_{\text{per}}) \in HC_2^{(2)}(A)$ is obtained by first writing the cocycle ω as a coboundary $\delta(\xi)$ in the Čech bicomplex of A . For $\xi = (\xi_+, \xi_-)$, $\xi_{\pm} \in \Gamma(\pi_X^{-1}(U_{\pm}, \Omega_A^1)$, the Čech coboundary is $\delta(\xi) = \xi_+ - \xi_- \in \Gamma(\pi_X^{-1}(U_+ \cap U_-, \Omega_A^1))$. Here one finds

$$\omega = \xi_+ - \xi_-, \quad \xi_+ = \bar{z}dz/(1+z\bar{z}), \quad \xi_- = -dz/(z(1+z\bar{z})),$$

where $\xi_{\pm} \in \Gamma(\pi_X^{-1}(U_{\pm}, \Omega_A^1))$. Hence the class $\pi_X^*([\omega]_{\text{per}}) \in HC_2^{(2)}(A)$ is represented by the 2-form $\omega_2 = -d\xi \in \Gamma(\text{Spec}(A), \Omega_A^2)$

$$\omega_2 = -dz \wedge d\bar{z}/(1+z\bar{z})^2 \in \Gamma(\text{Spec}(A), \Omega_A^2).$$

A similar computation performed in the Čech bicomplex of A , starting with $\pi_X^*([\omega]_{\text{cyc}})$, produces 0 because the coboundary d is 0 since there is no component Ω^2 in $J^{0,0}$ (see Figure 6).

(ii) In the above example (i), it suffices to adjoin the variable \bar{z} in order to obtain the existence of global affine coordinates such as $z/(1+z\bar{z})$, an affine scheme W and morphism $\pi : W \rightarrow X_{\mathbb{C}}$ so that $\pi^*([\omega]_{\text{per}}) \in HC_2^{(2)}(\mathcal{O}(W))$ is represented by a global 2-form. This illustrates the general fact, mentioned above, that the map π_X factors through an affine scheme W whose construction is entirely in the realm of algebraic geometry. The general construction of this factorization is provided in Appendix A.

(iii) Proposition 2.2 shows that the cyclic homology $HC_n(X_{\mathbb{C}})$ already stabilizes, *i.e.* becomes periodic, for $n \geq \dim(X)$. On the other hand, the cyclic homology $HC_n(A_{\text{top}})$ of the underlying smooth manifold only stabilizes for $n \geq 2\dim(X)$, and is infinite dimensional for lower values of n . This shows that these two homologies cannot coincide in the unstable region $n < 2\dim(X)$.

4.4. The Tate-twisted map $\tau : PC_{\text{real}}^*(X_{\mathbb{C}}) \rightarrow PC^*(X_{\mathbb{C}})$

We keep denoting with $X_{\mathbb{C}}$ a smooth, projective algebraic variety over \mathbb{C} and with X_{sm} the associated smooth manifold. Let $PC^*(X_{\mathbb{C}})$ and $PC^*(C^\infty(X_{\text{sm}}, \mathbb{C}))_{\text{top}}$ be cochain complexes which compute the periodic cyclic homologies as

$$\begin{aligned} HP_n(X_{\mathbb{C}}) &= H^{-n}(PC^*(X_{\mathbb{C}})), \\ HP_n(C^\infty(X_{\text{sm}}, \mathbb{C}))_{\text{top}} &= H^{-n}(PC^*(C^\infty(X_{\text{sm}}, \mathbb{C}))_{\text{top}}). \end{aligned} \quad (27)$$

In order to define the Tate twisted map

$$\tau : HP_*^{\text{real}}(X_{\mathbb{C}}) \rightarrow HP_*(X_{\mathbb{C}}) \quad (28)$$

at the level of complexes of cochains, we shall use a derived pullback of two maps of complexes which correspond to the morphisms of schemes

$$\pi_X : \text{Spec}(C^\infty(X_{\text{sm}}, \mathbb{C})) \rightarrow X_{\mathbb{C}}, \quad \iota : \text{Spec}(C^\infty(X_{\text{sm}}, \mathbb{C})) \rightarrow \text{Spec}(C^\infty(X_{\text{sm}}, \mathbb{R}))$$

where the morphism ι is induced by the natural inclusion $C^\infty(X_{\text{sm}}, \mathbb{R}) \subset C^\infty(X_{\text{sm}}, \mathbb{C})$. We recall that if $A \xrightarrow{f} C \xleftarrow{g} B$ is a diagram of cochain complexes in an abelian category, its derived pullback is defined as the cochain complex

$$A \times_C B := \text{Cone}(A \oplus B \xrightarrow{f-g} C)[1].$$

The two natural projections give rise to maps from $A \times_C B$ to A and B *resp.*, and the diagram

$$\begin{array}{ccc} A \times_C B & \xrightarrow{f'} & B \\ \downarrow g' & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

commutes up-to canonical homotopy. The short exact sequence of complexes

$$0 \rightarrow \text{Cone}(A \xrightarrow{f} C)[1] \rightarrow A \times_C B \xrightarrow{f'} B \rightarrow 0 \quad (29)$$

then determines an induced long exact sequence of cohomology groups. In particular when f is a quasi isomorphism, so is f' and the maps $g : B \rightarrow C$ and $g' : A \times_C B \rightarrow A$ are quasi isomorphic.

We now consider two morphism of complexes. The first is given by

$$\text{top} \circ \pi_X^* : PC^*(X_{\mathbb{C}}) \rightarrow PC^*(C^\infty(X_{\text{sm}}, \mathbb{C})_{\text{top}}), \quad (30)$$

where π_X^* is the map of complexes induced from the morphism π_X and top is defined in (61) of Appendix B. The second morphism of complexes is

$$(2\pi i)^{\Theta_0} \circ \iota : PC^*(C^\infty(X_{\text{sm}}, \mathbb{R})_{\text{top}}) \rightarrow PC^*(C^\infty(X_{\text{sm}}, \mathbb{C})_{\text{top}}) \quad (31)$$

where ι is the tautological morphism of complexes associated to the inclusion $C^\infty(X_{\text{sm}}, \mathbb{R}) \subset C^\infty(X_{\text{sm}}, \mathbb{C})$ and Θ_0 , which acts on $PC^*(C^\infty(X_{\text{sm}}, \mathbb{C})_{\text{top}})$, is defined as in (4), *i.e.* is the generator of the λ -operations.

Definition 4.6 We define $PC_{\text{real}}^*(X_{\mathbb{C}})$ to be the derived pullback of the two morphisms of complexes (30) and (31). We denote by $\tau : PC_{\text{real}}^*(X_{\mathbb{C}}) \rightarrow PC^*(X_{\mathbb{C}})$ the projection map of complexes on the first factor of the derived pullback.

Both morphisms of complexes are compatible with the λ -decomposition and one thus obtains a corresponding λ -decomposition of $PC_{\text{real}}^*(X_{\mathbb{C}})$ and an induced λ -decomposition in homology

$$HP_n^{\text{real}}(X_{\mathbb{C}}) = \bigoplus HP_n^{\text{real},(j)}(X_{\mathbb{C}}).$$

The map $\text{top} \circ \pi_X^* : PC^*(X_{\mathbb{C}}) \rightarrow PC^*(C^\infty(X_{\text{sm}}, \mathbb{C})_{\text{top}})$ is an isomorphism in cohomology (the composition with top takes into account the Frechet topology of $C^\infty(X_{\text{sm}}, \mathbb{C})$, see Appendix B). Thus, the map $\tau : HP_n^{\text{real}}(X_{\mathbb{C}}) \rightarrow HP_n^*(X_{\mathbb{C}})$ is quasi isomorphic to the map $(2\pi i)^{\Theta_0} \circ \iota : PC^*(C^\infty(X_{\text{sm}}, \mathbb{R})_{\text{top}}) \rightarrow PC^*(C^\infty(X_{\text{sm}}, \mathbb{C})_{\text{top}})$. This shows, using the component (j) in the λ -decomposition, that one has a commutative diagram

$$\begin{array}{ccc} HP_{n+2}^{\text{real},(j+1)}(X_{\mathbb{C}}) & \xrightarrow{S} & HP_n^{\text{real},(j)}(X_{\mathbb{C}}) \\ \downarrow \tau & & \downarrow \tau \\ HP_{n+2}^{(j+1)}(X_{\mathbb{C}}) & \xrightarrow{(2\pi i)^{-1}S} & HP_n^{(j)}(X_{\mathbb{C}}). \end{array} \quad (32)$$

In the simplest case when $X_{\mathbb{C}}$ is a single point, one has $HP_n^{(j)}(X_{\mathbb{C}}) \neq \{0\}$ only for $n = 2j$ and in that case the map $\tau : HP_n^{\text{real},(j)}(X_{\mathbb{C}}) \rightarrow HP_n^{(j)}(X_{\mathbb{C}})$ is the multiplication by $(2\pi i)^j$ from \mathbb{R} to \mathbb{C} . We denote by $\mathbb{R}(j)$ the subgroup $(2\pi i)^j \mathbb{R} \subset \mathbb{C}$.

Proposition 4.7 *The isomorphism ς as in Proposition 4.4 (i) lifts to an isomorphism*

$$HP_n^{\text{real},(j)}(X_{\mathbb{C}}) \xrightarrow[\sim]{\varsigma} H_B^{2j-n}(X(\mathbb{C}), \mathbb{R}(j))$$

such that the following diagram commutes

$$\begin{array}{ccc} HP_n^{\text{real},(j)}(X_{\mathbb{C}}) & \xrightarrow[\sim]{\varsigma} & H_B^{2j-n}(X(\mathbb{C}), \mathbb{R}(j)) \\ \downarrow \tau & & \downarrow \subset \\ HP_n^{(j)}(X_{\mathbb{C}}) & \xrightarrow[\sim]{\varsigma} & H_B^{2j-n}(X(\mathbb{C}), \mathbb{C}). \end{array} \quad (33)$$

Proof: Taking the cohomology H^{-n} and the component (j) of the λ -decomposition, the map

$$(2\pi i)^{\Theta_0} \circ \iota : PC^*(C^\infty(X_{\text{sm}}, \mathbb{R})_{\text{top}}) \rightarrow PC^*(C^\infty(X_{\text{sm}}, \mathbb{C})_{\text{top}})$$

is isomorphic to the map

$$H_B^{2j-n}(X(\mathbb{C}), \mathbb{R}(j)) \rightarrow H_B^{2j-n}(X(\mathbb{C}), \mathbb{C})$$

associated to the additive group homomorphism $\mathbb{R}(j) \subset \mathbb{C}$. □

5. Archimedean cyclic homology

In this section we give the definition and describe the first properties of archimedean cyclic homology. For the construction, we follow the parallel with the description of the real Deligne cohomology that can be interpreted (see §3.3) as the hypercohomology of a complex of sheaves on X_{sm} which is, up to a shift of degrees, the derived pullback of the two natural maps of complexes (§2.1)

$$\mathbb{R}(r) \rightarrow \Omega_{X(\mathbb{C})}^*, \quad F^r \Omega_{X(\mathbb{C})}^* \rightarrow \Omega_{X(\mathbb{C})}^*.$$

In cyclic homology, the Hodge filtration gets replaced by negative cyclic homology: see §5.1. In §5.2 we describe the archimedean cyclic homology of a smooth, projective complex algebraic variety. In §5.3, we give the corresponding definition in case the algebraic variety is defined over the reals, by using the action of the anti-linear Frobenius.

5.1. Negative cyclic homology

We recall (see [18]) that the negative cyclic homology HN_* (of a cyclic k -module over a field k) is defined by dualizing the cyclic cohomology HC^* with respect to the coefficient polynomial ring $k[v] = HC^*(k)$. Here, the variable v has degree two and corresponds to the periodicity map S in cyclic cohomology. At the level of the defining bi-complexes, the negative cyclic homology bi-complex $NC_{*,*}$ coincides, up-to a shift of degrees, with the kernel of the natural map of complexes connecting the periodic (b, B) bi-complex $PC_{*,*}$ and the (b, B) bi-complex $CC_{*,*}$. We set $u = v^{-1}$ and let (C_*, b, B) be a mixed complex of modules or sheaves. Then one finds (see [18])

$$\begin{aligned} NC_{*,*} &= k[u] \hat{\otimes}_k C_*, & PC_{*,*} &= k[u, u^{-1}] \hat{\otimes}_k C_*, \\ CC_{*,*} &= (k[u, u^{-1}]/uk[u]) \hat{\otimes}_k C_*. \end{aligned}$$

Here, the two boundary operators are b and uB and if M and N are graded k -modules, one defines $M \hat{\otimes} N$ by $(M \hat{\otimes} N)_n = \prod M_i \otimes N_{n-i}$.

In the context of the sheafified theories over a scheme X_k over a field k , it is convenient (see [34]) to re-index the above bi-complexes in negative degrees, so that they turn into cohomology bi-complexes. Figure 9 shows their behavior.

Then the hyper-cohomology \mathbb{H}^{-n} of the negative cochain bi-complex $NC^{*,*}$ is denoted by $HN_n(X_k)$, the hyper-cohomology \mathbb{H}^{-n} of the periodic cochain bi-complex $PC^{*,*}$ is denoted $HP_n(X_k)$ and the hyper-cohomology \mathbb{H}^{-n} of the quotient cochain bi-complex is $HC_{n-2}(X_k)$. One has a short exact sequence of

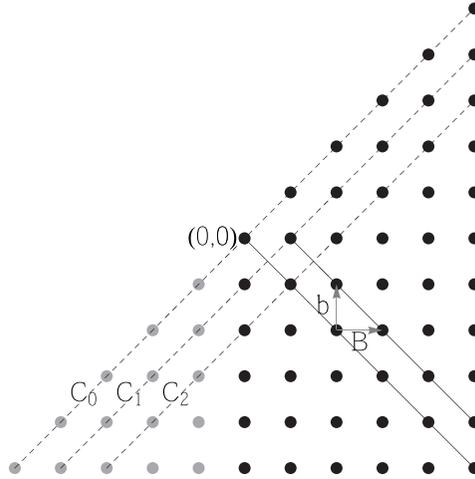


Figure 9: In black, the negative sub-complex $NC_{*,*}$ of the periodic (b, B) bi-complex $PC_{*,*}$ all re-indexed in negative degrees. The hypercohomology \mathbb{H}^{-n} of the black cochain bi-complex is HN_n . The hypercohomology \mathbb{H}^{-n} of the quotient (in gray) cochain bi-complex is HC_{n-2} .

cochain bi-complexes of the form

$$0 \rightarrow NC^{*,*} \xrightarrow{I} PC^{*,*} \rightarrow CC^{*,*}[-1, -1] \rightarrow 0 \tag{34}$$

For instance, one has $NC^{-1,-1} = 0$, $PC^{-1,-1} = CC^{0,0}$, $CC^{-1,-1}[-1, -1] = CC^{0,0}$. We let $PC^j = \text{Tot}^j(PC^{*,*})$ and $NC^j = \text{Tot}^j(NC^{*,*})$ be the total complexes and we denote with $I : NC^j \rightarrow PC^j$ the canonical inclusion. Then (34) yields the short exact sequence of total cochain complexes

$$0 \rightarrow NC^* \xrightarrow{I} PC^* \rightarrow CC^*[-2] \rightarrow 0. \tag{35}$$

This sequence induces the long exact sequence of hypercohomology groups \mathbb{H}^{-n}

$$\dots \rightarrow HN_n(X_k) \xrightarrow{I} HP_n(X_k) \xrightarrow{S} HC_{n-2}(X_k) \xrightarrow{B} HN_{n-1}(X_k) \rightarrow \dots \tag{36}$$

that sits on top of the *SBI* long exact sequence in a canonical commutative diagram (see [23], Proposition 5.1.5). The λ -decomposition is compatible with (36), therefore one also has the following exact sequence (see [23], Theorem 4.6.9, is the analogous statement for the *SBI* sequence)

$$\dots \rightarrow HN_n^{(j)}(X_k) \xrightarrow{I} HP_n^{(j)}(X_k) \xrightarrow{S} HC_{n-2}^{(j-1)}(X_k) \xrightarrow{B} HN_{n-1}^{(j)}(X_k) \rightarrow \dots$$

Note that the index j is fixed all along the sequence. The sequence (35) determines a canonical isomorphism

$$HC_{n-2}(X_k) \cong \mathbb{H}^{-n}(X_k, \text{Cone}(NC^* \xrightarrow{I} PC^*)). \quad (37)$$

5.2. The archimedean cyclic homology of $X_{\mathbb{C}}$

For a smooth complex projective variety $X_{\mathbb{C}}$ we consider the complex which is (up to a shift of degrees) the derived pullback of the two maps of complexes (see Definition 4.6)

$$I : NC^*(X_{\mathbb{C}}) \rightarrow PC^*(X_{\mathbb{C}}), \quad \text{and} \quad \tau : PC_{\text{real}}^*(X_{\mathbb{C}}) \rightarrow PC^*(X_{\mathbb{C}}).$$

Definition 5.1 The archimedean cyclic homology of a smooth projective complex variety $X_{\mathbb{C}}$ is defined as follows

$$HC_n^{\text{ar}}(X_{\mathbb{C}}) := \mathbb{H}^{-n} \left(X_{\mathbb{C}}, \text{Cone} \left(NC^*(X_{\mathbb{C}}) \oplus PC_{\text{real}}^*(X_{\mathbb{C}}) \xrightarrow{\beta} PC^*(X_{\mathbb{C}}) \right) [2] \right)$$

where the map β is given by

$$\beta(\omega, a) = I(\omega) - \tau(a).$$

The long exact sequence of cohomology groups associated to (29) produces in this context the following

Proposition 5.2 *There is a long exact sequence of the form*

$$\begin{aligned} \dots \rightarrow HP_{n+2}^{\text{real}}(X_{\mathbb{C}}) \xrightarrow{S \circ \tau} HC_n(X_{\mathbb{C}}) \rightarrow HC_n^{\text{ar}}(X_{\mathbb{C}}) \rightarrow HP_{n+1}^{\text{real}}(X_{\mathbb{C}}) \xrightarrow{S \circ \tau} \\ \rightarrow HC_{n-1}(X_{\mathbb{C}}) \rightarrow HC_{n-1}^{\text{ar}}(X_{\mathbb{C}}) \rightarrow \dots \end{aligned}$$

Proof: The exact sequence of complexes (29) takes here the form

$$\begin{aligned} 0 \rightarrow \text{Cone} \left(NC^*(X_{\mathbb{C}}) \xrightarrow{I} PC^*(X_{\mathbb{C}}) \right) \rightarrow \\ \rightarrow \text{Cone} \left(PC_{\text{real}}^*(X_{\mathbb{C}}) \oplus NC^*(X_{\mathbb{C}}) \xrightarrow{\beta} PC^*(X_{\mathbb{C}}) \right) \rightarrow \\ \rightarrow PC_{\text{real}}^*(X_{\mathbb{C}})[-1] \rightarrow 0. \end{aligned}$$

Moreover, (37) reads as $HC_{n-2}(X_{\mathbb{C}}) = \mathbb{H}^{-n}(X_{\mathbb{C}}, \text{Cone}(NC^*(X_{\mathbb{C}}) \xrightarrow{I} PC^*(X_{\mathbb{C}})))$. Passing to hypercohomology one gets the long exact sequence

$$\begin{aligned} \dots \rightarrow HC_{n-2}(X_{\mathbb{C}}) \rightarrow HC_{n-2}^{\text{ar}}(X_{\mathbb{C}}) \rightarrow \\ \rightarrow \mathbb{H}^{-n}(X_{\mathbb{C}}, PC_{\text{real}}^*(X_{\mathbb{C}})[-1]) \rightarrow HC_{n-3}(X_{\mathbb{C}}) \rightarrow \dots \end{aligned}$$

which takes the form

$$\cdots \rightarrow HC_{n-2}(X_{\mathbb{C}}) \rightarrow HC_{n-2}^{\text{ar}}(X_{\mathbb{C}}) \rightarrow HP_{n-1}^{\text{real}}(X_{\mathbb{C}}) \rightarrow HC_{n-3}(X_{\mathbb{C}}) \rightarrow \cdots$$

□

Next, we compare the exact sequence of Proposition 5.2 with the following exact sequence of real Deligne cohomology (*see* §3.3)

$$\begin{aligned} \rightarrow H_B^w(X(\mathbb{C}), \mathbb{R}(r)) \rightarrow H_{\text{dR,rel}}^w(X(\mathbb{C}), r) \rightarrow \\ \rightarrow H_{\mathcal{D}}^{w+1}(X_{\mathbb{C}}, \mathbb{R}(r)) \rightarrow H_B^{w+1}(X(\mathbb{C}), \mathbb{R}(r)) \rightarrow \cdots \end{aligned} \quad (38)$$

To achieve a correct comparison we implement the λ -decomposition in the exact sequence of Proposition 5.2 and deduce the following exact sequence

$$\begin{aligned} \rightarrow HP_{n+2}^{\text{real},(j+1)}(X_{\mathbb{C}}) \xrightarrow{S\sigma\tau} HC_n^{(j)}(X_{\mathbb{C}}) \rightarrow HC_n^{\text{ar},(j)}(X_{\mathbb{C}}) \rightarrow \\ \rightarrow HP_{n+1}^{\text{real},(j+1)}(X_{\mathbb{C}}) \xrightarrow{S\sigma\tau} HC_{n-1}^{(j)}(X_{\mathbb{C}}) \rightarrow \cdots \end{aligned} \quad (39)$$

Notice that in both exact sequences (38) and (39) the indices j and r are kept fixed, while the value of $w = 2j - n$ increases by 1 as n decreases by 1.

The following result and its Corollary 5.4 show that, after reindexing, the archimedean cyclic homology of a smooth complex projective variety is isomorphic to Deligne cohomology. The analogous result for varieties over \mathbb{R} will be established in Corollary 5.7.

Proposition 5.3 *For a smooth, complex projective variety $X_{\mathbb{C}}$ of dimension d , and for any pair of integers $(n, j) \in E_d$ (Lemma 3.1) there is a short exact sequence*

$$0 \rightarrow HP_{n+2}^{\text{real},(j+1)}(X_{\mathbb{C}}) \xrightarrow{S\sigma\tau} HC_n^{(j)}(X_{\mathbb{C}}) \rightarrow HC_n^{\text{ar},(j)}(X_{\mathbb{C}}) \rightarrow 0. \quad (40)$$

Proof: Let $(n, j) \in E_d$, then by Lemma 3.1 one derives that $(2j - n, j - n) \in A_d$. One then has the short exact sequence (14) (for $w = 2j - n$ and $r = j + 1$)

$$0 \rightarrow H_B^{2j-n}(X(\mathbb{C}), \mathbb{R}(j+1)) \rightarrow H_{\text{dR,rel}}^{2j-n}(X_{\mathbb{C}}, j+1) \rightarrow H_{\mathcal{D}}^{2j+1-n}(X_{\mathbb{C}}, \mathbb{R}(j+1)) \rightarrow 0. \quad (41)$$

Moreover, one has a canonical isomorphism (*see* Proposition 2.2)

$$HC_n^{(j)}(X_{\mathbb{C}}) \cong H_{\text{dR,rel}}^{2j-n}(X_{\mathbb{C}}, j+1).$$

One then uses (32) and Proposition 4.7 to identify the map $HP_{n+2}^{\text{real},(j+1)}(X_{\mathbb{C}}) \xrightarrow{S\sigma\tau} HC_n^{(j)}(X_{\mathbb{C}})$ with the map

$$H_B^{2j-n}(X(\mathbb{C}), \mathbb{R}(j+1)) \xrightarrow{\subset} H_{\text{dR,rel}}^{2j-n}(X_{\mathbb{C}}, j+1).$$

□

Corollary 5.4 *Let $X_{\mathbb{C}}$ be a smooth, complex projective variety of dimension d . Then for any pair of integers $(n, j) \in E_d$ one has*

$$H_{\mathcal{D}}^{2j+1-n}(X_{\mathbb{C}}, \mathbb{R}(j+1)) \cong HC_n^{\text{ar},(j)}(X_{\mathbb{C}})$$

Moreover $HC_n^{\text{ar},(j)}(X_{\mathbb{C}}) = \{0\}$ for $n \geq 0, j \in \mathbb{Z}$ and $(n, j) \notin E_d$.

Proof: The first statement follows by comparing the two exact sequences (41) and (40). We prove the second statement. By applying Proposition 3.1 of [34], one derives for $2j < n$ that $HC_n^{(j)}(X_{\mathbb{C}}) = 0$. Moreover the same vanishing also holds for $2j - n > 2d$ (*op.cit.* Proposition 4.1). Proposition 4.7 shows that for $2j - n > 2d$ one has (since $2(j+1) - (n+2) > 2d$ and $2(j+1) - (n+1) > 2d$)

$$HP_{n+2}^{\text{real},(j+1)}(X_{\mathbb{C}}) = 0, \quad HP_{n+1}^{\text{real},(j+1)}(X_{\mathbb{C}}) = 0. \quad (42)$$

Thus the exact sequence (39) shows that $HC_n^{\text{ar},(j)}(X_{\mathbb{C}}) = \{0\}$ for $2j - n > 2d$. For $2j < n$, if $2j < n - 1$ then Proposition 4.7 shows that (42) holds, thus (39) shows that $HC_n^{\text{ar},(j)}(X_{\mathbb{C}}) = \{0\}$. Let us consider now the limit case $n = 2j + 1$. In this case one has $2(j+1) - (n+1) = 0$ and thus $HP_{n+1}^{\text{real},(j+1)}(X_{\mathbb{C}}) = H_B^0(X(\mathbb{C}), \mathbb{R}(j+1)) \neq 0$ but the next map in (39) is the same as the map

$$H_B^0(X(\mathbb{C}), \mathbb{R}(j+1)) \rightarrow H_{\text{dR,rel}}^{2j-(n-1)}(X_{\mathbb{C}}, j+1) \cong HC_{n-1}^{(j)}(X_{\mathbb{C}}),$$

and this map is injective since $H_B^0 \cap F^{j+1} = 0$. Thus (39) gives $HC_n^{\text{ar},(j)}(X_{\mathbb{C}}) = \{0\}$. \square

5.3. The case of real schemes

Next, we consider the case of a smooth, complex projective variety defined over \mathbb{R} , $X_{\mathbb{R}}$. In this case following [34] $X_{\mathbb{R}}$ admits a cyclic homology theory over \mathbb{R} .

Lemma 5.5 *Let $X_{\mathbb{R}}$ be a smooth, real projective variety, and denote by $X_{\mathbb{C}} = X_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ the variety obtained by extension of scalars. Then*

$$HC_n(X_{\mathbb{C}}) = HC_n(X_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}, \quad \forall n \in \mathbb{Z}.$$

Proof: The result follows from Proposition 4.1 of [34] and the well known decomposition

$$H^q(X_{\mathbb{C}}, \Omega_{X_{\mathbb{C}}}^p) = H^q(X_{\mathbb{R}}, \Omega_{X_{\mathbb{R}}}^p) \otimes_{\mathbb{R}} \mathbb{C}.$$

\square

From Lemma 5.5 one derives the definition of a canonical anti-linear involution $\bar{F}_\infty = \text{id} \otimes \bar{}$ on $HC_n(X_{\mathbb{C}})$, by implementing the complex conjugation $\bar{}$ on \mathbb{C} . This involution should not be confused with the \mathbb{C} -linear involution F_∞ used in [10]. By Proposition 1.4 of [8], the anti-linear involution on the Betti cohomology $H^*(X_{\mathbb{R}}, \mathbb{C}) = H^*(X_{\mathbb{R}}, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ corresponding to \bar{F}_∞ is $F_\infty \otimes \bar{}$ (note that, unlike F_∞ , \bar{F}_∞ preserves the Hodge spaces $H^q(X_{\mathbb{R}}, \Omega_{X_{\mathbb{R}}}^p)$). It follows that the subspace of $HC_n^{(j)}$ which is the image of $HP_{n+2}^{\text{real},(j+1)}$ through the composite $S \circ \tau$ is globally invariant under \bar{F}_∞ . This is seen, as in the proof of Proposition 5.3, by identifying the map $HP_{n+2}^{\text{real},(j+1)}(X_{\mathbb{C}}) \xrightarrow{S \circ \tau} HC_n^{(j)}(X_{\mathbb{C}})$ with the map

$$H_B^{2j-n}(X(\mathbb{C}), \mathbb{R}(j+1)) \xrightarrow{\subset} H_{\text{dR,rel}}^{2j-n}(X_{\mathbb{C}}, j+1).$$

This implies that the operator \bar{F}_∞ descends to the quotient $HC_n^{\text{ar},(j)}$ of $HC_n^{(j)}$ by the image $(S \circ \tau)HP_{n+2}^{\text{real},(j+1)}(X_{\mathbb{C}})$.

We can now introduce the definition of archimedean cyclic homology for real varieties.

Definition 5.6 Let $X_{\mathbb{R}}$ be a smooth real projective variety and let $X_{\mathbb{C}} = X \otimes_{\mathbb{R}} \mathbb{C}$ be the associated complex variety. Define the archimedean cyclic homology of $X_{\mathbb{R}}$ as

$$HC_n^{\text{ar}}(X_{\mathbb{R}}) := HC_n^{\text{ar}}(X_{\mathbb{C}})^{\bar{F}_\infty = \text{id}}. \quad (43)$$

Note that one can re-write (43) in the equivalent form

$$HC_n^{\text{ar}}(X_{\mathbb{R}}) = HC_n(X_{\mathbb{R}}) / (HC_n(X_{\mathbb{R}}) \cap \text{Im}(S \circ \tau)) \quad (44)$$

since it is easy to see that the natural map from the right hand side of (44) to the left side is bijective.

Corollary 5.7 Let $X_{\mathbb{R}}$ be a smooth real projective variety of dimension d . Then for any pair of integers $(n, j) \in E_d$ we have

$$H_{\mathbb{D}}^{2j+1-n}(X_{\mathbb{R}}, \mathbb{R}(j+1)) = HC_n^{\text{ar},(j)}(X_{\mathbb{R}}).$$

Moreover $HC_n^{\text{ar},(j)}(X_{\mathbb{R}}) = \{0\}$ for $n \geq 0$, $j \in \mathbb{Z}$ and $(n, j) \notin E_d$.

Proof: The proof is easily adapted from that of Corollary 5.4. \square

Let us consider the simplest example of $X_{\mathbb{R}} = \text{Spec} \mathbb{R}$. In this case the exact sequence (40) gives for $X_{\mathbb{C}} = \text{Spec} \mathbb{C}$

$$0 \rightarrow HP_{n+2}^{\text{real},(j+1)}(\text{Spec} \mathbb{C}) \xrightarrow{S \circ \tau} HC_n^{(j)}(\text{Spec} \mathbb{C}) \rightarrow HC_n^{\text{ar},(j)}(\text{Spec} \mathbb{C}) \rightarrow 0,$$

which vanishes unless $n = 2j$. One therefore derives

$$HC_{2j}^{\text{ar}}(\text{Spec}\mathbb{C}) = HC_{2j}^{\text{ar},(j)}(\text{Spec}\mathbb{C}) = \mathbb{C}/((2\pi i)^{j+1}\mathbb{R})$$

while the odd homology groups all vanish. This is in agreement with the well-known result for real Deligne cohomology. For $X_{\mathbb{R}} = \text{Spec}\mathbb{R}$, the de Rham conjugation \bar{F}_{∞} is the ordinary complex conjugation, and we claim that it admits a non-zero fixed vector in $HC_{2j}^{\text{ar}}(\text{Spec}\mathbb{C})$ if and only if j is even, $j = 2k$. Indeed, in that case the quotient $\mathbb{C}/((2\pi i)^{j+1}\mathbb{R})$ coincides with \mathbb{R} , while it reduces to $i\mathbb{R}$ when j is odd. Thus we derive

$$HC_{4k}^{\text{ar}}(\text{Spec}\mathbb{R}) = \mathbb{C}/\left((2\pi i)^{2k+1}\mathbb{R}\right), \quad HC_n^{\text{ar}}(\text{Spec}\mathbb{R}) = \{0\}, \quad \text{for } n \neq 0(4). \quad (45)$$

Remark 5.8 The archimedean cohomology defined in [10] differs from the real Deligne cohomology in the case of real places and when the weight w is odd. Indeed, by [10] (*see* proof of Proposition 5.1) one gets in such case (we use the notation of *op.cit.*)

$$(H_{\text{ar}}^w)^{\Theta=m} = H_{\mathcal{D}}^{w+1}(X/\mathbb{C}, \mathbb{R}(n))^{\bar{F}_{\infty}=-id}.$$

This mismatch is understood in [11], Prop. 6.10, using a perfect pairing between the spaces in question. On the other hand the real archimedean cyclic homology HC^{ar} does not have this mismatch and it produces exactly the real Deligne cohomology.

6. Proof of Theorem 1.1

We refer to [27, 10] for the definition and properties of the regularized determinant det_{∞} . It is based on the definition of the regularized product of an infinite sequence of complex numbers λ_{ν} with chosen arguments α_{ν} . By definition and assuming that the Dirichlet series $\sum_{\nu > N} |\lambda_{\nu}|^{-s} e^{-is\alpha_{\nu}}$ converges for $\Re(s) \gg 0$ and admits an analytic continuation $\zeta_N(s)$ for $\Re(s) > -\epsilon$ one lets

$$\prod_{\nu} (\lambda_{\nu}, \alpha_{\nu}) := \left(\prod_1^N \lambda_{\nu} \right) \exp(-\zeta'_N(0)). \quad (46)$$

The regularized determinant is then defined as the regularized product of the eigenvalues. It agrees with the usual determinant in the finite dimensional case and is multiplicative, *i.e.* one has, assuming the existence of $det_{\infty}(S)$ and $det_{\infty}(T)$

$$det_{\infty}(S \oplus T) = det_{\infty}(S)det_{\infty}(T). \quad (47)$$

By [10], Proposition 2.1, the following formulas hold for the regularized products

$$\begin{aligned} \left(\prod_{\nu=0}^{\infty} \frac{s+\nu}{2\pi} \right)^{-1} &= \Gamma_{\mathbb{C}}(s) := (2\pi)^{-s} \Gamma(s), \\ \left(\prod_{\nu=0}^{\infty} \frac{s+2\nu}{2\pi} \right)^{-1} &= \Gamma_{\mathbb{R}}(s) := 2^{-1/2} \pi^{-s/2} \Gamma(s). \end{aligned} \quad (48)$$

Before giving the proof of Theorem 1.1, we treat the simplest example of $X_{\mathbb{R}} = \text{Spec} \mathbb{R}$. Let $\Theta = \Theta_0 - \Gamma$, where the linear operator Θ_0 acts by multiplication by j on $HC_n^{\text{ar},(j)}$ and Γ is the grading, *i.e.* acts by multiplication by n on HC_n^{ar} . Then, by (45), Θ acts on $HC_{4k}^{\text{ar}}(\text{Spec} \mathbb{R})$ by multiplication by $2k - 4k$ and hence its spectrum consists of all negative even integers $-2k$. One thus derives from (48) the equality:

$$\det_{\infty} \left(\frac{1}{2\pi} (s - \Theta) \right) |_{HC_{\text{even}}^{\text{ar}}(\text{Spec} \mathbb{R})} = \prod_{k \geq 0} \left(\frac{s + 2k}{2\pi} \right) = \Gamma_{\mathbb{R}}(s)^{-1},$$

while $HC_{\text{odd}}^{\text{ar}}(\text{Spec} \mathbb{R}) = \{0\}$ so that the formula (2) for the local (archimedean) Euler factor holds in this simple case.

We now consider the general case. We let $\Theta = \Theta_0 - \Gamma$ where the linear operator Θ_0 acts by multiplication by j on $HC_n^{\text{ar},(j)}$ and the linear operator Γ acts by multiplication by n on HC_n^{ar} . We set $HC_{\text{even}}^{\text{ar}}(X_{\nu}) = \bigoplus_{n=2k \geq 0} HC_n^{\text{ar}}(X_{\nu})$, $HC_{\text{odd}}^{\text{ar}}(X_{\nu}) = \bigoplus_{n=2k+1 \geq 1} HC_n^{\text{ar}}(X_{\nu})$.

Theorem 6.1 *Let X be a smooth, projective variety of dimension d over an algebraic number field K and let $\nu | \infty$ (*i.e.* $K_{\nu} = \mathbb{C}, \mathbb{R}$) be an archimedean place of K . Then the action of Θ on the archimedean cyclic homology of X_{ν} satisfies the following formula*

$$\prod_{0 \leq w \leq 2d} L_{\nu}(H^w(X), s)^{(-1)^{w+1}} = \frac{\det_{\infty} \left(\frac{1}{2\pi} (s - \Theta) \right) |_{HC_{\text{even}}^{\text{ar}}(X_{\nu})}}{\det_{\infty} \left(\frac{1}{2\pi} (s - \Theta) \right) |_{HC_{\text{odd}}^{\text{ar}}(X_{\nu})}}, \quad s \in \mathbb{R}. \quad (49)$$

The left-hand side of the formula describes the product of archimedean local factors ([29, 28]) in the complex L -function of X and \det_{∞} denotes the regularized determinant (see e.g. [27, 10]).

Proof: One has $HC_n^{\text{ar},(j)}(X_{\nu}) = \{0\}$ for $n \geq 0, j \in \mathbb{Z}$ and $(n, j) \notin E_d$, $d = \dim(X)$, (Corollary 5.4, Corollary 5.7). Thus (47) shows that, provided one shows that the regularized determinants exist in the right hand side, the following equality holds

$$\begin{aligned} \det_{\infty} \left(\frac{1}{2\pi} (s - \Theta) \right) |_{HC_{\text{even}}^{\text{ar}}(X_{\nu})} &= \det_{\infty} \left(\frac{1}{2\pi} (s - \Theta) \right) |_{\mathcal{E}}, \\ \mathcal{E} &= \bigoplus_{n \text{ even}, (n, j) \in E_d} HC_n^{\text{ar},(j)}(X_{\nu}). \end{aligned} \quad (50)$$

The same observation holds with "even" replaced by "odd" in (50). By Corollary 5.4 and Corollary 5.7 one has

$$H_{\mathcal{D}}^{2j+1-n}(X_\nu, \mathbb{R}(j+1)) = HC_n^{\text{ar},(j)}(X_\nu), \quad \forall (n, j) \in E_d. \quad (51)$$

By Lemma 3.1, the pairs $(n, j) \in E_d$, n even, correspond bijectively to the pairs $(w, m) \in A_d$ for $w = 2j - n$ even, and $m = j - n$. Thus the real vector space \mathcal{E} of (50) is the same as (see (12))

$$\bigoplus_{(m,w) \in A_d, w \text{ even}} H_{\mathcal{D}}^{w+1}(X_\nu, \mathbb{R}(w+1-m)) = \bigoplus_{n \text{ even}, (n,j) \in E_d} H_{\mathcal{D}}^{2j+1-n}(X_\nu, \mathbb{R}(j+1)). \quad (52)$$

Moreover the operator $\Theta := \Theta_0 - \Gamma$ becomes the linear operator M of multiplication by m on each component indexed by (m, w) in the left hand side of (52). Thus the equality

$$\prod_{0 \leq w \leq 2d, w \text{ even}} L_\nu(H^w(X), s)^{-1} = \det_\infty\left(\frac{1}{2\pi}(s - \Theta)\right)|_{HC_{\text{even}}^{\text{ar}}(X_\nu)}, \quad s \in \mathbb{R}, \quad (53)$$

will follow, using (47), provided one can show that for any w , $0 \leq w \leq 2d$ one has

$$L_\nu(H^w(X), s)^{-1} = \det_\infty\left(\frac{1}{2\pi}(s - M)\right)|_{\mathcal{D}_w},$$

$$\mathcal{D}_w := \bigoplus_{m \leq w/2} H_{\mathcal{D}}^{w+1}(X_\nu, \mathbb{R}(w+1-m)), \quad s \in \mathbb{R}. \quad (54)$$

To prove (54), we fix the Hodge weight w and consider the archimedean local factor $L_\nu(H^w(X), s)$ associated to the place ν of K . In [10] Theorem 4.1, C. Deninger constructed an operator $M_{\mathcal{D}}$ which is also a direct sum of multiplication operators by integers m with finite multiplicity and is such that (54) holds using $M_{\mathcal{D}}$ instead of M . Thus to show (54) it is enough to check that M and $M_{\mathcal{D}}$ give to each m the same multiplicity. The multiplicities of $M_{\mathcal{D}}$ are the multiplicities of the poles of $L_\nu(H^w(X), s)$ and it is well known ([1], (3.1), (3.3), [28] §1,2, [10], Prop. 5.1) that these latter only occur at integer values $s = m \leq \frac{w}{2} \leq d = \dim(X)$, and their multiplicity is described by the following formula

$$\text{ord}_{s=m} L_\nu(H^w(X), s)^{-1} = \dim_{\mathbb{R}} H_{\mathcal{D}}^{w+1}(X_\nu, \mathbb{R}(w+1-m)). \quad (55)$$

Thus (54) follows using [10] §5. For the denominator of (49) the same proof applies and gives

$$\prod_{0 \leq w \leq 2d, w \text{ odd}} L_\nu(H^w(X), s)^{-1} = \det_\infty\left(\frac{1}{2\pi}(s - \Theta)\right)|_{HC_{\text{odd}}^{\text{ar}}(X_\nu)}, \quad s \in \mathbb{R}. \quad (56)$$

Combining (53) and (56) yields the required equality (49). \square

Remark 6.2 (i) The operators Θ_0 and Γ commute and the proof of Theorem 6.1 shows that the decomposition of the alternating product on the left hand side of equation (49) into factors associated to specific weights corresponds to the spectral decomposition of the operator $2\Theta_0 - \Gamma$, which commutes with Θ . Thus, by restricting the right hand side of (49) to the spectral projection $2\Theta_0 - \Gamma = w$, one recovers precisely the factor $L_\nu(H^w(X), s)^{(-1)^{w+1}}$. In this way we refine (49).

(ii) The following formula only makes use of the grading operator Γ

$$\prod_{0 \leq w \leq 2d} L_\nu(H^w(X), s + \frac{w}{2})^{(-1)^{w+1}} = \frac{\det_\infty(\frac{1}{2\pi}(s + \frac{1}{2}\Gamma))|_{HC_{\text{even}}^{\text{ar}}(X_\nu)}}{\det_\infty(\frac{1}{2\pi}(s + \frac{1}{2}\Gamma))|_{HC_{\text{odd}}^{\text{ar}}(X_\nu)}}, \quad s \in \mathbb{R}. \quad (57)$$

To prove (57) one applies the same argument as above together with the equality

$$\Theta - \frac{1}{2}(2\Theta_0 - \Gamma) = -\frac{1}{2}\Gamma,$$

which shows that replacing Θ in (49) by $-\frac{1}{2}\Gamma$ generates a shift of $\frac{w}{2}$ in the argument of the L -functions. This shift occurs naturally in the functional equation and in the expected location of the zeros of the individual global L -functions $L(H^w(X), s)$. More precisely, as is well known, the function $L(H^w(X), s + \frac{w}{2})$ is conjectured to have its zeros on the line $\Re(s) = \frac{1}{2}$.

A. Weil restriction, the morphism $\pi_X : \text{Spec}(C^\infty(X_{\text{sm}}, \mathbb{C})) \rightarrow X$

Let X be a smooth, projective complex variety viewed as a scheme and X_{alg} its set of closed points. The results of Section 4 are based on the canonical morphism of locally ringed spaces $\mu : X_{\text{sm}} \rightarrow X_{\text{alg}}$ that links X_{alg} to the associated C^∞ -manifold X_{sm} , viewed as the locally ringed space with structure sheaf $\mathcal{O}_{X_{\text{sm}}}(U) := C^\infty(U, \mathbb{C})$ for every open subset $U \subset X_{\text{sm}}$. Although μ is very useful and plays an important role in our construction it does not belong properly to the realm of algebraic geometry. In particular the locally ringed space X_{sm} is not a scheme and differs substantially from the affine scheme $\text{Spec}(A)$, with $A = \mathcal{O}(X_{\text{sm}}) = C^\infty(X_{\text{sm}}, \mathbb{C})$. In this section we show how one may *extend* μ to a *morphism of schemes*

$$\pi_X : \text{Spec}(A) \rightarrow X, \quad (58)$$

which factors through an affine noetherian scheme.

A.1. Weil restriction

We first recall the notion of Weil restriction which we use for the conceptual understanding of the construction below. Let X be a scheme over \mathbb{C} and let

$Y_{\mathbb{R}} = \text{Res}_{\mathbb{C}/\mathbb{R}} X$ be the Weil restriction. As a contravariant functor from schemes S over \mathbb{R} to sets it is defined by the equality

$$(\text{Res}_{\mathbb{C}/\mathbb{R}} X)(S) := X(S \otimes_{\mathbb{R}} \mathbb{C}).$$

It is a scheme under a mild hypothesis on X ([9] I, §1 Proposition 6.6) which is fulfilled in the case of a quasi-projective scheme. Under this condition the extension of scalars $Y_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$ is the scheme Z defined in full generality as

$$Z := \prod_{\rho} X^{\rho}$$

where the product of schemes is over \mathbb{C} , ρ varies between the two \mathbb{R} -linear embeddings $\mathbb{C} \rightarrow \mathbb{C}$ (*i.e.* the identity and the complex conjugation σ) and X^{ρ} is the complex scheme obtained from X by extension of scalars using $\rho : \mathbb{C} \rightarrow \mathbb{C}$. To every pair (τ, ρ) of elements of $\text{Gal}(\mathbb{C} : \mathbb{R})$ corresponds a canonical bijection $X^{\rho} \rightarrow X^{\tau \circ \rho}$ since $X^{\tau \circ \rho}$ is deduced from X^{ρ} by extension of scalars $\tau : \mathbb{C} \rightarrow \mathbb{C}$. The resulting action of $\text{Gal}(\mathbb{C} : \mathbb{R})$ on the points of $Z = Y_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$ defined over \mathbb{C} is the action of $\text{Gal}(\mathbb{C} : \mathbb{R})$ on $Y(\mathbb{C})$. Its fixed points $Y(\mathbb{R}) = X(\mathbb{C})$ are given by the diagonal subset $\Delta = \{(z, \bar{z}) \mid z \in X\}$.

A.2. Affine open neighborhoods of $Y(\mathbb{R})$ in $Z = X \times \bar{X}$

To obtain the required morphism (58), one proceeds as follows. By construction the smooth projective complex variety X is a subvariety of $\mathbb{P}_{\mathbb{C}}^n$. For $z \in \mathbb{P}_{\mathbb{C}}^n$ we let $\bar{z} \in \mathbb{P}_{\mathbb{C}}^n$ have homogeneous coordinates $(\bar{z})_j = \bar{z}_j$. We let $\bar{X} = \{\bar{z} \mid z \in X\}$, it is a smooth projective complex subvariety $\bar{X} \subset \mathbb{P}_{\mathbb{C}}^n$ which as a scheme over \mathbb{C} is obtained from X by extension of scalars using complex conjugation $\sigma : \mathbb{C} \rightarrow \mathbb{C}$. The product $X \times \bar{X}$ is a subvariety $X \times \bar{X} \subset \mathbb{P}_{\mathbb{C}}^n \times \mathbb{P}_{\mathbb{C}}^n$. Let $\tau : \mathbb{P}_{\mathbb{C}}^n \times \mathbb{P}_{\mathbb{C}}^n \rightarrow \mathbb{P}_{\mathbb{C}}^{n^2+2n}$ be the Segre embedding given in terms of homogeneous coordinates by

$$\tau(x, y)_{i,j} := x_i y_j, \quad i, j \in \{0, \dots, n\}, \quad \forall x = (x_i), y = (y_i). \quad (59)$$

Theorem A.1 *Let $X \subset \mathbb{P}_{\mathbb{C}}^n$ be a projective complex variety. Let $\tilde{\tau}$ be the restriction of the Segre embedding (59) to $X \times \bar{X}$. Then*

(i) *The intersection $V = \tilde{\tau}(X \times \bar{X}) \cap H^c$ of the range of $\tilde{\tau}$ with the complement of the hyperplane $H = \{u = (u_{i,j}) \mid \sum_{i \in \{0, \dots, n\}} u_{i,i} = 0\}$ in $\mathbb{P}_{\mathbb{C}}^{n^2+2n}$ is an affine variety.*

(ii) *One has $V = W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$ where W is an affine scheme over \mathbb{R} such that $W(\mathbb{R}) = Y(\mathbb{R}) \sim X(\mathbb{C})$, where $Y_{\mathbb{R}} = \text{Res}_{\mathbb{C}/\mathbb{R}} X$ is the Weil restriction.*

(iii) *The restriction of the first projection $(X \times \bar{X}) \rightarrow X$ gives a morphism $p_X : W_{\mathbb{C}} \rightarrow X$ which is surjective on complex points.*

(iv) Assume that X is smooth. The morphism of locally ringed spaces $X_{\text{sm}} \rightarrow X_{\text{alg}}$ extends canonically to a morphism of schemes

$$\pi_X : \text{Spec}(\mathcal{O}(X_{\text{sm}})) \xrightarrow{\gamma^*} W_{\mathbb{C}} \xrightarrow{p_X} X$$

where $\gamma : \mathcal{O}(W_{\mathbb{C}}) \rightarrow \mathcal{O}(X_{\text{sm}}) = C^\infty(X_{\text{sm}}, \mathbb{C})$ is the Gelfand transform of the involutive algebra $\mathcal{O}(W_{\mathbb{C}})$ of global regular sections of the noetherian affine scheme $W_{\mathbb{C}}$:

$$\mathcal{O}(W_{\mathbb{C}}) \ni a \mapsto \gamma(a) = \hat{a}, \quad \hat{a}(\chi) := \chi(a), \quad \forall \chi \in \text{Hom}_{\mathbb{C}}(\mathcal{O}(W_{\mathbb{C}}), \mathbb{C}),$$

$$\chi(a^*) = \bar{\chi}(a) \quad \forall a \in \mathcal{O}(W_{\mathbb{C}}).$$

Proof: (i) Let V be the affine variety of rank one matrices $A = (A_{ij})$, $i, j \in \{0, \dots, n\}$, with trace equal to 1 whose columns fulfill the homogeneous equations defining X and whose rows fulfill the homogeneous equations defining \bar{X} . The range of the Segre embedding $\tau : \mathbb{P}_{\mathbb{C}}^n \times \mathbb{P}_{\mathbb{C}}^n \rightarrow \mathbb{P}_{\mathbb{C}}^{n^2+2n}$ is the space of rank one matrices A_{ij} up to scalar multiples. The range $\tilde{\tau}(X \times \bar{X})$ of its restriction to $X \times \bar{X}$ is the subspace of matrices (up to scalar multiples) whose columns fulfill the homogeneous equations defining X and whose rows fulfill the homogeneous equations defining \bar{X} . The intersection of $\tilde{\tau}(X \times \bar{X})$ with the complement of the hyperplane $H = \{u = (u_{i,j}) \mid \sum_{i \in \{0, \dots, n\}} u_{i,i} = 0\}$ in $\mathbb{P}_{\mathbb{C}}^{n^2+2n}$ is the affine variety V , which shows (i).

(ii) The complex algebra $\mathcal{O}(V) = B$ is generated by the coordinates $u_{i,j}$, $i, j \in \{0, \dots, n\}$ with $u_{i,j}(A) := A_{ij}$. It is endowed with the antilinear involution $f \mapsto f^*$ given by $f^*(A) := \bar{f}(A^*)$ where A^* is the adjoint of the matrix A , i.e. $(A^*)_{ij} := \bar{A}_{ji}$. This antilinear involution is uniquely determined by its action on the coordinates $u_{i,j}$ as follows $(u_{i,j})^* := u_{j,i}$.

The fixed points of this antilinear involution form a real algebra B_{sa} (where “sa” stands for “self-adjoint”) such that $B_{\text{sa}} \otimes_{\mathbb{R}} \mathbb{C} = B$. This shows that the scheme V is equal to $W \otimes_{\mathbb{R}} \mathbb{C}$ where $W = \text{Spec}(B_{\text{sa}})$. The points of $W = \text{Spec} B_{\text{sa}}$ defined over \mathbb{R} are given by the self-adjoint characters of B

$$\text{Hom}(B_{\text{sa}}, \mathbb{R}) = \{\chi \in \text{Hom}(B, \mathbb{C}) \mid \chi(f^*) = \overline{\chi(f)}, \quad \forall f \in B\}.$$

Let χ be such a character of B . Evaluating χ on the coordinates $u_{i,j}$ yields a self-adjoint matrix $E := (\chi(u_{i,j}))$ of rank one and trace one, and hence a self-adjoint idempotent whose columns (resp. rows) belong to X (resp. \bar{X}). Thus there exists $z \in X$ such that $\chi(u_{i,j}) = z_i \bar{z}_j$. Conversely the range $\tilde{\tau}(\Delta)$ of the diagonal subset $\Delta = \{(z, \bar{z}) \mid z \in X\}$ does not intersect the hyperplane $H = \{u = (u_{i,j}) \mid$

$\sum_{i \in \{0, \dots, n\}} u_{i,i} = 0$ in $\mathbb{P}_{\mathbb{C}}^{n^2+2n}$ since for $z \neq 0$, $z = (z_i)$,

$$\sum_{i \in \{0, \dots, n\}} z_i \bar{z}_i > 0.$$

(iii) It follows from (ii) that the first projection $(X \times \bar{X}) \rightarrow X$ gives a morphism $p_X : W_{\mathbb{C}} \rightarrow X$ which is surjective on complex points.

(iv) The affine scheme $W_{\mathbb{C}}$ is noetherian ([17], II, 3.2.1). Algebraic functions on $W_{\mathbb{C}}$ determine smooth functions on X_{sm} by restriction to Δ . This gives a morphism $\gamma : \mathcal{O}(W_{\mathbb{C}}) \rightarrow \mathcal{O}(X_{\text{sm}})$, and hence a corresponding morphism $\text{Spec}(A) \rightarrow W_{\mathbb{C}}$, $A = \mathcal{O}(X_{\text{sm}})$, whose composition with p_X gives the desired morphism $\pi_X : \text{Spec}(A) \rightarrow X$ of (58). The morphism γ coincides with the Gelfand transform

$$B \ni a \mapsto \hat{a}, \quad \hat{a}(\chi) = \chi(a), \quad \forall \chi \in \text{Hom}_{\mathbb{C}}(B, \mathbb{C}), \quad \chi = \chi^*$$

and is given in terms of the generators $u_{i,j}$ of B by

$$\gamma(u_{i,j}) = z_i \bar{z}_j / \left(\sum z_i \bar{z}_i \right).$$

Since we restrict to elements of $\text{Hom}_{\mathbb{C}}(B, \mathbb{C})$ which fulfill $\chi = \chi^*$ the homomorphism γ is compatible with involutions. It remains to show that the morphism of schemes

$$\nu := p_X \circ \gamma^* : \text{Spec}(\mathcal{O}(X_{\text{sm}})) \rightarrow X$$

extends the morphism of locally ringed spaces $\mu : X_{\text{sm}} \rightarrow X_{\text{alg}}$. Let $Z_i = V(z_i)$ be the Zariski closed subset of X associated to the coordinate z_i , then $p_X^{-1}(Z_i)$ is the closed subset $\cap_j V(u_{i,j})$ of $W_{\mathbb{C}}$. Its inverse image in X_{sm} is the closed set $V(z_i)$. The map $\nu := p_X \circ \gamma^*$ is covered by the Spec of the ring morphisms $f_i : \mathbb{C}[z_j/z_i] \rightarrow C^\infty(X_{\text{sm}} \setminus V(z_i))$. This allows one to check (iv). \square

Remark A.2 The above construction of the affine scheme $\text{Spec} B_{\text{sa}}$ over \mathbb{R} and of the morphism $p : \text{Spec} B_{\text{sa}} \otimes_{\mathbb{R}} \mathbb{C} \rightarrow X$ suffices for our purposes (*i.e.* the proof of Proposition 4.4) but depends on many choices such as the closed immersion of X in projective space. One can improve this construction and obtain a natural additive functor $X \mapsto *X$ from a category of schemes over \mathbb{C} to the category of schemes over \mathbb{R} and a natural transformation of functors

$$p_X : (*X)_{\mathbb{C}} \rightarrow X, \quad (*X)_{\mathbb{C}} := *X \otimes_{\mathbb{R}} \mathbb{C}$$

which fulfills the following properties when applied to projective varieties over \mathbb{C} :

(i) The *real* scheme $*X$ is affine.

(ii) The morphism p_X is bijective on complex points, *i.e.* p_X induces a bijection

$*X(\mathbb{C}) \cong X(\mathbb{C})$.

(iii) If X is also smooth, the morphism of locally ringed spaces $X_{\text{sm}} \rightarrow X_{\text{alg}}$ extends canonically to a morphism of schemes

$$\pi_X : \text{Spec}(\mathcal{O}(X_{\text{sm}})) \xrightarrow{\gamma^*} (*X)_{\mathbb{C}} \xrightarrow{p_X} X$$

where $\gamma : \mathcal{O}(*X) \rightarrow \mathcal{O}(X_{\text{sm}}) = C^\infty(X_{\text{sm}}, \mathbb{C})$ is the Gelfand transform ([13]) of the algebra $B = \mathcal{O}(*X)$ of global regular sections of the affine scheme $*X = \text{Spec}(B)$:

$$B = \mathcal{O}(*X) \ni a \mapsto \gamma(a) = \hat{a}, \quad \hat{a}(\chi) := \chi(a), \quad \forall \chi \in \text{Hom}_{\mathbb{R}}(B, \mathbb{C}).$$

A.3. Relation between the locally ringed space X_{alg} , X_{sm} and $\text{Spec}(\mathcal{O}(X_{\text{sm}}))$

We now describe in more details the relations between X_{alg} , X_{sm} and $\text{Spec}(\mathcal{O}(X_{\text{sm}}))$. The underlying topological space of the locally ringed space X_{alg} is the set of complex points $X(\mathbb{C})$ of the algebraic variety endowed with the Zariski topology. The sheaf $\mathcal{O} = \mathcal{O}_X$ associates to each Zariski open set U the ring $\mathcal{O}(U)$ of regular functions from U to \mathbb{C} . The weakest topology making these functions continuous is the ordinary topology and the map μ is the identity on points and is the inclusion $\mathcal{O}(U) \subset \mathcal{O}_{X_{\text{sm}}}(U)$ at the level of the structure sheaves. The locally ringed space X_{alg} is the set of closed points of the algebraic scheme X with the induced topology and the restriction of the structure sheaf. Note that the map which to an open set of X associates its intersection with X_{alg} gives an isomorphism on the categories of open sets so that the sheaves on X and on X_{alg} are in one to one correspondence. This simple relation fails when comparing X_{sm} with $\text{Spec}(\mathcal{O}(X_{\text{sm}}))$. Let $\iota : X_{\text{sm}} \rightarrow \text{Spec}(\mathcal{O}(X_{\text{sm}}))$ be the map which associates to $x \in X_{\text{sm}}$ the kernel of the character $\mathcal{O}(X_{\text{sm}}) \ni f \mapsto f(x)$. It is a bijection of X_{sm} with the set of closed points of $\text{Spec}(\mathcal{O}(X_{\text{sm}}))$. It is continuous since the open sets

$$D(f) := \{\mathfrak{p} \in \text{Spec}(\mathcal{O}(X_{\text{sm}})) \mid f \notin \mathfrak{p}\} \quad (60)$$

form a basis of the topology of $\text{Spec}(\mathcal{O}(X_{\text{sm}}))$, and $\iota^{-1}(D(f))$ is the open subset $\{x \in X_{\text{sm}} \mid f(x) \neq 0\} \subset X_{\text{sm}}$. Moreover ι defines a morphism of locally ringed spaces since for every open set $V \subset \text{Spec}(\mathcal{O}(X_{\text{sm}}))$ the elements of $\mathcal{O}(V)$ give, by composition with ι , elements of $C^\infty(V, \mathbb{C}) = \Gamma(V, \mathcal{O}_{X_{\text{sm}}})$. One then easily checks that the composition

$$X_{\text{sm}} \xrightarrow{\iota} \text{Spec}(\mathcal{O}(X_{\text{sm}})) \xrightarrow{\gamma^*} W_{\mathbb{C}} \xrightarrow{p_X} X$$

maps to the closed points of X and is equal to the morphism μ .

We end this appendix with the following result that shows how to recover the locally ringed space X_{sm} from the affine scheme $\text{Spec}(\mathcal{O}(X_{\text{sm}}))$.

Proposition A.3 *Let V be a smooth and compact manifold and V_{sm} the locally ringed space such that $\mathcal{O}_{V_{\text{sm}}}(U) = C^\infty(U, \mathbb{C})$ for every open subset $U \subset V$. Let V_{aff} be the affine scheme over \mathbb{C} , $V_{\text{aff}} = \text{Spec}(C^\infty(V, \mathbb{C}))$.*

(i) *The map ι which to $x \in V$ associates the kernel of the character $\mathcal{O}(V_{\text{sm}}) \ni f \mapsto f(x)$ is a bijection of V with the maximal ideals of the \mathbb{C} -algebra $\mathcal{O}(V_{\text{sm}})$.*

(ii) *For each point $\mathfrak{p} \in V_{\text{aff}}$ there exists a unique maximal ideal $m(\mathfrak{p}) \in V$ containing the prime ideal \mathfrak{p} .*

(iii) *The map $m : V_{\text{aff}} = \text{Spec}(\mathcal{O}(V_{\text{sm}})) \rightarrow V$, $\mathfrak{p} \mapsto m(\mathfrak{p})$ is continuous and the direct image sheaf $m_* (\mathcal{O}_{V_{\text{aff}}})$ of the structure sheaf of V_{aff} is the sheaf $\mathcal{O}_{V_{\text{sm}}}$.*

(iv) *The morphism of locally ringed spaces $m \circ \iota : V \rightarrow V$ is the identity.*

Proof: One checks (ii) by using a partition of unity argument (see [13], 7.15). Moreover (ii) implies (i).

(iii) We show that m is continuous. Let $U \subset V$ be an open set and let $\tilde{U} = m^{-1}(U)$. Each point $\mathfrak{p} \in \tilde{U}$ is a prime ideal contained in a maximal ideal $\mathfrak{m} = x \in U$. Let then $f \in \mathcal{O}(V_{\text{sm}})$ be a smooth function on V with support contained in U and such that $f(x) \neq 0$. One has $f \notin \mathfrak{m}$ and hence $f \notin \mathfrak{p} \subset \mathfrak{m}$. Thus $\mathfrak{p} \in D(f)$ using (60). For any prime ideal $\mathfrak{q} \in D(f) \subset V_{\text{aff}}$ one has $m(\mathfrak{q}) \in \text{Support}(f)$. Indeed, if that is not the case the germ of f around $m(\mathfrak{q})$ is zero and thus $f \in \mathfrak{q}$ which contradicts (60). Thus one has $m(\mathfrak{q}) \in \text{Support}(f) \subset U$ and $D(f) \subset \tilde{U} = m^{-1}(U)$ is an open neighborhood of $\mathfrak{p} \in \tilde{U}$. This shows that $\tilde{U} = m^{-1}(U)$ is open in $\text{Spec}(\mathcal{O}(V_{\text{sm}}))$. The elements $s \in \mathcal{O}_{V_{\text{aff}}}(\tilde{U})$ are obtained from a covering of $\tilde{U} = m^{-1}(U)$ by open subsets $D(f_j) \subset m^{-1}(U)$ and for each $D(f_j)$ an element s_j of the ring $\mathcal{O}(V_{\text{sm}})_{f_j}$ of fractions with denominator a power of f_j , with the condition that these elements agree on the pairwise intersections. Thus to such an $s \in \mathcal{O}_{V_{\text{aff}}}(\tilde{U})$ one can associate a function $s|_U$ by restriction to the subset of maximal ideals, and this function is smooth. The obtained map $\text{res} : \mathcal{O}_{V_{\text{aff}}}(\tilde{U}) \rightarrow C^\infty(U, \mathbb{C}) = \mathcal{O}_{V_{\text{sm}}}(U)$ is surjective. Indeed, one can cover U by open sets $U_j = \{x \in U \mid f_j(x) \neq 0\}$ where each $f_j \in C^\infty(U)$ has compact support in U , thus any $h \in C^\infty(U, \mathbb{C})$ is obtained from the $s_j = hf_j/f_j \in \mathcal{O}(V_{\text{sm}})_{f_j}$. The map $\text{res} : \mathcal{O}_{V_{\text{aff}}}(\tilde{U}) \rightarrow C^\infty(U, \mathbb{C})$ is injective since any element of $\mathcal{O}_{V_{\text{aff}}}(D(f)) = \mathcal{O}(V_{\text{sm}})_f$ is determined by its restriction to maximal ideals.

(iv) One just needs to check that $m \circ \iota$ is the identity at the level of structure sheaves and this follows from the proof of (iii). \square

Remark A.4 As soon as the dimension of X is non zero, the image of the morphism of schemes $\pi_X : \text{Spec}(\mathcal{O}(X_{\text{sm}})) \rightarrow X$ contains a non-closed point. This shows in particular that this morphism does not factor through the morphism of locally ringed spaces $X_{\text{sm}} \rightarrow X_{\text{alg}}$.

B. Taking into account the topology of $C^\infty(X_{\text{sm}}, \mathbb{C})$

One important point of this paper revolves around the role of the Frechet topology of the algebra $C^\infty(X_{\text{sm}}, \mathbb{C})$ in applying the results of [5] to its cyclic homology. In this appendix we clarify this point. We let V be a smooth compact manifold. The algebra $A = C^\infty(V, \mathbb{C})$ is naturally endowed with a locally convex vector space topology of Frechet nuclear space ([14]). Nuclear locally convex vector spaces share many properties with finite dimensional vector spaces and in particular there is only one way to take the completion of tensor products and for any locally convex topological vector space E the natural map from the projective to the injective tensor product of A and E is an isomorphism. In particular the topological tensor powers

$$C_n(A_{\text{top}}) = A^{\hat{\otimes} n} := A \hat{\otimes} A \hat{\otimes} \cdots \hat{\otimes} A$$

are unambiguously defined and give a covariant functor $C(A_{\text{top}})$ from the category \mathfrak{Fin} of finite sets to the category of complex (locally convex nuclear) vector spaces. By construction this functor $C(A_{\text{top}})$ is obtained by completion from the covariant functor $C(A)$ ([4], [23]) which suffices to obtain the cyclic homology of the algebra A as well as the λ -operations. By construction one has a natural transformation of functors

$$\text{top} : C(A) \rightarrow C(A_{\text{top}}), \quad C_n(A) = A^{\otimes n} \rightarrow C_n(A_{\text{top}}) = A^{\hat{\otimes} n} \quad (61)$$

given by the inclusion of the vector spaces $A^{\otimes n}$ in their completions $A^{\hat{\otimes} n}$. The simplest description of the functor $C(A_{\text{top}})$ is as the composition of the contravariant functor $C^\infty(\bullet, \mathbb{C})$ (from the category of compact smooth manifolds to that of complex vector spaces) with the contravariant functor from the category \mathfrak{Fin} (of finite sets) to the category of smooth manifolds which associates to a finite set F the manifold of maps from F to V , *i.e.* the product of $\#F$ copies of V indexed by elements of F .

By Lemma 45 of [5] one derives the description of the topological Hochschild homology groups as

$$HH_k(A_{\text{top}}) = C^\infty(V, \wedge^k T_{\mathbb{C}}^*)$$

where the map B is the de Rham differential. By [5] (*see* Theorem 46), the periodic cyclic homology $HP_*(A_{\text{top}})$ is computed as the de Rham cohomology of smooth differential forms, *i.e.* as the Betti cohomology with complex coefficients $H_B^*(V, \mathbb{C})$. We refer to [5] for the computation of the cyclic homology and in particular for the identification of the periodic cyclic homology with the de Rham cohomology of smooth differential forms. Moreover, by applying Theorem 3.3 of [34] (adapted to the topological case) one obtains the more precise identification

$$HP_n^{(j)}(A_{\text{top}}) = H_B^{2j-n}(V, \mathbb{C}), \quad \forall n, j \in \mathbb{Z}. \quad (62)$$

All the above results remain valid with the obvious modifications for the real algebra $A^{\text{real}} = C^\infty(V, \mathbb{R})$ and one has by construction a canonical morphism $HP_*(A_{\text{top}}^{\text{real}}) \rightarrow HP_*(A_{\text{top}})$ which corresponds to the inclusion $H_B^*(V, \mathbb{R}) \subset H_B^*(V, \mathbb{C})$ in terms of the Betti cohomologies. One has $HP_*(A_{\text{top}}) = HP_*(A_{\text{top}}^{\text{real}}) \otimes_{\mathbb{R}} \mathbb{C}$ since this already holds at the level of the functors from $\mathfrak{F}in$ to vector spaces.

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