

Cyclic theory and the pericyclic category

Alain Connes and Caterina Consani

*Pour moi le “paradis originel” pour l’algèbre topologique, n’est nullement la sempiternelle catégorie Δ semi-simpliciale, si utile soit-elle, et encore moins celle des espaces topologiques (qui l’une et l’autre s’envoient dans la 2-catégorie des topos, qui en est comme une enveloppe commune), mais bien la catégorie **Cat** des petites catégories, vue avec un oeil de géomètre par l’ensemble d’intuitions, étonnamment riche, provenant des topos. En effet, les topos ayant comme catégories des faisceaux d’ensembles les $\hat{\mathcal{C}}$, avec \mathcal{C} dans **Cat**, sont de loin les plus simples des topos connus, et c’est pour l’avoir senti que j’insiste tant sur ces topos “catégoriques” dans SGA IV.*

Alexander Grothendieck

ABSTRACT. We give a historical perspective on the role of the cyclic category Λ in the development of cyclic theory. This involves a continuous interplay between the extension in characteristic one and in \mathbb{S} -algebras, of the traditional development of cyclic theory, and the geometry of the toposes associated with several small categories involved. We clarify the link between various existing presentations of the cyclic and the epicyclic categories and we exemplify the role of the absolute coefficients by presenting the ring of the integers as polynomials in powers of 3, with coefficients in the spherical group ring $\mathbb{S}[C_2]$ of the cyclic group of order 2. Finally, we introduce the pericyclic category which unifies and refines two conflicting notions of epicyclic space existing in the literature.

1. Introduction

The introduction (by the first author of this article) of cyclic cohomology as the analog of de Rham cohomology in noncommutative geometry, was inspired by the development of the quantized calculus [3–6] where cyclic cohomology of noncommutative algebras is the natural receptacle for the Chern character of K-homology cycles. Both the periodicity operator $S : \mathrm{HC}^n(\mathcal{A}) \rightarrow \mathrm{HC}^{n+2}(\mathcal{A})$ and the long exact

2020 *Mathematics Subject Classification.* Primary 55U40, 18F10; Secondary 14C40, 18B40.

Key words and phrases. Cyclic categories, Grothendieck toposes, Γ -sets, characteristic one, pericyclic category.

The second author was partially supported by the Simons Foundation collaboration grant n. 691493.

sequence relating cyclic and Hochschild cohomologies of an algebra \mathcal{A} stem naturally from the analytic theory. After many repetitive, algebraic manipulations it became clear [4] that laying at the core of the theory there is a small category Λ : the cyclic category, encoding combinatorially the rules of computations for an algebra through a representation of Λ in the vector space \mathcal{A}^{\natural} (cyclic module) obtained as the direct sum $\mathcal{A}^{\natural} := \bigoplus \mathcal{A}^{\otimes n}$ of tensor powers of \mathcal{A} . This result shows, in particular, that cyclic cohomology and homology are two instances of derived functors in homological algebra. At the conceptual level, it determines an embedding of the category of algebras in the larger abelian category of cyclic modules, thus providing a hint for a motivic development of noncommutative geometry.

The cyclic category Λ is, by definition, an extension of the category Δ of finite ordinals. Its classifying space $B\Lambda$ is the infinite projective space $\mathbb{P}^{\infty}(\mathbb{C})$ [4]. This fact points out the similarity of Λ with the compact group $U(1)$, as it is well known that $BU(1) = \mathbb{P}^{\infty}(\mathbb{C})$. However, one also ought to keep in mind that while shifting our focus from a small category to the homotopy type of its classifying space we lose some relevant information. For instance, $B\Delta$ is contractible (the category Δ has a final object), while the restriction to Δ of the cyclic representation in \mathcal{A}^{\natural} is known to govern Hochschild homology.

A better way to store geometrically the relevant categorical information is using the associated presheaf topos, as explained in the quote of A. Grothendieck at the beginning of this article. A small category \mathcal{C} is geometrically represented by the (Grothendieck) topos $\widehat{\mathcal{C}}$ of contravariant functors $\mathcal{C} \rightarrow \mathbf{Sets}$ to sets. Both Hochschild and cyclic homologies are described naturally by derived functors on sheaves of abelian groups on the corresponding toposes $\widehat{\Delta}$ and $\widehat{\Lambda}$. The additional structure provided by the λ -operations determines two extensions of Δ and Λ , yielding to the epicyclic refinement $\widetilde{\Lambda}$ of the cyclic category [1, 10, 28] which is, in essence, the semi-direct product of Λ by an action of the multiplicative monoid \mathbb{N}^{\times} of non-zero positive integers. This action determines a functor $\mathbf{Mod} : \widetilde{\Lambda} \rightarrow \mathbb{N}^{\times}$, and at the topos level, a geometric morphism from the epicyclic topos to the topos $\widehat{\mathbb{N}^{\times}}$ [11]. The notion of “point” in a topos leads, even for toposes of presheaf type, to remarkable new spaces. For example, the presheaf topos $\widehat{\mathbb{N}^{\times}}$ [12] gave the first example of a Grothendieck topos whose space of points is noncommutative, *i.e.* a quotient of an ordinary space by an equivalence relation, without any Borel section. The relation between the space of points of $\widehat{\mathbb{N}^{\times}}$ and the adèle class space of the rationals $\mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}}$ [7] has culminated with the discovery of the topos $[0, \infty) \times \mathbb{N}^{\times}$ [14], and the result that the space of its points describes the “ ζ -sector” of $\mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}}$: *i.e.* the quotient $\mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}} / \widehat{\mathbb{Z}^{\times}}$ of the adèle class space of \mathbb{Q} by the right action of $\widehat{\mathbb{Z}^{\times}}$. The upgrading of $\widehat{\mathbb{N}^{\times}}$ to the topos $[0, \infty) \times \mathbb{N}^{\times}$ was motivated by the search for suitable structure sheaves on these toposes providing a precise geometric structure. As explained in Section 2, the algebras involved here are of “characteristic one” and these structures appear naturally in the study of the cyclic and epicyclic categories. In Section 3 we show how an independent study of these algebras in “characteristic one” and the discovery of the existence for these structures of an analog of the Frobenius endomorphisms, led us to the definition of the geometries of the Arithmetic and Scaling sites.

The search for a universal arithmetic then guided us to the study of algebras in the context of Segal’s Γ -rings. This development is reported in Section 4 and illustrates the versatility of the “paradis originel” for the topological algebras mentioned by

Grothendieck. Indeed, the fundamental objects of the relevant tensor category of \mathbb{S} -modules (*i.e.* Γ -sets) are defined as functors from a small category Γ° (a skeleton of the category of finite pointed sets) to sets. Moreover, the existence of the absolute base \mathbb{S} is dictated by the development of topological cyclic cohomology within the realm of ring spectra. As shown in [20], Segal’s Γ spaces provide the right framework for topological Hochschild and cyclic theories. By the Dold-Kan correspondence they give rise to the tools apt to develop homological algebra for \mathbb{S} -algebras. In Section 4 we explain how this point of view has brought us to the definition of the structure sheaf for the Arakelov compactification $\overline{\text{Spec}} \mathbb{Z}$ of the spectrum of the integers and the Riemann-Roch theorem for $\overline{\text{Spec}} \mathbb{Z}$. In Section 5 we focus on the resulting presentation of the ring \mathbb{Z} of the integers as polynomials with coefficients in the spherical group ring $\mathbb{S}[C_2] = \mathbb{S}[\pm 1]$, where C_2 is the cyclic group of order 2. This algebra also appears as global sections of the structure sheaf of $\overline{\text{Spec}} \mathbb{Z}$.

Finally, Section 7 introduces the main novelty of the present paper namely the *pericyclic category* Π . This category unifies and refines two conflicting notions of epicyclic space existing in the literature. The epicyclic category of Section 6 was originally introduced by T. Goodwillie in a letter to F. Waldhausen and it was implemented in [1] to define the corresponding notion of epicyclic space. Later on, Goodwillie introduced in [24] a topological category denoted $S\mathcal{R}\mathcal{F}$, also called “epicyclic”, which is at the root of topological cyclic homology. The pericyclic category Π (see §7.3) is the combinatorial version of a slightly different version $S'\mathcal{R}\mathcal{F}$ of $S\mathcal{R}\mathcal{F}$. We show that the two notions of epicyclic space mentioned above are encoded by two covariant functors $\Pi \longrightarrow \mathfrak{S}\text{ets}$ and we expect that the pericyclic category Π will play the role of a relevant backbone in our future development of a combinatorial version of cyclic homology (hinted at in Section 4).

2. Cyclic categories and projective geometry

Our original motivation for studying exotic algebraic structures of “characteristic one” was to achieve a good algebraic understanding of the cyclic and epicyclic categories, and in particular of the intriguing isomorphism between the cyclic category and its opposite Λ° . It turns out this isomorphism can be read as the transposition of linear maps in projective geometry, in the natural extension of linear algebra within the context of semifields. The conceptual reason behind the algebraic interpretation of the self-duality of the cyclic category requires some explanation. Originally, Λ has been defined in terms of homotopy classes of degree one, non-decreasing self-maps of the circle S^1 , which are compatible with roots of unity [4]. A better interpretation of it, (see [21]), is obtained by lifting these maps to the universal cover \mathbb{R} of S^1 , and then by restricting these maps to \mathbb{Z} . In this way, the morphisms between the objects in Λ are described by non-decreasing maps $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ fulfilling the equality, for some $n, m \in \mathbb{N}$

$$(2.1) \quad \phi(x+n) = \phi(x) + m \quad \forall x \in \mathbb{Z}.$$

An equivalence between two such maps ϕ, ϕ' is expressed by the following relation

$$(2.2) \quad \phi \sim \phi' \iff \exists k \in \mathbb{Z} \text{ s.t. } \phi'(x) = \phi(x) + km \quad \forall x \in \mathbb{Z}.$$

We shall see that one may interpret the non-decreasing condition together with (2.1) as a linearity property, and (2.2) as a projective equivalence. This is made possible by appealing to the theory of semirings and semifields which is the natural

development of the classical theory of rings and fields when one drops the existence of additive inverses. Semirings arise naturally as endomorphisms of any object in a category with a zero object (*i.e.* initial and final) in which the natural maps from direct sums to direct products are isomorphisms. Moving from fields to semifields introduces a few new objects of great interest: the first one is the Boolean semifield $\mathbb{B} = \{0, 1\}$, the unique finite semifield which is not a field. The addition in \mathbb{B} is idempotent ($1 + 1 := 1$) and a semiring contains \mathbb{B} if and only if it is also idempotent (aka of “characteristic one”). One also sees that there is only one semifield whose multiplicative group is infinite cyclic: this is the tropical semifield \mathbb{Z}_{\max} . More precisely, \mathbb{Z}_{\max} is the set $\mathbb{Z} \cup \{-\infty\}$ endowed with the operation

$$n \vee m := \max(n, m) \quad \forall n, m \in \mathbb{Z} \cup \{-\infty\}$$

replacing the classical addition in \mathbb{Z} , while the addition of the integers defines the multiplication in \mathbb{Z}_{\max} . In particular, the multiplicative group of \mathbb{Z}_{\max} is the infinite cyclic group \mathbb{Z} .

Next, for each integer $n > 0$, one defines a \mathbb{Z}_{\max} -module E_n whose underlying set is $\mathbb{Z} \cup \{-\infty\}$ with the same \vee operation, and where the action of \mathbb{Z}_{\max} on E_n is defined by

$$k \bullet m := m + kn \quad \forall k \in \mathbb{Z}_{\max}, m \in E_n.$$

One has the following interpretation for a map $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$

$$\phi \text{ is non decreasing} \iff \phi(n \vee m) = \phi(n) \vee \phi(m) \quad \forall n, m \in \mathbb{Z}.$$

Therefore, the non-decreasing maps $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ fulfilling (2.1) can be seen as the non-zero elements of the set $\text{Hom}_{\mathbb{Z}_{\max}}(E_n, E_m)$ which determine, in view of (2.2), projective equivalence classes of (\mathbb{Z}_{\max} -)linear maps. Thus, one concludes that the cyclic category Λ coincides with the category of projective morphisms between the modules E_n , and that the isomorphism $\Lambda \simeq \Lambda^\circ$ is expressed in terms of transpositions of such maps [9]. This algebraic description of the cyclic category hints at the extension from Λ to the epicyclic category $\tilde{\Lambda}$ which is known, by the work of J. L. Loday [28], to govern the λ -operations in cyclic homology. In classical projective geometry over fields, morphisms connecting two projective spaces over fields K and K' are constructed from linear maps $\phi : E \rightarrow E'$ of vector spaces, twisted by a morphism $\sigma : K \rightarrow K'$ of the fields, that satisfy the rule

$$\phi(\lambda\xi) = \sigma(\lambda)\phi(\xi) \quad \forall \lambda \in K, \xi \in E.$$

This suggests the extension of the cyclic category by implementing the endomorphisms of the semifield \mathbb{Z}_{\max} . In this respect, one of the striking features of an algebra in “characteristic one” is that for any idempotent semifield F (*i.e.* the addition of any element with itself gives itself) the map

$$(2.3) \quad \text{Fr}_a : F \rightarrow F, \quad \text{Fr}_a(x) := x^a \quad \forall x \in F$$

is an injective endomorphism, for any positive integer a ([22], Propositions 4.43-4.44). These endomorphisms are the analogs of the Frobenius endomorphism in finite characteristic, and for $F = \mathbb{Z}_{\max}$, they are the only non-trivial ones. Using these morphisms to allow for twisted maps between projective spaces amounts to extending (2.1) to

$$\phi(x + n) = \phi(x) + m a \quad \forall x \in \mathbb{Z}.$$

The epicyclic category $\tilde{\Lambda}$ is described precisely as the extension of the cyclic category obtained by keeping the same objects $\mathbb{P}(E_n)$ and implementing new morphisms

given by general (twisted) morphisms of projective spaces [9]. Furthermore, there is a natural “degree” functor $\mathbf{Mod} : \tilde{\Lambda} \rightarrow \mathbb{N}^\times$ which associates to a twisted morphism by Fr_a its degree $a \in \mathbb{N}^\times$. The emerging geometric picture involves geometric morphisms of the toposes dual to the categories. To promote, when the algebra \mathcal{A} is commutative, the cyclic module \mathcal{A}^{\natural} as a sheaf of vector spaces over the topos dual to the epicyclic category one proceeds as follows. The forgetful functor which associates to a map of projective spaces $\mathbb{P}(E_n) \rightarrow \mathbb{P}(E_m)$ the underlying morphism $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ of sets of points, connects $\tilde{\Lambda}$ to the category \mathfrak{Fin} of finite sets. When an algebra \mathcal{A} over a field k is commutative, the cyclic module \mathcal{A}^{\natural} extends to a covariant functor from \mathfrak{Fin} to vector spaces. To understand this functor as a sheaf of vector spaces of a topos, one considers the topos associated with the small category $\tilde{\Lambda}^\circ$ (the opposite category of $\tilde{\Lambda}$) which maps to \mathbb{N}^\times by the “degree” functor

$$\begin{array}{c} \tilde{\Lambda}^\circ \\ \mathbf{Mod} \downarrow \\ \mathbb{N}^\times \end{array}$$

This functor extending \mathcal{A}^{\natural} governs the λ -operations in cyclic homology and is discussed in details in §7.3. In [8], these operations are implemented to obtain the cyclic cohomological interpretation of Serre’s local L -factors of arithmetic varieties.

3. Geometric developments

The main role played by \mathbb{Z}_{\max} in the understanding of the cyclic and epicyclic categories, and the canonical action (2.3) of \mathbb{N}^\times by Frobenius endomorphisms on this semifield, led us to endow the “base” topos $\widehat{\mathbb{N}^\times}$ with the structure sheaf prescribed exactly by this action. The outcoming geometric structure is the *Arithmetic Site* [12] which, despite being defined using only countable data of combinatorial nature, admits a one-parameter family of correspondences whose composition law reflects a fundamental action of the multiplicative group \mathbb{R}_+^* of positive real numbers. This is the group of Galois automorphisms of the semifield \mathbb{R}_+^{\max} of positive real numbers under the two operations of max for addition and usual multiplication, which plays a central role both in the semiclassical limit of quantum mechanics *i.e.* in idempotent analysis, and tropical geometry. The space of points of the Arithmetic Site over \mathbb{R}_+^{\max} is canonically isomorphic to the ζ -sector of the adèle class space $\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}$ of the rationals [12], and the Galois action of \mathbb{R}_+^* on this space reflects geometrically the action of the idèle class group on the ζ -sector, in complete analogy with the geometric development in finite characteristic. The extension of scalars to \mathbb{R}_+^{\max} defines a new geometric object: the *Scaling Site* [14]. As a topos, this is the semi-direct product $[0, \infty) \rtimes \mathbb{N}^\times$, where $[0, \infty)$ is the euclidean half line $[0, \infty)$. The points of the topos $[0, \infty) \rtimes \mathbb{N}^\times$ coincide with the “ ζ -sector” already selected by the points of the Arithmetic Site over the same scalar extension. The new additional structure inherited by the Scaling Site is the structure sheaf made by piecewise affine, convex functions. A Riemann-Roch theorem analogous to that holding for elliptic curves is verified for the restriction of the Scaling Site to its periodic orbits [14]. A new striking feature is that the dimensions of the cohomologies involved are real numbers and they correspond, in the world of “characteristic one”, to the real dimensions of Murray and von Neumann. This Riemann-Roch

formula implements only H^0 and H^1 for divisors. One uses Serre duality to define H^1 , by-passing in this way, the development of a full cohomological machine. The module H^0 of global sections is naturally filtered using the p -adic valuation for the periodic orbit associated with the prime p , and this filtration gives rise to continuous dimensions.

The development in [15] of homological algebra in “characteristic one” has shown that the lack of additive inverses creates considerable difficulties in the process of passing from abelian categories to their counterpart, for idempotent semirings.

4. Universal cyclic theory

Surprisingly, one can circumvent the open problem of the elaboration of homological algebra for idempotent structures by developing cyclic homology in a world parallel to that of noncommutative geometry! This is the world of algebraic K -theory as promoted by D. Quillen and F. Waldhausen. We see it parallel to noncommutative geometry since it is transverse to the standard classification of mathematics into distinct branches. Cyclic homology was dragged into this context at the beginning of the 80’s, and then developed for spectra, in the world of “brave new rings”. At the Oberwolfach meeting in 1987, which was organized to sponsor interactions between functional analysis and algebraic topology, T. Goodwillie [23] emphasized that in the world of “brave new rings”, the ring \mathbb{Z} of the integers becomes an algebra over a more fundamental ring, namely the sphere spectrum \mathbb{S} . While the homotopical constructions are not particularly appealing to functional analysts, as iterated loop spaces may get quickly out of explicit control, a more concrete and basic version of the theory has gradually emerged and it is accounted for in [20]. The framework in question is that of G. Segal Γ -spaces which provides a workable model of connective spectra.

Our independent viewpoint on this subject is that the theory of Γ -spaces supplies the required constructions to produce the basic homological algebra, available through the Dold-Kan correspondence, for a very concrete algebraic set-up solely based on the Segal category Γ° . This category is a skeleton of the category \mathfrak{Fin}_* of pointed finite sets. Its main role is to provide the broadest generalization of a commutative and associative operation. These two properties combine to give a meaning to a “finite sum” and to obtain a covariant functor $\mathcal{F} : \mathfrak{Fin}_* \rightarrow \mathfrak{Sets}_*$, where the image of a map $f : X \rightarrow Y$ of finite pointed sets uses the finite sum over the inverse image $f^{-1}(y)$ to obtain the value at $y \in Y$. This construction generalizes broadly the Eilenberg-MacLane functor which replaces an abelian group A with the covariant functor $\mathrm{HA} : \mathfrak{Fin}_* \rightarrow \mathfrak{Sets}_*$ that assigns to a finite pointed set X the pointed set of A -valued divisors on X (taking the value 0 at the base point). Divisors push forward by the formula

$$(4.1) \quad f_*(D)(y) := \sum_{f(x)=y} D(x).$$

The Eilenberg-MacLane functor determines a fully faithful inclusion of abelian groups into the category of (covariant) functors $\Gamma^\circ \rightarrow \mathfrak{Sets}_*$, where morphisms are natural transformations of functors. Then, by implementing an idea of G. Lydakis, one defines the smash product of such functors starting from the closed structure determined by the smash product in \mathfrak{Fin}_* . The identity functor $\mathrm{Id} : \Gamma^\circ \rightarrow \mathfrak{Sets}_*$ defines the easiest example of such an algebra in this tensor category and it describes

the sphere spectrum \mathbb{S} in its most elementary form. This is the fundamental brave new ring advocated by Goodwillie in Oberwolfach.

In [13] we show that this theory of \mathbb{S} -algebras embodies all the previous attempts to extend ordinary algebra to reach a “universal arithmetic”.

The main difficulty that one encounters in trying to extend the structure sheaf of $\text{Spec } \mathbb{Z}$ by taking into account the point at ∞ (the archimedean place), comes from the lack of an analog, for the archimedean prime, of the subring of \mathbb{Q} which, for a non-archimedean prime p , is the localization of \mathbb{Z} at p . In p -adic terms this corresponds to elements of \mathbb{Q} which belong to the local ring \mathbb{Z}_p of p -adic integers (whose p -adic norm is ≤ 1). At the archimedean place, one meets the difficulty that the condition $|q| \leq 1$ does not define a subgroup of \mathbb{Q} . However, this problem is successfully solved by the theory of \mathbb{S} -algebras. Indeed, the Eilenberg-MacLane functor \mathbf{H} replaces faithfully a ring R by the \mathbb{S} -algebra $\mathbf{H}R$ and this applies in particular to the rings \mathbb{Z} and \mathbb{Q} . In this categorical universe the \mathbb{S} -algebra $\mathbf{H}\mathbb{Q}$ admits the non-trivial subalgebra $\mathcal{O}_\infty : \Gamma^\circ \rightarrow \mathbf{S}\mathbf{ets}_*$ defined by the condition on \mathbb{Q} -valued divisors:

$$(4.2) \quad \sum_{x \in X} |D(x)| \leq 1.$$

This condition is stable by push forward and as such it defines an \mathbb{S} -module. Moreover, since the product map

$$\mathcal{O}_\infty(X) \wedge \mathcal{O}_\infty(Y) \rightarrow \mathcal{O}_\infty(X \wedge Y)$$

is the restriction of the product map for $\mathbf{H}\mathbb{Q}$ which respects the basic condition (4.2), one sees that \mathcal{O}_∞ is an \mathbb{S} -subalgebra of $\mathbf{H}\mathbb{Q}$. This fact allows one to define the structure sheaf of $\overline{\text{Spec } \mathbb{Z}}$ as a subsheaf of the constant sheaf \mathbb{Q} [13].

A relevant application of this construction is the detection of the Gromov norm, which is obtained through a refinement of the ordinary homology with rational coefficients using \mathcal{O}_∞ as \mathbb{S} -module [16].

With the point at infinity taken into account, $\overline{\text{Spec } \mathbb{Z}}$ behaves like a projective curve, and the global sections of its structure sheaf form a very simple extension of the basic algebra \mathbb{S} . This is the spherical group ring $\mathbb{S}[C_2] = \mathbb{S}[\pm 1]$, which associates to a finite pointed set X the smash product of X with the pointed, multiplicative monoid $\{-1, 0, 1\}$. By working over this base, one gains the categorical interpretation of additive inverses! For an abelian group A , for example, the \mathbb{S} -module $\mathbf{H}A$ is naturally an $\mathbb{S}[\pm 1]$ -module using the rule $(-1) \times x = -x$ for all $x \in A$.

This construction seems to have the potential to go quite far. In the recent article [19] for example, we show how to parallel, on $\overline{\text{Spec } \mathbb{Z}}$, the adelic proof of the Riemann-Roch formula for function fields given by A. Weil, including the use of Pontryagin duality. The definition of the cohomologies is based on the universal arithmetic explained above and on an extension for \mathbb{S} -modules of the Dold-Kan correspondence. To an Arakelov divisor D on $\overline{\text{Spec } \mathbb{Z}}$ one associates a Γ -space $\mathbf{H}(D)$ using a short complex of $\mathbb{S}[\pm 1]$ -modules directly related to the divisor. The two cohomologies $H^0(D)$ and $H^1(D)$ are then defined *independently*, as well as the notion of their dimension over $\mathbb{S}[\pm 1]$. The Riemann-Roch formula also proven in *op.cit.* equates the *integer valued* Euler characteristic of a divisor D with a simple modification of the traditional expression (*i.e.* the degree of the divisor plus $\log 2$). The integer-valued topological side of the formula involves, besides the ceiling function, the division by $\log 3$. This result unveils the special role of the number 3

among the integers and the triadic expansion of the integers as polynomials $P(X)$, with coefficients in $\{-1, 0, 1\}$ evaluated at $X = 3$. In Section 5 we shall explore the extent to which one can view \mathbb{Z} as a polynomial ring over $\mathbb{S}[\pm 1]$.

A long-range goal is the computation of the universal cyclic cohomology for $\overline{\text{Spec } \mathbb{Z}}$.

5. $\mathbb{S}[\pm 1][3]$ and the ring of integers

In our recent work [19] on the arithmetic Riemann-Roch theorem for $\overline{\text{Spec } \mathbb{Z}}$, the number 3 plays the role of a natural generator, allowing one to label the integers $m \in \mathbb{Z}$ as polynomials in powers of 3 with coefficients in $\mathbb{S}[\pm 1]$

$$(5.1) \quad P(X) = \sum_{j=0}^k a_j X^j, \quad a_j \in \{-1, 0, 1\}, \quad \forall j.$$

The subtlety of the description of the ring structure of the integers \mathbb{Z} in this parametrization derives from the addition of polynomials with coefficients in the multiplicative monoid $\{-1, 0, 1\}$. Here $\{-1, 0, 1\}$ is endowed with the obvious multiplication rule, thus one knows how to multiply monomials *i.e.* $a X^n \times b X^m = (ab) X^{n+m}$, while the product of polynomials is obtained uniquely, using the distributive law, provided one knows how to add them. The sum of two monomials $a X^n, b X^m$ of different degrees is simply the polynomial $a X^n + b X^m$ and when the degrees are the same the distributivity law $a X^n + b X^n = (a + b)X^n$ reduces the reconstruction of the sum to polynomials of degree 0.

We shall make sure that addition is defined on polynomials of degree 0 in such a way that the following rule holds:

The sum of 3 polynomials of degree 0 is a polynomial of degree ≤ 1 .

Using this hypothesis it follows, by induction on the degree n , that the sum of two polynomials of degree $\leq n$ is a polynomial of degree $\leq n + 1$. The need to require that the sum of three polynomials of degree 0 is of degree ≤ 1 (rather than for just the sum of two), is due to the role of the carry over in adding arbitrary polynomials. Let us now consider the addition of two elements of $\{-1, 0, 1\}$. If one of them is 0 or if they have opposite signs, then the sum is the obvious one. Thus, using the symmetry $x \mapsto -x$, we can assume that they are both equal to 1. The only new rule that we add is the following

$$(5.2) \quad 1 + 1 = X - 1.$$

By applying (5.2), one adds two polynomials of degree 0 using the following table

+	-1	0	1
-1	1 - X	-1	0
0	-1	0	1
1	0	1	X - 1

To specify the sum of three polynomials of degree 0, it is enough to know how to add to the polynomial $X - 1$ the elements of $\{-1, 0, 1\}$. Using the associativity of the sum and (5.2), one has

$$(X - 1) + 1 = X, \quad (X - 1) - 1 = X - (1 + 1) = X - (X - 1) = 1.$$

It follows then that, as required, the sum of 3 polynomials of degree 0 is a polynomial of degree ≤ 1 . The following result determines the sum of any pair of polynomials under (5.2)

PROPOSITION 5.1. *Let $P(X)$ and $Q(X)$ be two polynomials as in (5.1), of degree at most n . Then there exists a unique polynomial of degree at most $n + 1$ which coincides, using the rule (5.2), with the sum $P(X) + Q(X)$.*

PROOF. Let assume (by induction) that the statement holds for any two polynomials of degree $< n$ and let consider a pair $P(X), Q(X)$ both of degree n . We decompose $P(X) = P_1(X) + \alpha_n X^n, Q(X) = Q_1(X) + \beta_n X^n$, with $\alpha_n, \beta_n \in \{-1, 0, 1\}$. Then

$$P(X) + Q(X) = P_1(X) + Q_1(X) + (\alpha_n + \beta_n)X^n,$$

where $P_1(X) + Q_1(X)$ is, by induction, the sum of a polynomial of degree $n - 1$ with a term $\gamma X^n, \gamma \in \{-1, 0, 1\}$. Then, from the specification of the sum of three polynomials of degree zero we have $\alpha_n + \beta_n + \gamma = \delta + \epsilon X$, with $\delta, \epsilon \in \{-1, 0, 1\}$. Thus the sum $P(X) + Q(X)$ can be written as a polynomial of degree at most $n + 1$, where the terms of degree $\geq n$ are $\delta X^n + \epsilon X^{n+1}$. \square

We obtain the following intriguing conclusion

PROPOSITION 5.2. *The set of polynomials as in (5.1), under the addition stated in Proposition 5.1, and the unique associated product, forms a ring isomorphic to \mathbb{Z} .*

PROOF. The map

$$\theta : \sum \alpha_j X^j \mapsto \sum \alpha_j 3^j$$

is a bijection from the set of polynomials as in (5.1) with the integers. This map is compatible with addition and multiplication in view of Proposition 5.1. \square

There is a function that allows one to compute effectively sums of polynomials as above. This is a function of three variables α, β, γ in $\{-1, 0, 1\}$ which specifies their sum as a polynomial of degree at most 1 in X . To determine $\alpha + \beta + \gamma$ one can think of it as a sum of 3 real numbers and write the set $\{-1, 0, 1\} + \gamma$ where this sum belongs, to obtain the coefficient of X . At the algebraic level, $\alpha + \beta + \gamma$ can be written explicitly using the finite field $\mathbb{F}_3 = \{-1, 0, 1\}$ as follows

$$\alpha + \beta + \gamma = \overline{\alpha + \beta + \gamma} + \overline{\sigma(\alpha, \beta, \gamma)} X$$

where \overline{m} is the residue of m modulo 3 and

$$\sigma(\alpha, \beta, \gamma) := \alpha\beta\gamma - \alpha^2\beta - \alpha^2\gamma - \alpha\beta^2 - \alpha\gamma^2 - \beta^2\gamma - \beta\gamma^2.$$

This determines the algebraic expression for the sum of three polynomials of degree 0 as in the proof of Proposition 5.1. Then, by induction on n , one obtains an algebraic expression for the coefficients of X^n in the sum $\sum_j \alpha_j X^j + \sum_j \beta_j X^j$ of two polynomials. Indeed, one defines by induction polynomials $s_n(\alpha_0, \dots, \alpha_{n-1}, \beta_0, \dots, \beta_{n-1})$ with coefficients in \mathbb{F}_3 as follows

$$s_0 := 0, \quad s_n = \sigma(\alpha_{n-1}, \beta_{n-1}, s_{n-1}).$$

Then the coefficient of X^n in the sum $\sum_j \alpha_j X^j + \sum_j \beta_j X^j = \sum_j \gamma_j X^j$ is given by

$$(5.3) \quad \gamma_n = \alpha_n + \beta_n + s_n.$$

This construction provides a sequence of polynomials $\gamma_n(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)$ and finally one has

LEMMA 5.3. *The polynomial $\gamma_n(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)$ vanishes provided the components α_n, α_{n-1} and β_n, β_{n-1} all vanish.*

PROOF. It is enough to show that $s_n = 0$ if $\alpha_{n-1} = 0$ and $\beta_{n-1} = 0$. This follows from the inductive definition of $s_n = \sigma(\alpha_{n-1}, \beta_{n-1}, s_{n-1})$. \square

Since one works over \mathbb{F}_3 $x^n = x$ if n is an odd number, while $x^n = x^2$ if n is a non-zero even number. Thus one can reduce the exponents of the variables in the polynomials γ_n to be at most 2. One has $s_1(\alpha_0, \beta_0) = -\alpha_0\beta_0^2 - \alpha_0^2\beta_0$, while s_2 in reduced form is given by

$$\begin{aligned} s_2(\alpha_0, \alpha_1, \beta_0, \beta_1) &= \alpha_1\alpha_0^2\beta_0^2 + \alpha_0^2\beta_0\beta_1^2 + \alpha_1^2\alpha_0^2\beta_0 + \alpha_0^2\beta_0^2\beta_1 - \alpha_1\alpha_0^2\beta_0\beta_1 \\ &\quad + \alpha_1^2\alpha_0\beta_0^2 + \alpha_0\beta_0^2\beta_1^2 + \alpha_1\alpha_0\beta_0 - \alpha_1\alpha_0\beta_0^2\beta_1 + \alpha_0\beta_0\beta_1 - \alpha_1\beta_1^2 - \alpha_1^2\beta_1 \end{aligned}$$

To write in full $s_3(\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2)$ would take several lines.

Conceptually, one sees that the above construction is described by the addition rule of two Witt vectors over \mathbb{F}_3 . The number 3 is the only prime for which the Witt vectors with only finitely many non-zero components form an additive subgroup of the Witt ring. More precisely one can state the following

PROPOSITION 5.4. *Witt vectors with only finitely many non-zero components form a subring of the ring $\mathbf{W}(\mathbb{F}_3)$. This subring is isomorphic to $\mathbb{Z} \subset \mathbb{Z}_3$.*

PROOF. For any prime p the map

$$\mathbf{W}(\mathbb{F}_p) \ni \xi = (\xi_j) \mapsto \tilde{\tau}(\xi) := \sum_{j=0}^{\infty} \tau(\xi_j)p^j,$$

where τ is the Teichmüller lift, is known to be a ring isomorphism between the Witt ring $\mathbf{W}(\mathbb{F}_p)$ and the ring \mathbb{Z}_p of p -adic integers. For $p = 3$, $\tau(\mathbb{F}_3) = \{-1, 0, 1\} \subset \mathbb{Z} \subset \mathbb{Z}_3$, thus $\tilde{\tau}$ maps Witt vectors with only finitely many non-zero components into the subring $\mathbb{Z} \subset \mathbb{Z}_3$. This map is an injective ring homomorphism and it surjects onto \mathbb{Z} since by Proposition 5.2 any integer can be written as a finite sum of powers of 3 with coefficients in $\{-1, 0, 1\}$. \square

6. The cyclic and epicyclic categories as subcategories of \mathbf{Cat}

In Section 2 we reviewed the cyclic and the epicyclic categories using concepts of linear algebra in characteristic one, in particular we have described the objects of these categories as projective spaces over the semifield \mathbb{Z}_{\max} . Given these results it seems natural to wonder about the existence of further presentations of the same categories: in particular whether their objects and morphisms may be realized within the “paradis originel” provided by the category \mathbf{Cat} of small categories. This idea was developed by D. Kaledin in [27] and it turns out to be a slight variation of our earlier viewpoint developed in [10]. In that paper we described the cyclic and epicyclic categories in terms of oriented groupoids: this result is reviewed here below. This section is dedicated to comparing our viewpoint with that of [27] that has the advantage to interpret the collection of the objects as small categories (and as such objects of \mathbf{Cat}) and the morphisms as functors, thus presenting the two categories as subcategories of \mathbf{Cat} .

6.1. Oriented groupoids. We recall that a groupoid G can be seen as a small category in which every morphism is invertible. We denote by $G^{(0)}$ the set of objects of G and by $r, s : G \rightarrow G^{(0)}$ resp. the range and the source of the morphisms of G . We view $G^{(0)} \subset G$ as the subset of the identity morphisms of G .

The following definition is a direct generalization of the notion of the right ordered group for a groupoid G . Given $X \subset G$, we let $X^{-1} := \{\gamma^{-1} \mid \gamma \in X\}$.

DEFINITION 6.1. An oriented groupoid (G, G_+) is a groupoid G endowed with a subcategory $G_+ \subset G$, fulfilling the following relations

$$(6.1) \quad G_+ \cap G_+^{-1} = G^{(0)}, \quad G_+ \cup G_+^{-1} = G.$$

Let H be a group acting on a set X . The semi-direct product $G := X \rtimes H$ is a groupoid with source, range and composition law defined respectively as follows

$$s(x, h) := x, \quad r(x, h) := hx, \quad (x, h) \circ (y, k) := (y, hk).$$

As it happens in any groupoid, the composite $\gamma \circ \gamma'$ of two elements of G is only defined when $s(\gamma) = r(\gamma')$: here this holds if and only if $x = ky$.

There is a canonical homomorphism of groupoids $\rho : G \rightarrow H$, $\rho(x, h) = h$. We recall from [10] (Lemma 2.2) the following

LEMMA 6.2. *Let (H, H_+) be a right ordered group. Assume that H acts on a set X . Then the semi-direct product $G = X \rtimes H$, with $G_+ := \rho^{-1}(H_+)$, is an oriented groupoid.*

We shall consider, in particular, the right ordered group $(H, H_+) = (\mathbb{Z}, \mathbb{Z}_+)$ acting by translation on the set $X = \mathbb{Z}/(m + 1)\mathbb{Z}$ of integers modulo $m + 1$. Then, by applying the above lemma one obtains the oriented groupoid

$$\mathbf{g}(m) := (\mathbb{Z}/(m + 1)\mathbb{Z}) \rtimes \mathbb{Z}.$$

The relation with the epicyclic category is provided by the following statement (see [10] Corollary 3.11)

PROPOSITION 6.3. *The epicyclic category $\tilde{\Lambda}$ is canonically isomorphic to the category whose objects are the oriented groupoids $\mathbf{g}(m)$, $m \geq 0$, and the morphisms are the non-trivial morphisms of oriented groupoids.*

The functor which associates to a morphism of oriented groupoids its class up-to equivalence (of small categories) corresponds, under the canonical isomorphism, with the functor $\mathbf{Mod} : \tilde{\Lambda} \rightarrow \mathbb{N}^\times$ sending a semilinear map of semimodules over $\mathbb{F} = \mathbb{Z}_{\max}$ to the corresponding injective endomorphism $\text{Fr}_n \in \text{End}(\mathbb{F})$ cf. [9].

6.2. Comparison with [27]. We recall the presentation of the cyclic category given in [27]: the following notations and the three definitions are taken literally from *op.cit.*

Let $[1]_\Lambda$ be the category with one object and whose endomorphisms are determined by non-negative integers $a \in \mathbb{Z}_+$. The inclusion $\mathbb{Z}_+ \subset \mathbb{Z}$ induces a functor $[1]_\Lambda \rightarrow \mathbf{pt}_{\mathbb{Z}}$. By the Grothendieck construction, sets with a \mathbb{Z} -action correspond to small categories discretely bi-fibered over $\mathbf{pt}_{\mathbb{Z}}$, thus a \mathbb{Z} -set S induces, by pull-back, a discrete bi-fibration $[1]_\Lambda^S \rightarrow [1]_\Lambda$. For any integer $n \geq 1$, let $[n]_\Lambda$ be the category corresponding to the \mathbb{Z} -set $\mathbb{Z}/n\mathbb{Z}$. The category $[n]_\Lambda$ is equivalent to the additive monoid \mathbb{N} of non-negative integers. The degree of a functor is defined in terms of the corresponding morphism of monoids

DEFINITION 6.4. A functor $f : [n]_\Lambda \rightarrow [m]_\Lambda$ is said to be vertical if it is a discrete bi-fibration, horizontal if $\text{deg}(f) = 1$ and non-degenerate if $\text{deg}(f) \neq 0$.

DEFINITION 6.5. The *cyclotomic category* ΛR is the small category whose objects $[n]$ are indexed by positive integers and whose morphisms $[n] \rightarrow [m]$ are given by non-degenerate functors $f : [n]_\Lambda \rightarrow [m]_\Lambda$.

DEFINITION 6.6. The cyclic category Λ is the subcategory of ΛR with the same set of objects and whose morphisms are the horizontal ones.

Next, we relate the above definitions (taken literally from [27]) the description of the epicyclic category given in [10] and here recalled in §6.1. There is a natural functor \mathcal{P} which associates to an oriented groupoid G the subcategory $G_+ = \mathcal{P}(G)$, viewed as a small category. For $G = \mathfrak{g}(n)$, one has

$$(6.2) \quad \mathcal{P}(\mathfrak{g}(n)) = [n + 1]_\Lambda \quad \forall n \geq 0.$$

The link with the epicyclic category is then provided by the following

PROPOSITION 6.7. *The functor \mathcal{P} identifies the epicyclic category $\tilde{\Lambda}$ (as described in Proposition 6.3) with the cyclotomic category ΛR . From the commutativity of the diagram*

$$\begin{array}{ccc} \tilde{\Lambda} & \xrightarrow{\mathcal{P}} & \Lambda R \\ \mathbf{Mod} \downarrow & & \downarrow \text{deg} \\ \mathbb{N}^\times & \xrightarrow{\text{Id}} & \mathbb{N}^\times \end{array}$$

the degree as in [27] corresponds to the module as in Proposition 6.3.

Since the cyclic category Λ is the subcategory of $\tilde{\Lambda}$ with the same objects, and morphisms fulfilling $\mathbf{Mod}(f) = 1$, it follows from the above identification that Definition 6.6 determines the cyclic category Λ inside $\tilde{\Lambda}$.

7. The pericyclic category Π

In §6.2 it has been shown that the epicyclic category $\tilde{\Lambda}$ can be identified with the subcategory ΛR of \mathbf{Cat} (the name cyclotomic category for ΛR seems, therefore, to be redundant and possibly also creating confusion). Another source of confusion is that the name *epicyclic* is used in the literature with two different meanings. The epicyclic category $\tilde{\Lambda}$ of Section 6 was originally introduced by T. Goodwillie in a letter to F. Waldhausen, however later on Goodwillie introduced in [24] another category, also called “epicyclic” (denoted with $S\mathcal{R}\mathcal{F}$ cf. [20]) and here described in §7.1. In the same §7.1 we also correct a “typo”, sometimes reported in the literature, regarding the composition rule for morphisms in the category $S\mathcal{R}\mathcal{F}$. In §7.2 we show that using the category $S'\mathcal{R}\mathcal{F}$ (with the modified rule implemented), a cyclotomic spectrum gives rise to a functor from $S'\mathcal{R}\mathcal{F}$ to spectra. In §7.3 we introduce the *pericyclic category* Π , as the combinatorial version of $S'\mathcal{R}\mathcal{F}$ and prove that the two notions of epicyclic space (in relation to the two epicyclic categories mentioned above) are encoded in terms of (covariant) functors $\Pi \rightarrow \mathfrak{S}\mathfrak{c}\mathfrak{t}\mathfrak{s}$. Finally, §7.4 investigates the points of the topos dual to a small category \mathcal{R} constituent of the pericyclic category.

7.1. The category $S\mathcal{R}\mathcal{F}$. The small category $\mathcal{R}\mathcal{F}$ (see [20] Definition 6.2.3.2) has one object a for each integer $a > 0$ and for any pair of objects a, b the morphisms connecting them are

$$\text{Hom}_{\mathcal{R}\mathcal{F}}(a, b) := \{f_{r,s} \mid r, s \in \mathbb{N}^\times, a = rbs\}.$$

In particular one has: $\text{Hom}_{\mathcal{R}\mathcal{F}}(a, b) = \emptyset$ unless $b|a$. The composition of morphisms is assigned by

$$f_{r,s} \circ f_{p,q} = f_{rp,sq}.$$

Thus one sees that the following maps define functors sending all the objects to the unique object \bullet of \mathbb{N}^\times and acting on morphisms as follows

$$(7.1) \quad \text{Res} : \mathcal{RF} \longrightarrow \mathbb{N}^\times, \quad \text{Res}(f_{p,q}) := p \in \mathbb{N}^\times, \quad \text{Fr} : \mathcal{RF} \longrightarrow \mathbb{N}^\times, \quad \text{Fr}(f_{p,q}) := q \in \mathbb{N}^\times.$$

The small category $S\mathcal{RF}$ (see *op.cit.* Remark 6.2.3.5) is topological and has the same objects as \mathcal{RF} but implements an angular parameter θ in the morphisms (of \mathcal{RF}), viewed as an element of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ or equivalently understood as the element $e(\theta) := \exp(2\pi i\theta)$ of $U(1) := \{z \in \mathbb{C} \mid |z| = 1\}$. An *epicyclic space* is a functor from $S\mathcal{RF}$ to topological spaces. In this framework the first index r in the morphisms $f_{r,s}$ corresponds to restriction maps (see [20], 6.2.3) while the second index labels the *inclusions of fixed points*.

Let us carefully review the role of such inclusions. Given a \mathbb{T} -space X , *i.e.* a space X endowed with an action $\theta \mapsto \rho_\theta \in \text{Aut}(X)$ of \mathbb{T} on X , one associates to an integer $q > 0$ the cyclic subgroup $C_q = \{\theta \in \mathbb{T} \mid q\theta = 0\}$ (equivalently the group of q -th roots of unity in $U(1)$) and the subspace X^{C_q} of the C_q fixed points. The canonical inclusion $i_q : X^{C_q} \rightarrow X$ is \mathbb{T} -equivariant by construction, however, because of the canonical isomorphism

$$\alpha : \mathbb{T} \rightarrow \mathbb{T}/C_q, \quad \alpha(\theta) = \frac{1}{q}\theta, \quad \alpha^{-1} : \mathbb{T}/C_q \rightarrow \mathbb{T}, \quad \alpha^{-1}(\theta) = q\theta$$

one modifies the action of \mathbb{T} on X^{C_q} by composing with α . For this new action $\rho' = \rho \circ \alpha$ on X^{C_q} fulfilling $\rho'_{q\theta} = \rho \circ \alpha(q\theta) = \rho_\theta$, one has

$$(7.2) \quad \rho_\theta i_q(x) = i_q(\rho'_{q\theta}(x)).$$

Next, we review in detail the notion of an epicyclic space given in [20] (Definition 6.2.3.1). For any integer $k > 0$, the edgewise subdivision is an endofunctor $\text{sd}^k : \Delta \rightarrow \Delta$ obtained by concatenating k copies of an ordinal: it multiplies the cardinality by k . Given a simplicial set Y one lets $\text{sd}_k(Y) := Y \circ \text{sd}^k$ be the simplicial set obtained by pre-composition with these endofunctors. If moreover Y is a cyclic set then the cyclic structure extends to $\text{sd}_k(Y)$ provided one replaces the cyclic category Λ with the k -cyclic category Λ_k defined by replacing the cyclic group C_n involved as automorphisms of the object of cardinality n , by the group C_{nk} with generator $t_{(n,k)}$, while keeping the same relations of the generator with faces and degeneracies. The fixed points Z^{C_k} in a k -cyclic set Z form a cyclic set. Thus, given a cyclic set Y and an integer $k > 0$ the fixed points $\text{sd}_k(Y)^{C_k}$ in the edgewise subdivision $\text{sd}_k(Y)$ form a cyclic set.

DEFINITION 7.1. An *epicyclic space* (Y, ϕ) is a pointed cyclic space Y equipped with pointed cyclic maps $\phi_q : \text{sd}_q(Y)^{C_q} \rightarrow Y$, $\phi_1 = \text{Id}$, so that the following diagrams commute

$$\begin{array}{ccc} \text{sd}_k(\text{sd}_r(Y)^{C_r})^{C_k} & \xrightarrow{\text{id}} & \text{sd}_{kr}(Y)^{C_{kr}} \\ \text{sd}_k(\phi_r) \downarrow & & \downarrow \phi_{kr} \\ \text{sd}_k(Y)^{C_k} & \xrightarrow{\phi_k} & Y \end{array}$$

The geometric realization $|Z|$ of a k -cyclic set Z admits a canonical action of the group $\mathbb{R}/k\mathbb{Z}$, and the fixed points of this action under the subgroup $\mathbb{Z}/k\mathbb{Z} \sim C_k$

give rise to the geometric realization of the cyclic set Z^{C_k} . Moreover, one has a canonical homeomorphism of geometric realizations ([20] Lemmas 6.2.2.1, 6.2.2.2)

$$D_a : |\mathrm{sd}_a(Y)| \rightarrow |Y|, \quad D_a(s + a\mathbb{Z})(y) = \left(\frac{s}{a} + \mathbb{Z}\right) D_a(y).$$

Given an epicyclic space (Y, ϕ) , one introduces for each integer $a > 0$ the \mathbb{T} -space

$$Y(a) := |\mathrm{sd}_a(Y)^{C_a}| \simeq |Y|^{C_a}$$

and the maps $i_q : Y(qa) \rightarrow Y(a)$ corresponding to the inclusion of the fixed points

$$Y(qa) \simeq |Y|^{C_{qa}} \subset |Y|^{C_a} \simeq Y(a).$$

Then it follows from (7.2) that the compatibility of the maps i_q with the \mathbb{T} -action is expressed by the following formula

$$(7.3) \quad \rho_\theta i_q(x) = i_q(\rho_{q\theta}(x)).$$

Now we come to the point of issue. The composition of the morphisms is given in [24] by the formula

$$(7.4) \quad (\theta, f_{r,s}) \circ (\tau, f_{p,q}) = (\theta + s\tau, f_{rp,sq}).$$

However, one should associate to the pair $(\theta, f_{r,s})$ the morphism $\phi_r i_s \rho_\theta$, instead of $\phi_r \rho_\theta i_s$, the reason being that the latter labeling is redundant since ρ_θ applied to the range of i_s is unchanged when θ is multiplied by s -torsion. Then, standing (7.3), (7.4) ought to be replaced with

$$(7.5) \quad (\theta, f_{r,s}) \circ (\tau, f_{p,q}) = (q\theta + \tau, f_{rp,sq})$$

which corresponds to

$$i_s \rho_\theta i_q \rho_\tau = i_s i_q \rho_{q\theta} \rho_\tau.$$

Next, we rely on (7.5) to define the variant $S'\mathcal{RF}$ of the category $S\mathcal{RF}$. Thus by construction $S'\mathcal{RF}$ is an extension of \mathcal{RF} by $U(1)$, and there is a natural forgetful functor $\sigma : S'\mathcal{RF} \rightarrow \mathcal{RF}$ which is the identity on objects and maps $(z, f_{r,s}) \mapsto f_{r,s}$. We now compare this category with the monoid $U(1) \times \mathbb{N}^\times$ considered in [1], whose elements are pairs $(z, s) \in U(1) \times \mathbb{N}^\times$, and the composition law is defined by

$$(z_1, s) \circ (z_2, q) = (z_1 z_2^s, sq).$$

One obtains the following commutative diagram

$$(7.6) \quad \begin{array}{ccc} (U(1) \times \mathbb{N}^\times)^o & \xleftarrow{\mathrm{Id} \times \mathrm{Fr}} & S'\mathcal{RF} \\ p \downarrow & & \downarrow \sigma \\ \mathbb{N}^\times & \xleftarrow{\mathrm{Fr}} & \mathcal{RF} \end{array}$$

7.2. Cyclotomic spectra. The notion for spectra corresponding to the construction reviewed above is that of *cyclotomic spectrum*. The general notion of G -spectrum is here used in the special case $G = \mathbb{T}$, where the compact Lie group \mathbb{T} has the special property that it is canonically isomorphic to its quotient \mathbb{T}/C by any non-trivial closed subgroup $C \subset \mathbb{T}$. To an integer $n \in \mathbb{N}^\times$ corresponds the subgroup C_n of n -th roots of unity; there is an action Ψ of the monoid \mathbb{N}^\times on \mathbb{T} -spectra, by restriction to the fixed points of C_n and then viewing the corresponding \mathbb{T}/C_n -spectrum as a \mathbb{T} -spectrum (see [25] Section 2, [26] page 9).

DEFINITION 7.2. A cyclotomic spectrum E is a \mathbb{T} -spectrum together with morphisms of \mathbb{T} -spectra $\epsilon_k : \Psi^k(E) \rightarrow E$ such that the following diagrams commute

$$\begin{array}{ccc} \Psi^k(\Psi^r(E)) & \xrightarrow{\simeq} & \Psi^{kr}(E) \\ \Psi^k(\epsilon_r) \downarrow & & \downarrow \epsilon_{kr} \\ \Psi^k(E) & \xrightarrow{\epsilon_k} & E \end{array}$$

By implementing (7.5) to define the variant $S'\mathcal{RF}$ of the category $S\mathcal{RF}$, we obtain

LEMMA 7.3. A cyclotomic spectrum E defines a (covariant) functor from $S'\mathcal{RF}$ to the category of $RO(\mathbb{T})$ -graded topological spaces.

PROOF. Let consider the inclusion of fixed points, i.e. for $a = sb$ the map $(\theta, f_{1,s}) : a \rightarrow b$. Let i_s^a be the inclusion $E^{C_a} \subset E^{C_{a/s}}$. The action of $(\theta, f_{1,s})$ is $i_s^a \circ E(\frac{\theta}{a})$. Thus (except for the inclusions) the action of $(\theta, f_{1,s}) : a \rightarrow b$ composed with the action of $(\tau, f_{1,q}) : qa \rightarrow a$ is

$$E\left(\frac{\theta}{a}\right) \circ E\left(\frac{\tau}{aq}\right) = E\left(\frac{\theta}{a} + \frac{\tau}{aq}\right) = E\left(\frac{q\theta + \tau}{aq}\right)$$

which agrees with (7.5). The action of the maps $f_{r,1}$ commutes with the action of \mathbb{T} and corresponds to the composition of the map $\beta : E^{C_r} \rightarrow \Phi^{C_r}(E)$ with the structure map $\Phi^{C_r}(E) \rightarrow E$ of the cyclotomic spectrum. These are all maps of \mathbb{T} -spectra by construction. Thus the parameters r, q do not interfere with the validity of (7.5). \square

7.3. The pericyclic category Π . There is a combinatorial version of the topological category $S'\mathcal{RF}$: this is the pericyclic category Π that we shall introduce here below.

In the diagram (7.6) we replace the monoid $(U(1) \times \mathbb{N}^\times)^\circ$ with the category $\tilde{\Lambda}^\circ$. Indeed, on the geometric realization of an epicyclic space in the sense of [1] (i.e. as a covariant functor from $\tilde{\Lambda}^\circ$ to spaces) one has, by applying Theorem A of *op.cit.* a natural *right* action of the monoid $U(1) \times \mathbb{N}^\times$, i.e. a left action of $(U(1) \times \mathbb{N}^\times)^\circ$. This shows that $\tilde{\Lambda}^\circ$ is the combinatorial version of the monoid $(U(1) \times \mathbb{N}^\times)^\circ$. In the following, we describe the missing entries (i.e. the ?'s) in the pullback diagram

$$(7.7) \quad \begin{array}{ccc} \tilde{\Lambda}^\circ & \xleftarrow{?} & ? \\ \text{Mod} \downarrow & & \downarrow ? \\ \mathbb{N}^\times & \xleftarrow{\text{Fr}} & \mathcal{RF}. \end{array}$$

DEFINITION 7.4. The pericyclic category Π is the subcategory of the product category $\tilde{\Lambda}^\circ \times \mathcal{RF}$ sharing the same objects and whose morphisms

$$(7.8) \quad \text{Hom}_\Pi((n, a), (m, b)) \subset \text{Hom}_{\tilde{\Lambda}^\circ}(n, m) \times \text{Hom}_{\mathcal{RF}}(a, b)$$

are given by those pairs $(h, f) \in \text{Hom}_{\tilde{\Lambda}^\circ}(n, m) \times \text{Hom}_{\mathcal{RF}}(a, b)$ fulfilling the equality

$$(7.9) \quad \text{Fr}(f) = \text{Mod}(h).$$

Thus an object of Π is a pair (n, a) with $n \geq 0$, $n \in \mathbb{N}$ and $a \in \mathbb{N}^\times$. Then we complete (7.7) to a pullback diagram as follows

$$\begin{array}{ccc} \tilde{\Lambda}^\circ & \xleftarrow{\lambda} & \Pi \\ \text{Mod} \downarrow & & \downarrow \pi \\ \mathbb{N}^\times & \xleftarrow{\text{Fr}} & \mathcal{RF} \end{array}$$

by implementing the two projections arising from (7.8), namely the functors $\lambda : \Pi \longrightarrow \tilde{\Lambda}^\circ$ and $\pi : \Pi \longrightarrow \mathcal{RF}$ defined by

$$\lambda(n, a) := n, \quad \lambda(h, f) := h, \quad \pi(n, a) := a, \quad \pi(h, f) := f.$$

To obtain a natural section of λ we introduce the following further morphisms in Π . For $n \geq 0$, $s > 0$ and $d > 0$ integers, with $m = s(n + 1) - 1$, one lets

$$\chi(n, s, d) := (\text{Id}_n^{s,\circ}, f_{1,s}) \in \text{Hom}_\Pi((n, sd), (m, d)).$$

This is meaningful since $(\text{Id}_n^{s,\text{op}}, f_{1,s}) \in \text{Hom}_{\tilde{\Lambda}^\circ}(n, m) \times \text{Hom}_{\mathcal{RF}}(sd, d)$ fulfills (7.9). Let W be the set of morphisms of the form $\text{Id} \times f_{r,1} : (n, a) \rightarrow (n, b)$ (for $br = a$). We localize the category Π on W : this is the category $\Pi[W^{-1}]$ obtained by inverting the morphisms in W . One obtains in this way a small category equivalent to $\tilde{\Lambda}^\circ$

LEMMA 7.5. *The functor $\theta : \tilde{\Lambda}^\circ \longrightarrow \Pi[W^{-1}]$*

$$\theta(n) := (n, 1), \quad \theta(\ell) = \ell \times f_{1,1}, \quad \forall \ell \in \text{Hom}_\Lambda(n, m), \quad \theta(\text{Id}_n^{s,\circ}) = \chi(n, s, 1) \circ f_{s,1}^{-1}$$

defines an equivalence of categories.

PROOF. By construction, the restriction of θ to the cyclic category Λ is an isomorphism with a subcategory of Π . Let $p_n^s := \chi(n, s, 1) \circ f_{s,1}^{-1} : (n, 1) \rightarrow (n, s) \rightarrow (s(n + 1) - 1, 1)$. One easily checks that these morphisms of $\Pi[W^{-1}]$ fulfill the presentation of $\tilde{\Lambda}^\circ$ as an extension of the cyclic category. Indeed, this follows from the construction of Π as a subcategory of $\tilde{\Lambda}^\circ \times \mathcal{RF}$, so that one can check the relations in $\tilde{\Lambda}^\circ \times \mathcal{RF}$ using the fact that they hold in $\tilde{\Lambda}^\circ$. This proves that θ is a functor. Moreover, the composite $\lambda \circ \theta$ is the identity thus θ is faithful. Finally, after inverting the morphisms in W one sees that an object (n, a) of Π is isomorphic to the object $(n, 1)$ which is in the range of θ . Thus θ is an equivalence of categories. \square

Lemma 7.5 shows that covariant functors $\Pi[W^{-1}] \longrightarrow \mathfrak{Sets}$ correspond to epicyclic sets in the sense of [1]. For an epicyclic space in the sense of [20] (Definition 6.2.3.1), *i.e.* of Definition 7.1 we have

PROPOSITION 7.6. *Let (Y, ϕ) be an epicyclic space as in Definition 7.1. The following equalities define a covariant functor:*

$$\beta : \Pi \longrightarrow \mathfrak{Sets}, \quad \beta(n, a) = sd_a(Y)^{C_a}(n)$$

$$(7.10) \quad \beta((\ell, \text{Id}_a)) = sd_a(Y)^{C_a}(\ell) : \beta(n, a) \rightarrow \beta(m, a), \quad \forall \ell \in \text{Hom}_\Lambda(n, m)$$

$$(7.11) \quad \beta(\chi(n, u, v)) : \beta(n, uv) \subset \beta(u(n + 1) - 1, v)$$

$$\beta((\text{Id}, f_{q,1})) = \phi_q : \beta(n, qa) = sd_{aq}(Y)^{C_{aq}}(n) \rightarrow \beta(n, a) = sd_a(Y)^{C_a}(n).$$

The functor β encodes faithfully the epicyclic space (Y, ϕ) .

PROOF. Let us first explain the meaning of the inclusion in (7.11). By construction, one has for all $a > 0$ and $n \geq 0$

$$\text{sd}_a(Y)(n) = Y(\text{sd}^a(n)) = Y(a(n+1) - 1),$$

so that

$$\text{sd}_{uv}(Y)(n) = Y(uv(n+1) - 1) = \text{sd}_v(Y)(u(n+1) - 1).$$

The fixed points under the cyclic subgroups involve the action of the subgroup C_{uv} for $\beta(n, uv) = \text{sd}_{uv}(Y)^{C_{uv}}(n)$, and its subgroup $C_v \subset C_{uv}$ for $\beta(u(n+1) - 1, v) = \text{sd}_v(Y)^{C_v}(u(n+1) - 1)$. Thus the inclusion in (7.11) is the inclusion of fixed points. The structure maps ϕ_q are cyclic maps $\text{sd}_{aq}(Y)^{C_{aq}} \rightarrow \text{sd}_a(Y)^{C_a}$ and this shows that $\beta((\text{Id}, f_{q,1}))$ commutes with the maps $\beta((\ell, \text{Id}_a))$ of (7.10) defining the cyclic action. Moreover $\beta((\text{Id}, f_{q,1}))$ commutes with the inclusion of fixed points (7.11), *i.e.* the following diagram is commutative for any n, q, u, v

$$\begin{array}{ccc} \beta(n, quv) & \xrightarrow{\beta(\chi(n,u,qv))} & \beta(u(n+1) - 1, qv) \\ \beta((\text{Id}, f_{q,1})) \downarrow & & \downarrow \beta((\text{Id}, f_{q,1})) \\ \beta(n, uv) & \xrightarrow{\beta(\chi(n,u,v))} & \beta(u(n+1) - 1, v). \end{array}$$

It remains to check that the inclusion of fixed points (7.11) fulfills the relation of $\text{Id}_n^{s,\circ}$ in the category $\tilde{\Lambda}^\circ$ with respect to the cyclic action (7.10). This follows at the combinatorial level from the above discussion of the inclusion of fixed points. \square

Proposition 7.6 applies, in particular, to the cyclic nerve of a small category \mathcal{C} (*cf.* [20] Example 6.2.3.3). This is the cyclic set

$$B_n^{\text{cy}}\mathcal{C} = \{c_n \xleftarrow{f_0} c_0 \xleftarrow{f_1} c_1 \leftarrow \dots \xleftarrow{f_n} c_n\}$$

formed by $n + 1$ composable morphisms of \mathcal{C} . The face maps are defined by composition, and the degeneracies by insertion of identities, while the cyclic structure is given by cyclic permutations. In [20] (Example 6.2.3.3), one gets an epicyclic space (in the sense of [20]), using the isomorphisms

$$(7.12) \quad \phi_k : (\text{sd}_k B^{\text{cy}}\mathcal{C})^{C_k} \simeq B^{\text{cy}}\mathcal{C}.$$

Proposition 7.6 thus defines a covariant functor $B^{\text{epi}}\mathcal{C} : \Pi \rightarrow \mathfrak{Sets}$. Moreover, since the ϕ_k are invertible, Lemma 7.5 applies and one gets an epicyclic set in the sense of [1]. This functor is the same as the epicyclic set associated to the cyclic nerve in *op.cit.* Indeed, the latter is the functor $\tilde{\Lambda}^\circ \rightarrow \mathfrak{Sets}$ which extends the canonical cyclic structure of $B^{\text{cy}}\mathcal{C}$ by the additional map $p_n^k : B_n^{\text{cy}} \rightarrow B_{k(n+1)-1}^{\text{cy}}$ associated to the morphism $\text{Id}_n^{k,\circ} \in \text{Hom}_{\tilde{\Lambda}^\circ}(n, k(n+1) - 1)$. The map p_n^k is the k -fold amplification, mapping the element $(c_n \xleftarrow{f_0} c_0 \xleftarrow{f_1} c_1 \leftarrow \dots \xleftarrow{f_n} c_n)$ of B_n^{cy} into its repetition k -times

$$c_n \xleftarrow{f_0} c_0 \xleftarrow{f_1} c_1 \leftarrow \dots \xleftarrow{f_n} c_n \xleftarrow{f_0} c_0 \xleftarrow{f_1} c_1 \leftarrow \dots \xleftarrow{f_n} c_n \xleftarrow{f_0} c_0 \xleftarrow{f_1} c_1 \leftarrow \dots \xleftarrow{f_n} c_n$$

The map p_n^k is the inverse of ϕ_k in (7.12) and this corresponds exactly to the equality $\theta(\text{Id}_n^{s,\text{op}}) = \chi(n, s, 1) \circ f_{s,1}^{-1}$ of Lemma 7.5.

7.4. The small category \mathcal{R} and points of the topos $\widehat{\mathcal{R}}$. To some extent, one may view the kernel of either one of the two functors in (7.1) as the category \mathcal{R} whose objects are the same as for \mathcal{RF} , and where there exists a unique morphism $f(a, b) \in \text{Hom}_{\mathcal{R}}(a, b)$ if $b|a$, while otherwise there is no morphism. Since the opposite category \mathcal{R}° has a morphism $a \rightarrow b$ exactly when $a|b$, it would seem that after a process of completion providing the points of the topos $\widehat{\mathcal{R}}$ as colimits of objects of \mathcal{R}° , these limits ought to be naturally interpreted in terms of supernatural numbers. It is natural to think of the objects of \mathcal{RF} by associating to the object $n \in \mathbb{N}^\times$ the cyclic subgroup $C_n \subset \mathbb{Q}/\mathbb{Z}$ given by the n -torsion:

$$C_n := \{r \in \mathbb{Q}/\mathbb{Z} \mid nr = 0\} = \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

LEMMA 7.7. *The category of points of the topos $\widehat{\mathcal{R}}$ is equivalent to the category of subgroups of \mathbb{Q} containing \mathbb{Z} , with morphisms given by inclusion.*

PROOF. Let $F : \mathcal{R} \rightarrow \mathfrak{Sets}$ be a flat functor. Let X_a be the image by F of the object a of \mathcal{R} . One has a unique map $F(a, b) : X_a \rightarrow X_b$ for any $a, b \in \mathbb{N}^\times$ with $b|a$. The filtering conditions for the category $\int_{\mathcal{R}} F$ read as

- (1) $X_a \neq \emptyset$ for some $a \in \mathbb{N}^\times$.
- (2) For any $x \in X_a, y \in X_b \exists c \in \mathbb{N}^\times$ and $z \in X_c$ such that $a|c, b|c$ and $F(c, a)z = x, F(c, b)z = y$.

The third filtering condition is automatically fulfilled since for objects i, j of $\int_{\mathcal{R}} F$ there is at most one morphism $i \rightarrow j$, thus $\alpha, \beta : i \rightarrow j$ implies $\alpha = \beta$. The second condition implies, by taking $a = b$, that there is at most one element in X_a for any $a \in \mathbb{N}^\times$. Let $J := \{a \in \mathbb{N}^\times \mid X_a \neq \emptyset\}$. One then has

$$(7.13) \quad a \in J, \quad b|a \implies b \in J, \quad a, b \in J \implies \exists c \in J, \quad a|c, \quad b|c.$$

The first implication follows from the existence of the morphism $F(a, b) : X_a \rightarrow X_b$ while the second one follows from (2). Thus $1 \in J$ and the flat functors $F : \mathcal{R} \rightarrow \mathfrak{Sets}$ correspond to subsets $J \subset \mathbb{N}^\times$ fulfilling (7.13). A morphism of functors $F \rightarrow F'$ exists if and only if $J \subset J'$, and in that case is unique since both the sets involved have one element. We use the following correspondence between subsets $J \subset \mathbb{N}^\times$ fulfilling (7.13) and subgroups H such that $\mathbb{Z} \subset H \subset \mathbb{Q}$

$$H^J := \bigcup_{n \in J} \frac{1}{n}\mathbb{Z} \subset \mathbb{Q}, \quad J_H := \{n \in \mathbb{N}^\times \mid \frac{1}{n} \in H\}.$$

The conditions (7.13) show that H^J is a filtering union of subgroups and is hence a subgroup of \mathbb{Q} which contains \mathbb{Z} since $1 \in J$. Conversely, for $\mathbb{Z} \subset H \subset \mathbb{Q}$, the subset $J_H \subset \mathbb{N}^\times$ fulfills the conditions (7.13). Finally, it is easy to see that the maps $J \mapsto H^J$ and $H \mapsto J_H$ are inverse of each other. \square

In [12] we proved that the non-trivial subgroups of \mathbb{Q} are parameterized by the quotient space $\mathbb{A}_{\mathbb{Q}}^f/\widehat{\mathbb{Z}}^*$ (where $\mathbb{A}_{\mathbb{Q}}^f$ are the finite adèles of \mathbb{Q}), while the subgroups of \mathbb{Q} containing \mathbb{Z} are parameterized by the subset $\widehat{\mathbb{Z}}/\widehat{\mathbb{Z}}^* \subset \mathbb{A}_{\mathbb{Q}}^f/\widehat{\mathbb{Z}}^*$. This latter subset surjects onto the quotient $\mathbb{Q}_+^\times \backslash \mathbb{A}_{\mathbb{Q}}^f/\widehat{\mathbb{Z}}^*$ which parametrizes the points of $\widehat{\mathbb{N}}^\times$. Here, one has a natural geometric morphism of toposes associated with the functor

$$\rho : \mathcal{R} \rightarrow \mathbb{N}^\times, \quad \rho(f(a, b)) := a/b$$

The next proposition provides a topos theoretic interpretation of the quotient map $\mathbb{A}_{\mathbb{Q}}^f/\widehat{\mathbb{Z}}^* \rightarrow \mathbb{Q}_+^\times \backslash \mathbb{A}_{\mathbb{Q}}^f/\widehat{\mathbb{Z}}^*$.

PROPOSITION 7.8. *The geometric morphism of toposes $\widehat{\mathcal{R}} \longrightarrow \widehat{\mathbb{N}^\times}$ associated with the functor ρ maps the category of points of $\widehat{\mathcal{R}}$ to the category of points of $\widehat{\mathbb{N}^\times}$ as the quotient map*

$$(7.14) \quad \widehat{\mathbb{Z}}/\widehat{\mathbb{Z}}^* \subset \mathbb{A}_{\mathbb{Q}}^f/\widehat{\mathbb{Z}}^* \rightarrow \mathbb{Q}_+^\times \setminus \mathbb{A}_{\mathbb{Q}}^f/\widehat{\mathbb{Z}}^*.$$

PROOF. For a point p of $\widehat{\mathcal{R}}$, let $J \subset \mathbb{N}^\times$ be the associated subset. The pullback p^* associates to a contravariant functor $Z : \mathcal{R} \longrightarrow \mathfrak{Sets}$ the set

$$(7.15) \quad \left(\prod_{a \in J} Z_a \right) / \sim = \varinjlim_{a \in J} Z_a$$

as the colimit of the filtering diagram of sets $Z_a \rightarrow Z_b$ for $a|b$ (Z is contravariant). The contravariant functor $y(\bullet) : \mathbb{N}^\times \rightarrow \mathfrak{Sets}$ which associates to the unique object \bullet of \mathbb{N}^\times the set $\mathbb{N}^\times = \text{Hom}_{\mathbb{N}^\times}(\bullet, \bullet)$ on which \mathbb{N}^\times acts by multiplication, once composed with ρ defines the following contravariant functor

$$(7.16) \quad Z = y(\bullet) \circ \rho : \mathcal{R} \longrightarrow \mathfrak{Sets}, \quad Z(a) = \mathbb{N}^\times, \quad Z(f(a, b)) = y(\bullet)(a/b).$$

Thus the flat functor $F : \mathbb{N}^\times \longrightarrow \mathfrak{Sets}$ associated with the image of the point p by the geometric morphism, is given by the action of \mathbb{N}^\times on the filtering colimit (7.15) for the functor Z of (7.16). The computation of this colimit gives

$$\varinjlim_{a \in J} \mathbb{N}^\times = \bigcup_{a \in J} \frac{1}{a} \mathbb{Z}_+ = (H_J)_+.$$

This shows that at the level of the subgroups of \mathbb{Q} the map of points associates with H ($\mathbb{Z} \subset H \subset \mathbb{Q}$) the group H viewed as an abstract ordered group. This corresponds precisely to the map (7.14). \square

References

- [1] D. Burghilea, Z. Fiedorowicz, and W. Gajda, *Power maps and epicyclic spaces*, J. Pure Appl. Algebra **96** (1994), no. 1, 1–14, DOI 10.1016/0022-4049(94)90081-7. MR1297435
- [2] Olivia Caramello and Nicholas Wentzlaff, *Cyclic theories*, Appl. Categ. Structures **25** (2017), no. 1, 105–126, DOI 10.1007/s10485-015-9414-y. MR3606497
- [3] A. Connes, *Spectral sequence and homology of currents for operator algebras*, Math. Forschungsinst. Oberwolfach Tagungsber., 41/81, Funktionalanalysis und C*-Algebren, 27-9/3-10, 1981.
- [4] Alain Connes, *Cohomologie cyclique et foncteurs Ext^n* (French, with English summary), C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), no. 23, 953–958. MR777584
- [5] Alain Connes, *Noncommutative differential geometry*, Inst. Hautes Études Sci. Publ. Math. **62** (1985), 257–360. MR823176
- [6] Alain Connes, *Noncommutative geometry*, Academic Press, Inc., San Diego, CA, 1994. MR1303779
- [7] Alain Connes, *Trace formula in noncommutative geometry and the zeros of the Riemann zeta function*, Selecta Math. (N.S.) **5** (1999), no. 1, 29–106, DOI 10.1007/s000290050042. MR1694895
- [8] Alain Connes and Caterina Consani, *Cyclic homology, Serre’s local factors and λ -operations*, J. K-Theory **14** (2014), no. 1, 1–45, DOI 10.1017/is014006012jkt270. MR3238256
- [9] Alain Connes and Caterina Consani, *Projective geometry in characteristic one and the epicyclic category*, Nagoya Math. J. **217** (2015), 95–132, DOI 10.1215/00277630-2887960. MR3343840
- [10] Alain Connes and Caterina Consani, *The cyclic and epicyclic sites*, Rend. Semin. Mat. Univ. Padova **134** (2015), 197–237, DOI 10.4171/RSMUP/134-5. MR3428418
- [11] Alain Connes and Caterina Consani, *Cyclic structures and the topos of simplicial sets*, J. Pure Appl. Algebra **219** (2015), no. 4, 1211–1235, DOI 10.1016/j.jpaa.2014.06.002. MR3282133

- [12] Alain Connes and Caterina Consani, *Geometry of the arithmetic site*, Adv. Math. **291** (2016), 274–329, DOI 10.1016/j.aim.2015.11.045. MR3459019
- [13] Alain Connes and Caterina Consani, *Absolute algebra and Segal’s Γ -rings: au dessous de $\text{Spec}(\mathbb{Z})$* , J. Number Theory **162** (2016), 518–551, DOI 10.1016/j.jnt.2015.12.002. MR3448278
- [14] Alain Connes and Caterina Consani, *Geometry of the scaling site*, Selecta Math. (N.S.) **23** (2017), no. 3, 1803–1850, DOI 10.1007/s00029-017-0313-y. MR3663595
- [15] Alain Connes and Caterina Consani, *Homological algebra in characteristic one*, High. Struct. **3** (2019), no. 1, 155–247. MR3939048
- [16] Alain Connes and Caterina Consani, *$\text{Spec} \mathbb{Z}$ and the Gromov norm*, Theory Appl. Categ. **35** (2020), Paper No. 6, 155–178. MR4066805
- [17] Alain Connes and Caterina Consani, *Segal’s gamma rings and universal arithmetic*, Q. J. Math. **72** (2021), no. 1-2, 1–29, DOI 10.1093/qmath/haaa042. MR4271378
- [18] Alain Connes and Caterina Consani, *On absolute algebraic geometry the affine case*, Adv. Math. **390** (2021), Paper No. 107909, 44, DOI 10.1016/j.aim.2021.107909. MR4291468
- [19] A. Connes, C. Consani, *Riemann Roch for $\overline{\text{Spec}} \mathbb{Z}$* , Preprint (2022). Available at [arXiv:2205.01391](https://arxiv.org/abs/2205.01391)
- [20] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy, *The local structure of algebraic K-theory*, Algebra and Applications, vol. 18, Springer-Verlag London, Ltd., London, 2013. MR3013261
- [21] B. L. Feigin and B. L. Tsygan, *Additive K-theory, K-theory, arithmetic and geometry* (Moscow, 1984), Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987, pp. 67–209, DOI 10.1007/BFb0078368. MR923136
- [22] Jonathan S. Golan, *Semirings and their applications*, Kluwer Academic Publishers, Dordrecht, 1999. Updated and expanded version of *The theory of semirings, with applications to mathematics and theoretical computer science* [Longman Sci. Tech., Harlow, 1992; MR1163371 (93b:16085)], DOI 10.1007/978-94-015-9333-5. MR1746739
- [23] T. Goodwillie, *A Topologist View of Cyclic Homology*, Math. Forschungsinst. Oberwolfach Tagungsber., 26/87, Zyklische Kohomologie und ihre Anwendungen, 14-6/20-6, 1987.
- [24] T. Goodwillie, *Notes on the cyclotomic trace* Lecture notes for a series of seminar talks at MSRI, 1990-1991.
- [25] Lars Hesselholt and Ib Madsen, *On the K-theory of finite algebras over Witt vectors of perfect fields*, Topology **36** (1997), no. 1, 29–101, DOI 10.1016/0040-9383(96)00003-1. MR1410465
- [26] Lars Hesselholt and Ib Madsen, *On the K-theory of local fields*, Ann. of Math. (2) **158** (2003), no. 1, 1–113, DOI 10.4007/annals.2003.158.1. MR1998478
- [27] D. Kaledin, *Hochschild-Witt complex*, Adv. Math. **351** (2019), 33–95, DOI 10.1016/j.aim.2019.05.007. MR3949985
- [28] Jean-Louis Loday, *Cyclic homology*, 2nd ed., Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 301, Springer-Verlag, Berlin, 1998. Appendix E by María O. Ronco; Chapter 13 by the author in collaboration with Teimuraz Pirashvili, DOI 10.1007/978-3-662-11389-9. MR1600246
- [29] Kenichi Miyazaki, *Distinguished elements in a space of distributions*, J. Sci. Hiroshima Univ. Ser. A **24** (1960), 527–533. MR131752

COLLÈGE DE FRANCE AND INSTITUT DES HAUTES ÉTUDES SCIENTIFIQUES, FRANCE
 Email address: alain@connes.org

DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218
 Email address: cconsan1@jhu.edu