On Absolute Algebraic Geometry the affine case

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To our dearest mothers, in loving memory.

ABSTRACT

We develop algebraic geometry for general Segal's Γ-rings and show that this new theory unifies two approaches we had considered earlier on (for a geometry under $\operatorname{Spec}\mathbb{Z}$). The starting observation is that the category obtained by gluing together the category of commutative rings and that of pointed commutative monoids, that we used in [3] to define $\mathbb{F}_1$-schemes, is naturally a full subcategory of the category of Segal's Γ-rings (equivalently of $\mathbb{S}$-algebras). In this paper we develop the affine case of this general algebraic geometry: one distinctive feature is that the spectrum $\mathfrak{Spec}(A)$ of an $\mathbb{S}$-algebra is in general a Grothendieck site rather than a point set endowed with a topology. Two striking features of this new geometry are that it is the natural domain for cyclic homology and for homological algebra as in [22], and that new operations, which do not make sense in ordinary algebraic geometry, are here available. For instance, in this new context, the quotient of a ring by a multiplicative subgroup is still an $\mathbb{S}$-algebra to which our general theory applies. Thus the adele class space gives rise naturally to an $\mathbb{S}$-algebra. Finally, we show that our theory is not a special case of the Töen-Vaquié general theory [24] of algebraic geometry under $\operatorname{Spec}\mathbb{Z}$.

Contents

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1. Introduction

The theory of Segal's Γ-rings provides a natural extension of the theory of rings (and semirings): in [5] we used this fact to extend the structure sheaf of the affine scheme $\operatorname{Spec}\mathbb{Z}$ to the Arakelov compactification $\overline{\operatorname{Spec}\mathbb{Z}}$, as a sheaf of Γ-rings. While the idea implemented in this construction is similar to Durov's [12], the clear advantage of working with Γ-rings is that this category forms the natural groundwork where cyclic homology is rooted. In particular, this means that de Rham theory is here naturally available. In fact, much more holds, since the simplicial version of modules over Γ-rings (*i.e.* the natural set-up for homological algebra in this context) forms the core of the local structure of algebraic K-theory [11]. Moreover, in [6] we showed that the new Γ-ring arising as the stalk of the structure sheaf of $\overline{\operatorname{Spec}\mathbb{Z}}$ at the archimedean place, is intimately related to hyperbolic geometry and the Gromov norm. In *op.cit.* we also explained that the theory of $\mathbb{S}$-modules (aka Γ-sets, $\mathbb{S}$ is the smallest Γ-ring, it corresponds to the sphere spectrum) can be thought of as the Kapranov-Smirnov theory of vector spaces over $\mathbb{F}_1$ [18], provided one works in the presheaf topos on the Segal category Γ, rather than in the topos of sets.
In the present paper we clarify the relation between our recent work [5, 6] and our previous developments on the theory of schemes over $\mathbb{F}_1$, originally based on some ideas of C. Soulé [23]. More precisely, in [3] we promoted, after the initial approach in [2], a theory of schemes (of finite type) over $\mathbb{F}_1$ using the category $\mathfrak{MR}$ obtained by gluing together the category $\mathfrak{Mo}$ of commutative pointed monoids [7, 19, 20, 24] with the category $\mathfrak{Ring}$ of commutative rings. This process implements a natural pair of adjoint functors relating $\mathfrak{Mo}$ to $\mathfrak{Ring}$ and follows an idea we learnt from P. Cartier. Using the category $\mathfrak{MR}$ we determined in [3] a natural notion of a variety (and also of a scheme) $\mathcal{X}$ over $\mathbb{F}_1$, as a covariant functor $\mathcal{X} : \mathfrak{MR} \longrightarrow \mathfrak{Sets}$ to the category of sets. Such a functor determines a scheme (of finite type) over $\mathbb{F}_1$ if it also fulfills three properties here recalled in the Appendix 9.
It is well known that the category $\mathfrak{S}$ of commutative $\mathbb{S}$-algebras (commutative Segal's Γ-rings) contains as full subcategories the category $\mathfrak{Ring}$ (through the Eilenberg-MacLane functor) and

the category $\mathfrak{Mo}$ (by the natural functor $M \mapsto \mathbb{S}M$: the spherical monoid algebra [11]). The first result of the present paper is Theorem 3.1 stating that the category $\mathfrak{MR}$ appears naturally as the full subcategory of $\mathfrak{S}$, with objects either in $\mathfrak{Ring}$ or in $\mathfrak{Mo}$. Thus, the a priori artificial process of glueing $\mathfrak{MR} = \mathfrak{Ring} \cup_{\beta,\beta^*} \mathfrak{Mo}$ is now fully justified and suggests, in view of the results of [3], to develop algebraic geometry directly in $\mathfrak{S}$. This is what we achieve in this paper, as far as the affine case is concerned, by extending to the general case of an arbitrary $\mathbb{S}$-algebra A, the construction of its spectrum as a topos $\mathfrak{Spec}(A)$, endowed with a (structure) sheaf of $\mathbb{S}$-algebras. We emphasize here that the construction of the spectrum $\mathfrak{Spec}(A)$ of an $\mathbb{S}$-algebra A yields a *Grothendieck site* endowed with a *presheaf* of $\mathbb{S}$-algebras: this datum is more refined than that of the associated topos and sheaf, as exhibited by some examples in §7.6. This construction determines the prime spectrum, when $A = HR$ for a semiring R, while it hands Deitmar's spectrum [7] when $A = \mathbb{S}M$ for a monoid M. In particular, it shows that our first development in [3] becomes a special case of this new "absolute" algebraic geometry.

The distinctive feature of this theory is to give significance to operations which are meaningless in ordinary algebraic geometry, such as taking the quotient of a ring by a subgroup of its multiplicative group, or by restricting to the unit ball in normed rings. These operations make sense in full generality for $\mathbb{S}$-algebras and one can analyze their effect on the associated spectra.

The paper is organized as follows: in Section 2 we investigate morphisms of $\mathbb{S}$-algebras with the goal of proving Theorem 3.1. The proof of this theorem is done in §3.2, after recalling in §3.1 the original construction of the category $\mathfrak{MR}$.

The first appearance of topoi is in §4.1, where we describe a preliminary construction taking place at the level of monoids and that is applied later in the paper, for an arbitrary $\mathbb{S}$-algebra A, to the specific monoid $A(1_+)$. This preliminary construction assigns to an object M of $\mathfrak{Mo}$ a small category $C(M)$ and a corresponding topos of presheaves $\mathfrak{Root}(M) := C(M)^\wedge$.

In §4.2 we compute the points of $\mathfrak{Root}(\mathbb{Z})$ and exhibit their relation with the Arithmetic Site [4]. In Section 5, we construct the spectrum $\mathfrak{Spec}(A)$ of an arbitrary $\mathbb{S}$-algebra A, as the Grothendieck site $(C^\infty(A), \mathcal{J}(A))$. The small category $C^\infty(A)$ is defined in §5.1 by localizing the category $C(A(1_+))$, using the points of the topos $\mathfrak{Root}(A(1_+))$. The idea driving this localization is to encode the expected properties of the localizations of the $\mathbb{S}$-algebra A, with respect to multiplicative subsets of $A(1_+)$, a process later developed in §7.1 for the construction of the structure sheaf. The main step in the construction of $\mathfrak{Spec}(A)$ is the selection of the Grothendieck topology $\mathcal{J}(A)$. This construction is based on the notion of partition of an object of $C^\infty(A)$, as developed in §5.2, and it depends crucially on the higher levels of the $\mathbb{S}$-algebra structure, where the n-th higher level is used to define partitions into n elements. In §5.3 we use partitions to generate multi-partitions based on rooted trees; while the Grothendieck topology is then constructed in §5.4. Finally, in §5.5 we show that the construction of $\mathfrak{Spec}(A)$ has the expected functoriality.

Section 6 is entirely devoted to get one familiar with the construction of Section 5, by treating examples of explicit computations of $\mathfrak{Spec}(A)$. The tropical case of the semiring $R = C_{\mathrm{Conv}}(I)$ of convex continuous piecewise affine functions on an open bounded interval $I \subset \mathbb{R}$ is considered in Proposition 6.4 and the points of the spectrum $\mathfrak{Spec}(HR)$ are shown to correspond to convex subsets of I. The main result is Theorem 6.5 which identifies, for a general semiring R, the spectrum $\mathfrak{Spec}(HR)$ with the Zariski site on the prime spectrum of R.

The construction of the structure sheaf on $\mathfrak{Spec}(A)$ for a general $\mathbb{S}$-algebra is developed in Section 7. We first explain in §7.1 that the localization of rings (or semirings) and of monoids is a special case of a general operation of localization for $\mathbb{S}$-algebras A with respect to a multiplicative subset of $A(1_+)$. When applied to $\mathbb{S}$-algebras this process provides a natural presheaf of $\mathbb{S}$-algebras on the category $C^\infty(A)$, and in §7.2 we construct the structure sheaf as the associated sheaf on

$\mathfrak{Spec}(A)$. We show that this construction determines a sheaf of $\mathbb{S}$-algebras. §7.3 checks that our general construction agrees with the standard one in the case of rings. The functoriality of the construction is shown in §7.4. The proof is not as straightforward as in the case of rings since the construction of the structure sheaf as in §7.2 requires passing to the associated sheaf.
Finally, in §7.5 we investigate the spectrum of the quotient of a general $\mathbb{S}$-algebra A by a subgroup of $A(1_+)$. This applies in particular to the quotient of the ring $\mathbb{A}_K$ of adeles of a global field K by the subgroup $K^\times$, *i.e.* the adele class space of [1]. In Proposition 7.11 we show that the site $\mathfrak{Spec}(A/G)$ is isomorphic to $\mathfrak{Spec}(A)$. In Remark 7.13 we explain that to obtain the structure sheaf on $\mathfrak{Spec}(A/G)$ one needs to pass to the associated sheaf of the natural presheaf. In Proposition 7.14 we prove that this sheafification is not needed for $A = HR$, when the ring R has no zero divisors. §7.6 describes the spectrum of the $\mathbb{S}$-subalgebra $|H\mathbb{Q}|_1$ of $H\mathbb{Q}$ that provides, locally at ∞, the structure of $\overline{\operatorname{Spec}\mathbb{Z}}$ [5].
In Section 8 we discuss the relations between the above developments and the theory of Töen-Vaquié [24] that is a general theory of algebraic geometry applying to any symmetric monoidal closed category that is complete and cocomplete like the category $\mathbb{S}-\operatorname{Mod}$ of $\mathbb{S}$-modules. We prove that the theory of [24], when implemented to the category $\mathbb{S}-\operatorname{Mod}$, does *not* agree with ordinary algebraic geometry already in the simplest case of the two point space corresponding to the spectrum of the product of two fields. The argument is based on Lemma 8.1 showing that the condition for a Zariski covering in the sense of [24] (Definition 2.10) is not fulfilled by the candidate cover given by the two points.
For completeness, we recall in Appendix 9 the construction of the glueing of two categories using a pair of adjoint functors. In the same appendix we also briefly recall the notion of a scheme over $\mathbb{F}_1$, as developed in [3].

2. Morphisms of $\mathbb{S}$-algebras

In this section we establish some preliminary results on morphisms of $\mathbb{S}$-modules and $\mathbb{S}$-algebras which are used in Section 3 to identify the category $\mathfrak{MR}$ with a full subcategory of the category $\mathfrak{S}$ of commutative $\mathbb{S}$-algebras.

2.1 Morphisms of $\mathbb{S}$-modules

We recall that $\mathbb{S}$-modules (equivalently Γ-sets) are by definition (covariant) functors $\Gamma^{\mathrm{op}} \longrightarrow \mathfrak{Sets}_*$ between pointed categories, where Γ^{op} is the small, full subcategory of the category $\mathfrak{Fin}_*$ of pointed finite sets, whose objects are pointed sets $k_+ := \{0,\ldots,k\}$, for each integer $k \geq 0$ (0 is the base point) and with morphism the sets $\Gamma^{\mathrm{op}}(k_+, m_+) = \{f : \{0,1,\ldots,k\} \to \{0,1,\ldots,m\} \mid f(0) = 0\}$. The morphisms in the category $\mathbb{S}-\operatorname{Mod}$ of $\mathbb{S}$-modules are natural transformations. The category $\mathbb{S}-\operatorname{Mod}$ is a closed symmetric monoidal category (see [6]).

Lemma 2.1. (i) *Let X be an $\mathbb{S}$-module. The map that associates to $x \in Hom_{\mathbb{S}}(\mathbb{S}, X)$ its value $x(1_+)(1) \in X(1_+)$ on $1 \in 1_+$ defines a bijection of sets $\epsilon : Hom_{\mathbb{S}}(\mathbb{S}, X) \stackrel{\sim}{\to} X(1_+)$.*
(ii) *Let A be an abelian monoid denoted additively, with a zero element and HA the associated $\mathbb{S}$-module. There is no non-trivial morphism of $\mathbb{S}$-modules from HA to $\mathbb{S}$.*

Proof. (i) Let $j \in k_+$ and $\phi_j : 1_+ \to k_+$ the map $\phi_j(1) = j$. By naturality of $x \in \operatorname{Hom}_{\mathbb{S}}(\mathbb{S}, X)$ one has the commutative diagram

$$\begin{tikzcd} 1_+ \arrow[r, "\phi_j"] \arrow[d, "x(1_+)"'] & k_+ \arrow[d, "x(k_+)"] \\ X(1_+) \arrow[r, "X(\phi_j)"] & X(k_+) \end{tikzcd} \tag{1}$$

It follows that $x(k_+)(j) = X(\phi_j)(x(1_+)(1))$. Thus, the natural transformation x is uniquely determined by $\epsilon(x) = x(1_+)(1)$. Let us show that $\epsilon : \mathrm{Hom}_{\mathbb{S}}(\mathbb{S}, X) \to X(1_+)$ is surjective. Let $a \in X(1_+)$. Define $x_a : \mathbb{S} \to X$ as follows: to each $j \in k_+$ assign the element $x_a(k_+)(j) = X(h_{k,j})(a) \in X(k_+)$, where $h_{k,j} : 1_+ \to k_+$ is such that $h_{k,j}(1) = j$. One needs to check that x_a is a natural transformation from the identity functor to X. Let $f : k_+ \to n_+$ be a morphism in Γ^{op}. One has $f \circ h_{k,j} = h_{n,f(j)}$ and the naturality follows since

$$x_a(n_+)(f(j)) = X(h_{n,f(j)})(a) = X(f \circ h_{k,j})(a) = X(f) \circ X(h_{k,j})(a) = X(f)(x_a(k_+)(j)).$$

(ii) The wedge sum $k_+ \vee \ell_+$ of two objects of Γ^{op} is defined using the inclusion maps

$$k_+ \to (k+\ell)_+, \quad j \mapsto j, \quad \ell_+ \to (k+\ell)_+, \quad i \mapsto k+i.$$

Let ψ be a natural transformation from HA to $\mathbb{S}$. Let $\phi \in HA(k_+) = A^k$. Then $(\phi, \phi) \in A^k \times A^k = A^{2k}$ defines an element $\phi' \in HA(2k_+) = A^{2k}$ that is invariant under the permutation of the two terms in the wedge sum $k_+ \vee k_+$. It follows that $\psi(\phi') = *$, since the base point is the only element of $k_+ \vee k_+$ invariant under the permutation of the two terms. Let then $p : k_+ \vee k_+ \to k_+$ be the morphism in Γ^{op} which is the identity on the first factor and the base point on the second factor. Then one has $HA(p)(\phi') = \phi$ and one concludes by naturality of ψ that $\psi(\phi) = *$. $\square$

2.2 $\mathbb{S}$-algebras and monoids

Let M be an object of $\mathfrak{Mo}$ *i.e.* a commutative multiplicative monoid with a unit 1 and a 0 element. We denote by $\mathbb{S}M$ the $\mathbb{S}$-algebra whose underlying $\mathbb{S}$-module is $M \wedge \mathbb{S}$ and whose multiplicative structure is given by

$$\mathbb{S}M(k_+) \wedge \mathbb{S}M(\ell_+) \to \mathbb{S}M(k_+ \wedge \ell_+), \quad (u,i) \wedge (v,j) \mapsto (uv, (i,j)). \tag{2}$$

For an $\mathbb{S}$-algebra A, the product operation restricted to $A(1_+)$ defines a multiplicative monoid with a zero (given by the base point $*$). When A is commutative, the monoid $A(1_+)$ is an object of $\mathfrak{Mo}$ and one thus obtains a functor $\mathcal{B}^* : \mathfrak{S} \longrightarrow \mathfrak{Mo}$, $\mathcal{B}^*(A) = A(1_+)$ where $\mathfrak{S}$ denotes the category of $\mathbb{S}$-algebras.
Next proposition shows that the functor $\mathcal{B}^*$ is right adjoint to the (fully faithful) functor $\mathcal{B} : \mathfrak{Mo} \longrightarrow \mathfrak{S}$, $\mathcal{B}(M) = \mathbb{S}M$.

Proposition 2.2. (i) *Let A be an $\mathbb{S}$-algebra and M be an object of $\mathfrak{Mo}$. The map which associates to a morphism of $\mathbb{S}$-algebras $\phi \in \mathrm{Hom}_{\mathfrak{S}}(\mathbb{S}M, A)$ its restriction to 1_+*

$$\mathcal{B}^*(\phi) = \phi(1_+) : \mathbb{S}M(1_+) = M \to A(1_+)$$

defines a bijection of sets $\nu : \mathrm{Hom}_{\mathfrak{S}}(\mathbb{S}M, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{Mo}}(M, A(1_+))$.
(ii) *Let R be a semiring. There is no non-trivial morphism of $\mathbb{S}$-algebras from HR to $\mathbb{S}M$.*

Proof. (i) The functoriality of $\mathcal{B}^* : \mathfrak{S} \longrightarrow \mathfrak{Mo}$ shows that $\mathcal{B}^*(\phi) = \phi(1_+)$ is a morphism in $\mathfrak{Mo}$. By Lemma 2.1, the $\mathbb{S}$-module morphism underlying ϕ is uniquely determined by $\phi(1_+)$. This shows that the map ν is injective. Let us show that it is also surjective. Let $\psi \in \mathrm{Hom}_{\mathfrak{Mo}}(M, A(1_+))$.

By applying again Lemma 2.1 one sees that there exists a unique $\mathbb{S}$-module morphism $\tilde{\psi} \in \mathrm{Hom}_{\mathbb{S}}(\mathbb{S}M, A)$ which extends ψ. With the notations of the proof of Lemma 2.1 one has, for $u \in M$ and $i \in k_+$,

$$\tilde{\psi}((u,i)) = A(h_{k,i})\psi(u) \in A(k_+).$$

Let us show that $\tilde{\psi}$ is a morphism of $\mathbb{S}$-algebras. The naturality of the product in A determines for any pair of morphisms $f : X \to X'$, $g : Y \to Y'$ in Γ^{op}, a commutative diagram

$$\begin{tikzcd} A(X) \wedge A(Y) \arrow[r, "m_A"] \arrow[d, "A(f)\wedge A(g)"'] & A(X \wedge Y) \arrow[d, "A(f\wedge g)"] \\ A(X') \wedge A(Y') \arrow[r, "m_A"'] & A(X' \wedge Y') \end{tikzcd} \tag{3}$$

Let $X = Y = 1_+$ and $X' = k_+$, $Y' = \ell_+$, while $f = h_{k,i}$, $g = h_{\ell,j}$. One then obtains from (3) that

$$m_A(A(f) \wedge A(g)(\psi(u) \wedge \psi(v)) = A(f \wedge g)\psi(uv).$$

This equality provides, using (2), the required multiplicativity.

(ii) It follows easily from Lemma 2.1 (ii). ☐

Next corollary uses the notation β^* introduced in (5) below.

Corollary 2.3. *(i) Let M be an object of $\mathfrak{Mo}$, R an object of $\mathfrak{Ring}$ and $f \in Hom_{\mathfrak{Mo}}(M, \beta^*(R))$. Then there exists a unique morphism of $\mathbb{S}$-algebras $\tilde{f} \in Hom_{\mathfrak{S}}(\mathbb{S}M, HR)$ such that*

$$\nu(\tilde{f}) = f.$$

(ii) Let A be an $\mathbb{S}$-algebra. The map of sets $\nu : Hom_{\mathfrak{S}}(\mathbb{S}[T], A) \to A(1_+)$ defines a canonical bijection $Hom_{\mathfrak{S}}(\mathbb{S}[T], A) \simeq A(1_+)$, where $\mathbb{S}[T] := \mathbb{S}M$ for $M := \{0\} \cup \{T^n \mid n \geq 0\}$ the free monoid with a single generator.

Proof. (i) follows from Proposition 2.2 (i) and the equality $HR(1_+) = R$.

(ii) With M as in (ii) one has $\mathrm{Hom}_{\mathfrak{Mo}}(M, A(1_+)) = A(1_+)$ so the result follows from Proposition 2.2 (i). ☐

3. The category $\mathfrak{MR} = \mathfrak{Ring} \cup_{\beta,\beta^*} \mathfrak{Mo}$

Next, we prove that the category $\mathfrak{MR} = \mathfrak{Ring} \cup_{\beta,\beta^*} \mathfrak{Mo}$ obtained, as recalled in §3.1, by glueing together the categories $\mathfrak{Mo}$ and $\mathfrak{Ring}$ using an appropriate pair of adjoint functors, is in fact a full subcategory of the category $\mathfrak{S}$ of $\mathbb{S}$-algebras.

3.1 The category $\mathfrak{MR}$

In [3] we developed the theory of schemes (of finite type) over $\mathbb{F}_1$ by implementing the category obtained by gluing together the category $\mathfrak{Mo}$ of commutative monoids [7, 19, 20, 24] with the category $\mathfrak{Ring}$ of commutative rings. This process uses a natural pair of adjoint functors relating $\mathfrak{Mo}$ to $\mathfrak{Ring}$ and follows an idea we learnt from P. Cartier. The resulting category $\mathfrak{MR}$ defines a framework in which the original definition of a variety over $\mathbb{F}_1$ as in [23] is applied to a covariant functor $\mathcal{X} : \mathfrak{MR} \longrightarrow \mathfrak{Sets}$ to the category of sets. Such a functor determines a scheme (of finite type) over $\mathbb{F}_1$ if it also fulfills the three properties recalled in Appendix 9.

The category $\mathfrak{MR} = \mathfrak{Ring} \cup_{\beta,\beta^*} \mathfrak{Mo}$ is obtained by applying the construction described in the appendix (Section 9) to the following pair of adjoint covariant functors β and β^*. The functor

$$\beta : \mathfrak{Mo} \longrightarrow \mathfrak{Ring}, \quad M \mapsto \beta(M) = \mathbb{Z}[M] \tag{4}$$

associates to a monoid M the convolution ring $\mathbb{Z}[M]$ (the 0 element of M is sent to 0). The adjoint functor β^*

$$\beta^* : \mathfrak{Ring} \longrightarrow \mathfrak{Mo} \quad R \mapsto \beta^*(R) = R \tag{5}$$

associates to a ring R the ring itself viewed as a multiplicative monoid (forgetful functor). The adjunction relation states that

$$\mathrm{Hom}_{\mathfrak{Ring}}(\beta(M), R) \cong \mathrm{Hom}_{\mathfrak{Mo}}(M, \beta^*(R)). \tag{6}$$

We apply Proposition 9.1 to construct the category $\mathfrak{MR} = \mathfrak{Ring} \cup_{\beta,\beta^*} \mathfrak{Mo}$ obtained by gluing $\mathfrak{Mo}$ and $\mathfrak{Ring}$. The collection[1] of objects of $\mathfrak{MR}$ is obtained as the *disjoint union* of the collection of objects of $\mathfrak{Mo}$ and $\mathfrak{Ring}$. For $R \in \mathrm{Obj}(\mathfrak{Ring})$ and $H \in \mathrm{Obj}(\mathfrak{Mo})$, one sets $\mathrm{Hom}_{\mathfrak{MR}}(R, H) = \emptyset$. On the other hand, one defines

$$\mathrm{Hom}_{\mathfrak{MR}}(H, R) := \mathrm{Hom}_{\mathfrak{Ring}}(\beta(H), R) \cong \mathrm{Hom}_{\mathfrak{Mo}}(H, \beta^*(R)). \tag{7}$$

The morphisms between objects contained in a same category are unchanged. The composition of morphisms in $\mathfrak{MR}$ is defined as follows. For $\phi \in \mathrm{Hom}_{\mathfrak{MR}}(H, R)$ and $\psi \in \mathrm{Hom}_{\mathfrak{Mo}}(H', H)$, one defines $\phi \circ \psi \in \mathrm{Hom}_{\mathfrak{MR}}(H', R)$ as the composite

$$\phi \circ \beta(\psi) \in \mathrm{Hom}_{\mathfrak{Ring}}(\beta(H'), R) = \mathrm{Hom}_{\mathfrak{MR}}(H', R). \tag{8}$$

Using the commutativity of the diagram (42), one obtains

$$\Phi(\phi \circ \beta(\psi)) = \Phi(\phi) \circ \psi \in \mathrm{Hom}_{\mathfrak{Mo}}(H', \beta^*(R)). \tag{9}$$

Similarly, for $\theta \in \mathrm{Hom}_{\mathfrak{Ring}}(R, R')$ one defines $\theta \circ \phi \in \mathrm{Hom}_{\mathfrak{MR}}(H, R')$ as the composite

$$\theta \circ \phi \in \mathrm{Hom}_{\mathfrak{Ring}}(\beta(H), R') = \mathrm{Hom}_{\mathfrak{MR}}(H, R') \tag{10}$$

and using again the commutativity of (42) one obtains that

$$\Phi(\theta \circ \phi) = \beta^*(\theta) \circ \Phi(\phi) \in \mathrm{Hom}_{\mathfrak{Mo}}(H, \beta^*(R')). \tag{11}$$

3.2 The category $\mathfrak{MR}$ as a category of $\mathbb{S}$-algebras

The first result that motivates the present paper is part (iii) of the following theorem exhibiting $\mathfrak{MR}$ as a full subcategory of the category $\mathfrak{S}$ of $\mathbb{S}$-algebras. In the statement we retain the notations of §2.2.

Theorem 3.1. (i) *The functor $\mathfrak{Mo} \longrightarrow \mathfrak{S}$, $M \mapsto \mathbb{S}M$ identifies $\mathfrak{Mo}$ as a full subcategory of $\mathfrak{S}$.*
(ii) *The functor $\mathfrak{Ring} \longrightarrow \mathfrak{S}$, $R \mapsto HR$ identifies $\mathfrak{Ring}$ as a full subcategory of $\mathfrak{S}$.*
(iii) *The above two functors extend uniquely to a fully faithful functor $\mathcal{F} : \mathfrak{MR} \longrightarrow \mathfrak{S}$ such that for any object M of $\mathfrak{Mo}$, and R of $\mathfrak{Ring}$ and any $f \in \mathrm{Hom}_{\mathfrak{Mo}}(M, \beta^*(R)) = \mathrm{Hom}_{\mathfrak{MR}}(M, R)$ one has*

$$\mathcal{F}(f) = \tilde{f} \in \mathrm{Hom}_{\mathfrak{S}}(\mathbb{S}M, HR).$$

[1] It is not a set: we refer for details to the discussion contained in the preliminaries of [10]

Proof. (i) Let M, N be two objects of $\mathfrak{Mo}$. By Proposition 2.2 the morphisms of $\mathbb{S}$-algebras from $\mathbb{S}M$ to $\mathbb{S}N$ correspond canonically to elements of $\mathrm{Hom}_{\mathfrak{Mo}}(M, N)$.
(ii) follows from [11] Section 2.1.
(iii) By Proposition 2.2 (ii) there exists no morphism of $\mathbb{S}$-algebras from HR to $\mathbb{S}M$ independently of the choice of the objects M of $\mathfrak{Mo}$, and R of $\mathfrak{Ring}$. Proposition 2.2 (i) and Corollary 2.3 (i) show that the morphisms in $\mathrm{Hom}_{\mathfrak{S}}(\mathbb{S}M, HR)$ (with the above notations) correspond canonically to elements of $\mathrm{Hom}_{\mathfrak{Mo}}(M, \beta^*(R)) = \mathrm{Hom}_{\mathfrak{MR}}(M, R)$. One easily checks that the composition of morphisms in $\mathfrak{MR}$ corresponds to the composition of morphisms in $\mathfrak{S}$. $\square$

4. The topos $\mathfrak{Root}(M)$ for a monoid M

In this section we introduce the category $C(M)$ canonically associated to a commutative pointed monoid M. The topos $\mathfrak{Root}(M)$ dual to this small category, *i.e.* the topos of contravariant functors $C(M) \longrightarrow \mathfrak{Sets}$, plays, in Section 5, an important role for the construction of the Grothendieck site spectrum $\mathfrak{Spec}(A)$ associated to an arbitrary $\mathbb{S}$-algebra A and encoding localization. The objects of $C(M)$ label the denominators in the localization process. After defining the category $C(M)$ in §4.1, we show in §4.2 that even in the simplest case of the (multiplicative) monoid $\mathbb{Z}$ of the integers the computation of the points of the topos $\mathfrak{Root}(M)$ is quite involved.

4.1 The category $C(M)$

Let M be an object of $\mathfrak{Mo}$. Next definition introduces the small category $C(M)$ canonically associated to M.

Definition 4.1. *Let M be an object of $\mathfrak{Mo}$.*
(i) We let $C(M)$ be the (small) category with one object $r(f)$ for any element $f \in M$; the morphisms are defined as follows

$$Hom_{C(M)}(r(f), r(g)) := \{u \in M \mid f = ug\}.$$

The composition of morphisms is given by the product in M.
(ii) We set $\mathfrak{Root}(M) := C(M)^\wedge$ to be the topos of contravariant functors $C(M) \longrightarrow \mathfrak{Sets}$ to the category of sets.

Since for $u \in \mathrm{Hom}_{C(M)}(r(f), r(g))$ and $v \in \mathrm{Hom}_{C(M)}(r(g), r(k))$ the composition law is meaningful, one has $f = ug$, $g = vk$ so that $f = uvk$ and $uv = vu \in \mathrm{Hom}_{C(M)}(r(f), r(k))$.
A *sieve* on an object $r(f)$ of $C(M)$ is by definition a collection of morphisms with codomain $r(f)$ and stable under precomposition. A morphism of $C(M)$ with codomain $r(f)$ is of the form $u \in \mathrm{Hom}_{C(M)}(r(fu), r(f))$ for a unique $u \in M$. We shall use the shorthand notation $\overset{u}{\to} r(f)$ for $u : r(fu) \to r(f)$ since the domain of the morphism u with codomain $r(f)$ is automatically $r(fu)$. The map $u \mapsto (\overset{u}{\to} r(f))$ from M to morphisms with codomain $r(f)$ is bijective and compatible with the product, in particular one has the following uniform description of the sieves with fixed codomain independently on this codomain.

Lemma 4.2. *The map which to an ideal $I \subset M$ of the object M of $\mathfrak{Mo}$ associates the set of morphisms of $C(M)$ with codomain $r(f)$ of the form $\overset{u}{\to} r(f)$ for some $u \in I$, defines a bijection between the sets of the ideals of M and of sieves on $r(f)$.*

When A is a (commutative) $\mathbb{S}$-algebra, there is an interesting Grothendieck topology on $C(A(1_+))$

obtained by applying Lemma 4.2 and the notion of partition of unity as in the following Definition 5.5. For the purpose of constructing the spectrum of an arbitrary $\mathbb{S}$-algebra we first need to "localize" the category $C(A(1_+))$ by inverting all morphisms of the form $f^k \in \mathrm{Hom}_{C(M)}(r(f^n), r(f^m))$ for $n = m + k$. The objects of this localized category are best described as the range of a natural map from the monoid $M = A(1_+)$ to the points of the topos $\mathfrak{Root}(M) := C(M)^\wedge$. Next, we show that already in the simplest case of the multiplicative monoid $M = \mathbb{Z}$ of the integers the computation of the points of the topos $\mathfrak{Root}(\mathbb{Z})$ involves the adeles of $\mathbb{Q}$ and the Arithmetic Site [4].

4.2 The points of $\mathfrak{Root}(\mathbb{Z})$

We consider the topos $\mathfrak{Root}(\mathbb{Z}) := C(\mathbb{Z})^\wedge$. The objects of $C(\mathbb{Z})$ are the relative integers and the morphisms are

$$\mathrm{Hom}_{C(\mathbb{Z})}(a, b) := \{n \in \mathbb{Z} \mid nb = a\}.$$

For $b \neq 0$, there exists a unique morphism $f(a, b) \in \mathrm{Hom}_{C(\mathbb{Z})}(a, b)$ if $a \in b\mathbb{Z}$, while otherwise there is no morphism.

For $b = 0$, $\mathrm{Hom}_{C(\mathbb{Z})}(a, b) = \varnothing$ unless $a = 0$. In that case, $\mathrm{Hom}_{C(\mathbb{Z})}(0, 0) = \mathbb{Z}$. We encode symbolically these morphisms in $C(\mathbb{Z})$ as $\stackrel{n}{\to} 0$.

Since the opposite category $C(\mathbb{Z})^{\mathrm{op}}$ has a morphism $a \to b$ exactly when $a|b$, one can expect that in the process of completion–that provides the points of the topos $C(\mathbb{Z})^\wedge$ as colimits of objects of $C(\mathbb{Z})^{\mathrm{op}}$– these points should be naturally interpreted in terms of supernatural numbers. Indeed the following proposition holds

Proposition 4.3. *The category $\mathcal{C}$ of points of the topos $C(\mathbb{Z})^\wedge$ contains as full subcategories the category $\mathcal{C}_1$ of subgroups of $\mathbb{Q}$ containing $\mathbb{Z}$, with morphisms given by inclusion, and the category $\mathcal{C}_2$ of groups isomorphic to subgroups of $\mathbb{Q}$, with morphisms of groups.*
The objects of $\mathcal{C}$ form the disjoint union of the objects of the two categories $\mathcal{C}_j$, $j = 1, 2$. There is no morphism from an object of $\mathcal{C}_2$ to any object of $\mathcal{C}_1$ and there is a single morphism from an object of $\mathcal{C}_1$ to any object of $\mathcal{C}_2$.

Proof. By [22] (Chapter VII, Theorem 2), the points of the topos $C(\mathbb{Z})^\wedge$ correspond to flat functors $F : C(\mathbb{Z}) \longrightarrow \mathfrak{Sets}$. Moreover by *op.cit.* (Chapter VII, Theorem 3) the flatness of F is equivalent to the following three filtering properties of the category $I = \int_{C(\mathbb{Z})} F$

1) I is non empty.
2) For any two objects i, j of I there exist an object k and morphisms $k \to i$, $k \to j$.
3) For any two morphisms $\alpha, \beta : i \to j$, there exists an object k and a morphism $\gamma : k \to i$ such that $\alpha \circ \gamma = \beta \circ \gamma$.

Let then $F : C(\mathbb{Z}) \longrightarrow \mathfrak{Sets}$ be a flat functor and $X_a = F(a)$ be the image by F of the object a of $C(\mathbb{Z})$. We let $F(a, b) := F(f(a, b)) : X_a \to X_b$ for $a \in b\mathbb{Z}$. The filtering conditions mean that:

(i) $X_a \neq \varnothing$ for some $a \in \mathbb{Z}$.
(ii) For any $x \in X_a$, $y \in X_b$ there exist $c \in \mathbb{Z}$ and $z \in X_c$ such that $a|c$, $b|c$ and $F(c, a)z = x$, $F(c, b)z = y$.

We first assume that $X_0 = \varnothing$. Then condition 3) is automatically satisfied since (using $X_0 = \varnothing$) for objects i, j of $\int_{\mathcal{D}} F$ there is at most one morphism from i to j, thus $\alpha, \beta : i \to j$ implies $\alpha = \beta$. The condition (ii) implies, by taking $a = b$, that there is at most one element in X_a for any

$a \in \mathbb{Z}, a \neq 0$. The functor F is thus uniquely determined by the set $J(F) := \{a \in \mathbb{Z}^\times \mid X_a \neq \emptyset\}$. The subsets $J \subset \mathbb{Z}$, $J \neq \emptyset$, thus corresponding to flat functors are those which fulfill

$$a \in J,\ b|a \implies b \in J, \quad a, b \in J \implies \exists c \in J, \quad a|c,\ b|c. \tag{12}$$

Indeed, the first implication follows from the existence of the morphism $F(a,b) : X_a \to X_b$, while the second implication derives from (ii). Conversely, let $J \subseteq \mathbb{Z}^\times$, $J \neq \emptyset$, fulfill the conditions (12). The first implication in (12) shows that there exists a functor $F(J)$ such that $F(J)(a)$ is empty for $a \notin J$ and $F(J)(a)$ is reduced to a single element for $a \in J$. One has $1 \in J$ so that (i) holds. Moreover (ii) follows from the second implication in (12). Thus we have shown that flat functors $F : C(\mathbb{Z}) \longrightarrow \mathfrak{Sets}$ with $F(0) = \emptyset$ correspond to subsets $J \subseteq \mathbb{Z}^\times$, $J \neq \emptyset$, fulfilling the conditions (12).

A morphism of functors $F(J) \to F(J')$ exists if and only if $J \subseteq J'$ and when it exists is unique since each of the sets X_a involved has one element. Next we use the following correspondence between subsets $J \subseteq \mathbb{Z}^\times$ fulfilling the conditions (12) and subgroups H of $\mathbb{Q}$ with $\mathbb{Z} \subseteq H \subseteq \mathbb{Q}$,

$$H^J := \bigcup_{n \in J} \frac{1}{n}\mathbb{Z} \subseteq \mathbb{Q}, \qquad J_H := \{n \in \mathbb{Z}^\times \mid \frac{1}{n} \in H\}. \tag{13}$$

The conditions (12) show that H^J is a filtering union of subgroups of $\mathbb{Q}$ and hence, since $1 \in J$, it is a subgroup of $\mathbb{Q}$ that contains $\mathbb{Z}$. Conversely, for a group H with $\mathbb{Z} \subseteq H \subseteq \mathbb{Q}$, the subset $J_H \subseteq \mathbb{Z}^\times$ fulfills the conditions (12). It is straightforward to check that the maps $J \mapsto H^J$ and $H \mapsto J_H$ are inverse of each other. This proves that the full subcategory of the category of points of $C(\mathbb{Z})^\wedge$ whose associated flat functors fulfill $F(0) = \emptyset$ is the category $\mathcal{C}_1$.

Let us now assume that $X_0 = F(0) \neq \emptyset$. First we show that each X_a, for $a \neq 0$, contains exactly one element. It is non-empty since there exists a morphism $F(0,a) : X_0 \to X_a$. Let $x, y \in X_a$, then using (ii) and the existence of a unique morphism in $C(\mathbb{Z})$ with codomain a one derives $x = y$. We can then reduce the determination of flat functors $F : C(\mathbb{Z}) \longrightarrow \mathfrak{Sets}$ such that $X_0 = F(0) \neq \emptyset$ by restricting F to the full subcategory of $C(\mathbb{Z})$ with the single object 0. Conversely one uniquely extends a flat functor defined on this full subcategory to a functor $F : C(\mathbb{Z}) \longrightarrow \mathfrak{Sets}$ by requiring that $F(a) = X_a$ is a single element for any $a \neq 0$. As in Theorem 2.4 in [4] one easily derives that the category $\mathcal{C}_2$ of points of this full subcategory of $C(\mathbb{Z})$ is isomorphic to the category of subgroups of $\mathbb{Q}$ with morphisms of groups. There is no morphism from an object of $\mathcal{C}_2$ to any object of $\mathcal{C}_1$ since the corresponding natural transformation of (flat) functors cannot be defined on the object 0 of $C(\mathbb{Z})$. Finally, we show that there is a single morphism from an object of $\mathcal{C}_1$ to any object of $\mathcal{C}_2$. Let F_j, $j = 1, 2$, be flat functors corresponding to objects of $\mathcal{C}_j$. Then, whenever $F_1(a) \neq \emptyset$ one has $a \neq 0$ and hence the set $F_2(a)$ has a single element and is thus the final object of $\mathfrak{Sets}$, hence the existence and uniqueness of the natural transformation $F_1 \to F_2$. $\square$

It follows from the results of [4] that the non-trivial subgroups of $\mathbb{Q}$ are parametrized by the quotient space $\mathbb{A}^{\mathrm{f}}_{\mathbb{Q}}/\hat{\mathbb{Z}}^\times$ of the finite adeles of $\mathbb{Q}$, while the subgroups of $\mathbb{Q}$ containing $\mathbb{Z}$ are parametrized by the subset $\hat{\mathbb{Z}}/\hat{\mathbb{Z}}^\times \subset \mathbb{A}^{\mathrm{f}}_{\mathbb{Q}}/\hat{\mathbb{Z}}^\times$. This subset surjects onto the quotient $\mathbb{Q}^\times_+\backslash\mathbb{A}^{\mathrm{f}}_{\mathbb{Q}}/\hat{\mathbb{Z}}^\times$ which parametrizes the points of $\widehat{\mathbb{N}^\times}$.

We denote by $\mathcal{D}$ the full subcategory of $C(\mathbb{Z})$ whose objects are the positive integers $a > 0$. By construction one has

$$\mathrm{Hom}_{\mathcal{D}}(a,b) = \begin{cases} \{f(a,b)\} & \text{if } b|a \\ \emptyset & \text{otherwise.} \end{cases}$$

The proof of Proposition 4.3 then shows that the points of the topos $\widehat{\mathcal{D}}$, dual to $\mathcal{D}$, form the category $\mathcal{C}_1$. One has a natural geometric morphism from $\widehat{\mathcal{D}}$ to the topos $\widehat{\mathbb{N}^\times}$ (where $\mathbb{N}^\times$ is considered here as a category with a single object $\bullet$) associated to the functor

$$\rho : \mathcal{D} \longrightarrow \mathbb{N}^\times, \;\; \rho(a) = \bullet, \;\; \rho(f(a,b)) := a/b.$$

It is well known (see Theorem VII, 2.2 in [22]) that a functor $\phi : \mathcal{C} \longrightarrow \mathcal{C}'$ between small categories induces a geometric morphism $\hat{\phi} : \hat{\mathcal{C}} \to \hat{\mathcal{C}}'$, whose inverse image $\hat{\phi}^*$ takes a presheaf $\mathcal{F}$ on $\mathcal{C}'$ to the composite $\mathcal{F} \circ \phi^{\mathrm{op}}$. To determine the action of the geometric morphism $\hat{\phi}$ on a point $\mathfrak{p}$ of the topos $\hat{\mathcal{C}}$, we consider the inverse image functor

$$(\hat{\phi} \circ \mathfrak{p})^* = \mathfrak{p}^* \circ \hat{\phi}^*.$$

One then identifies the flat functor $F' : \mathcal{C}' \longrightarrow \mathfrak{Sets}$ associated to the image $\mathfrak{p}'$ of $\mathfrak{p}$ using the Yoneda embedding $y : \mathcal{C}' \longrightarrow \mathrm{Sheaves}(\hat{\mathcal{C}}')$, and derives

$$F' = (\hat{\phi} \circ \mathfrak{p})^* \circ y = \mathfrak{p}^* \circ \hat{\phi}^* \circ y = \mathfrak{p}^* \circ (y \circ \phi^{\mathrm{op}}).$$

The map ρ provides a topos theoretic meaning of the quotient map from rational finite adeles to adele classes. Precisely, one has the following (*cf.* Theorem 2 of [21]).

Proposition 4.4. *The geometric morphism $\hat{\rho} : \widehat{\mathcal{D}} \longrightarrow \widehat{\mathbb{N}^\times}$ associated to the functor ρ maps the category of points of the topos $\widehat{\mathcal{D}}$ to the category of points of $\widehat{\mathbb{N}^\times}$ as the restriction to $\hat{\mathbb{Z}}/\hat{\mathbb{Z}}^\times \subset \mathbb{A}^{\mathrm{f}}_{\mathbb{Q}}/\hat{\mathbb{Z}}^\times$ of the quotient map*

$$\mathbb{A}^{\mathrm{f}}_{\mathbb{Q}}/\hat{\mathbb{Z}}^\times \to \mathbb{Q}^\times_+ \backslash \mathbb{A}^{\mathrm{f}}_{\mathbb{Q}}/\hat{\mathbb{Z}}^\times. \tag{14}$$

Proof. Given a point $\mathfrak{p} = F : \mathcal{D} \longrightarrow \mathfrak{Sets}$ of the topos $\widehat{\mathcal{D}}$, let $J = J(F) \subseteq \mathbb{N}^\times$ be the associated subset. Then the pullback $\mathfrak{p}^*$ of the geometric morphism $\mathfrak{p}$ associates to any contravariant functor $Z : \mathcal{D} \longrightarrow \mathfrak{Sets}$ the set

$$\left(\coprod_{a \in J} Z_a \right) / \sim \;\; = \varinjlim_{a \in J} Z_a \tag{15}$$

which is the colimit of the filtering diagram of sets $Z_a \to Z_b$ for $a|b$ (remember that Z is contravariant). The Yoneda embedding is given by the contravariant functor $y(\bullet) : \mathbb{N}^\times \longrightarrow \mathfrak{Sets}$ that associates to the singleton $\bullet$ the set $\mathbb{N}^\times = \mathrm{Hom}_{\mathbb{N}^\times}(\bullet, \bullet)$ on which $\mathbb{N}^\times$ acts by multiplication. Once composed with ρ it determines the following contravariant functor

$$Z = y(\bullet) \circ \rho : \mathcal{D} \longrightarrow \mathfrak{Sets}, \quad Z(a) = \mathbb{N}^\times, \quad Z(f(a,b)) = y(\bullet)(a/b). \tag{16}$$

Thus, the flat functor $F : \mathbb{N}^\times \longrightarrow \mathfrak{Sets}$ associated to the image of the point $\mathfrak{p}$ by the geometric morphism $\hat{\rho}$ is given by the action of $\mathbb{N}^\times$ on the filtering colimit (15) applied to Z of (16). This colimit gives

$$\varinjlim_{a \in J} \mathbb{N}^\times = \bigcup_{a \in J} \frac{1}{a} \mathbb{Z}_+ = (H_J)_+.$$

This argument shows that at the level of the subgroups of $\mathbb{Q}$ the map of points associates to H, with $\mathbb{Z} \subseteq H \subseteq \mathbb{Q}$, the group H viewed as an abstract ordered group. This corresponds to the map (14). ☐

5. The spectrum $\mathfrak{Spec}(A)$ of an $\mathbb{S}$-algebra

This section provides the basis to develop algebraic geometry over $\mathbb{S}$. The main construction, in the affine case, is to associate to a (commutative) $\mathbb{S}$-algebra A its spectrum $\mathfrak{Spec}(A)$ understood as the pair of a Grothendieck site and a structure presheaf of $\mathbb{S}$-algebras, whose associated sheaf

is a sheaf of $\mathbb{S}$-algebras. When $A = HR$, for an ordinary commutative ring R, $\mathfrak{Spec}(A)$ reproduces the classical affine spectrum $\operatorname{Spec} R$. In this section we define the Grothendieck site as the pair $(C^\infty(A), \mathcal{J}(A))$ of a small category $C^\infty(A)$ and a Grothendieck topology $\mathcal{J}$ on $C^\infty(A)$. The definitions of the structure presheaf and the associated sheaf are postponed to Section 7. The natural structure on $A(1_+)$ of a multiplicative pointed monoid (an object of $\mathfrak{Mo}$) provides, by the theory developed in §4.1, a small category $C(A(1_+))$ that only depends on $A(1_+)$ as an object of $\mathfrak{Mo}$. The role of the category $C^\infty(A)$, obtained from $C(A(1_+))$ by applying a localization process (turning the natural morphisms between powers of the same element into isomorphisms), is to label the morphisms of localization which generalize, in the context of $\mathbb{S}$-algebras, the classical localization morphisms $R \to R_f$ at multiplicative subsets generated by elements $f \in R$. This construction is described in §5.1 where we also show that the category $C^\infty(A)$ has all finite limits.
The definition of the Grothendieck topology $\mathcal{J}$ on $C^\infty(A)$ involves all the higher levels in the structure of the $\mathbb{S}$-algebra A which serve as a substitute for the missing additive structure on the multiplicative monoid $A(1_+)$. Here, the key notion that we implement is that of a partition of an object of $C^\infty(A)$ into n pieces : it is this process that involves the n-th level of the $\mathbb{S}$-algebra A. This construction is developed in §5.2 and refined in §5.3. In §5.4 we prove that one obtains in this way a Grothendieck topology $\mathcal{J}$ on $C^\infty(A)$. Proposition 5.10 provides a useful tool, used later in the paper, to compare our geometric construction with the standard one in the context of semirings. Finally, in §5.5 we prove Theorem 5.11, stating that the Grothendieck site $(C^\infty(A), \mathcal{J}(A))$ depends functorially (and contravariantly) on the $\mathbb{S}$-algebra A.

5.1 The category $C^\infty(A)$

Given an $\mathbb{S}$-algebra A, the product

$$m_A : A(X) \wedge A(Y) \to A(X \wedge Y)$$

together with the canonical isomorphism $1_+ \wedge Y \simeq Y$ provides a meaning to the expression $uy \in A(Y)$, for $u \in A(1_+)$ and $y \in A(Y)$. We shall use freely this notation below. It follows from the commutativity of the diagram (3) that

$$A(f)(ux) = uA(f)(x)\,, \quad \forall f \in \operatorname{Hom}_{\Gamma^{op}}(X,Y),\ x \in A(X), u \in A(1_+). \tag{17}$$

Next we construct, starting with the category $C(A(1_+))$, a small category $C^\infty(A)$ which is essentially a localization of $C(A(1_+))$, turning the natural morphisms between powers of the same element of $A(1_+)$ into isomorphisms. The first step in the construction consists of associating to an element $b \in A(1_+)$ a point b^∞ of the presheaf topos $C(A(1_+))^\wedge$. The point b^∞ is defined as the colimit of the sequence of morphisms in the opposite category $C(A(1_+))^{op}$

$$b \to b^2 \to b^3 \to \cdots$$

The understanding here is that, in general, the points of the topos which is the dual of a small category $\mathcal{C}$ are obtained as filtering colimits of objects of the opposite category $\mathcal{C}^{op}$. The flat functor $F_b : C(A(1_+)) \longrightarrow \mathfrak{Sets}$ associated to b^∞ is defined as

$$F_b(a) := \varinjlim_n \operatorname{Hom}_{C(A(1_+))}(b^n, a). \tag{18}$$

One may think of $F_b(a)$ as the set of multiplicative inverses of a in the localized $\mathbb{S}$-algebra of A with respect to the multiplicative monoid $M(b) := \{b^n \mid n \in \mathbb{N}, n > 0\}$. The proof of the first statement of the next proposition uses implicitly the uniqueness of the multiplicative inverse, when it exists.

Proposition 5.1. *Let A be an $\mathbb{S}$-algebra.*
(i) For any pair $a,b\in A(1_+)$ the set $F_b(a)$ contains at most one element.
(ii) One has

$$F_b(a)\neq\varnothing \iff \exists n\in\mathbb{N},\ b^n\in aA(1_+). \tag{19}$$

(iii) For any pair $b,c\in A(1_+)$ the existence of a morphism of functors $F_b\to F_c$ is equivalent to

$$\exists n\in\mathbb{N},\ c^n\in bA(1_+). \tag{20}$$

Moreover if (20) is fulfilled there exists a single morphism of functors $F_b\to F_c$.

Proof. (i) By construction one has

$$\mathrm{Hom}_{C(A(1_+))}(b^n,a)=\{c\in A(1_+)\mid ac=b^n\}$$

and the maps involved in the (filtering) colimit are, for $n,k>0$,

$$\beta_{n+k,n}:\mathrm{Hom}_{C(A(1_+))}(b^n,a)\to\mathrm{Hom}_{C(A(1_+))}(b^{n+k},a),\quad \beta_{n+k,n}(c):=b^kc.$$

Let $c,c'\in\mathrm{Hom}_{C(A(1_+))}(b^n,a)$: we show then that $\beta_{2n,n}(c)=\beta_{2n,n}(c')$. This equality follows from

$$b^nc'=acc'=ac'c=b^nc.$$

Thus the filtering colimit (18) contains at most one element.
(ii) If $b^n\in aA(1_+)$ then $ac=b^n$ for some $c\in A(1_+)$ and the sequence $x_{n+k}=\beta_{n+k,n}(c)=b^kc$ defines an element of $F_b(a)$. Conversely, if $\mathrm{Hom}_{C(A(1_+))}(b^n,a)=\varnothing$ for all n, then $F_b(a)=\varnothing$.
(iii) Let $\psi:F_b\to F_c$ be a morphism of functors. Then, since $F_b(b)\neq\varnothing$ (by (ii)), one has $F_c(b)\neq\varnothing$ and by (ii) one derives (20). Conversely, assume that (20) holds, *i.e.* $c^n=bu$ for some $n>0$ and $u\in A(1_+)$. We prove that

$$F_b(a)\neq\varnothing\Rightarrow F_c(a)\neq\varnothing.$$

Using (ii) one gets $b^k=av$ for some $k>0$ and $v\in A(1_+)$. Thus $c^{nk}=b^ku^k=avu^k\in aA(1_+)$ and by (ii) one has $F_c(a)\neq\varnothing$. Finally by applying (i) one obtains the existence and uniqueness of a morphism of functors $\psi:F_b\to F_c$. ∎

Definition 5.2. *Let A be an $\mathbb{S}$-algebra. We denote $C^\infty(A)$ the opposite of the full subcategory of the category of points of $C(A(1_+))^\wedge$ of the form b^∞, for $b\in A(1_+)$.*

Next proposition describes $C^\infty(A)$, when $A=HR$ for a commutative ring R.

Proposition 5.3. *Let R be a commutative unital ring, and* $\operatorname{Spec}R$ *its prime spectrum.*
(i) For any two elements $b,c\in R=HR(1_+)$ one has

$$\exists F_b\to F_c\iff V(b)\subseteq V(c)$$

with $V(b):=\{\mathfrak{p}\in\operatorname{Spec}R\mid b\in\mathfrak{p}\}$.
(ii) The map $D(b)\mapsto F_b$ defines an isomorphism of the category of open sets of $\operatorname{Spec}R$ *of the form $D(b)$, $b\in R$, and morphisms given by inclusions, with the category $C^\infty(HR)$.*

Proof. (i) It is well known (see [16], Lemma II, 2.1. (c)) that

$$\sqrt{bR}\supseteq\sqrt{cR}\iff V(b)\subseteq V(c).$$

One has $\sqrt{bR}\supseteq\sqrt{cR}\iff\exists n\in\mathbb{N},\ c^n\in bR$. Hence one derives by applying Proposition 5.1 (iii), that

$$V(b)\subseteq V(c)\iff\exists F_b\to F_c.$$

(ii) follows from (i) and the uniqueness statement of Proposition 5.1 (iii), together with the equality $D(b) := V(b)^c$, which defines the open sets $D(b) \subseteq \operatorname{Spec} R$. □

Note in particular that if $c = 0$ one always has a morphism of functors $F_b \to F_c$, for any $b \in A(1_+)$, since $F_c(a) \neq \varnothing$ for all $a \in A(1_+)$. This corresponds, in algebraic geometry, to the fact that the empty open set is contained in any other open set ($V(b) \subseteq V(0), \forall b \Rightarrow \varnothing = D(0) \subseteq D(b), \forall b$).

Lemma 5.4. *Let A be an $\mathbb{S}$-algebra. The category $C^\infty(A)$ has all finite limits.*

Proof. First we show that 1^∞ is a terminal object in $C^\infty(A)$. By Definition 5.2, this amounts to show that for any $b \in A(1_+)$ there exists a single morphism $F_1 \to F_b$. This fact follows from Proposition 5.1 (iii). In fact the same proposition also shows that there exists a morphism $b^\infty \to c^\infty$ in $C^\infty(A)$ if and only if there exists a morphism from some power of b to some power of c in the category $C(A(1_+))$. Moreover, if a morphism $b^\infty \to c^\infty$ exists it is unique. In particular note that $(b^k)^\infty = b^\infty$ for any $k > 0$. Also, note that for $f = a^\infty$, $g = b^\infty$ two objects of $C^\infty(A)$, the object $(ab)^\infty$ only depends on f an g. In fact, as we show now, $(ab)^\infty$ corresponds to the pullback of the morphisms to the terminal object. More generally, for $\phi \in \operatorname{Hom}_{C^\infty(A)}(a^\infty, c^\infty)$ and $\psi \in \operatorname{Hom}_{C^\infty(A)}(b^\infty, c^\infty)$, we prove that their pullback is given by $(ab)^\infty$ and the unique morphisms $\phi' : (ab)^\infty \to b^\infty$, $\psi' : (ab)^\infty \to a^\infty$. By Proposition 5.1 (iii) these morphisms exist and are unique. Let then x^∞ be an object of $C^\infty(A)$ and morphisms $\alpha : x^\infty \to a^\infty$, $\beta : x^\infty \to b^\infty$ such that $\phi \circ \alpha = \psi \circ \beta$. By Proposition 5.1 (iii) there exists a power x^n divisible by a, and x^m divisible by b. Thus x^{n+m} is divisible by ab and one obtains a (unique) morphism $x^\infty \to (ab)^\infty$. The uniqueness of morphisms when they exist thus shows that $(ab)^\infty$ is the pullback. □

5.2 Partitions in $C^\infty(A)$

In classical algebra the operation of "sum" of two elements can be iterated by applying the associative rule to obtain the sum of n elements. Then one defines a partition of unity as n elements whose sum is equal to 1. In the context of $\mathbb{S}$-algebras the sum of n elements cannot be well defined using only the first two levels of an $\mathbb{S}$-algebra A: at the best, the first two levels give a meaning to the statement $1 \in a + b$ for $a, b \in A(1_+)$. In general we know that, for $a, b \in A(1_+)$, $a + b$, when it is defined, is a hypersum. Thus one uses the level n (*i.e.* $A(n_+)$) to give a meaning to the statement $1 \in \sum_{j=1}^n a_j$ and this defines, in an $\mathbb{S}$-algebra A, a partition of unity in n pieces. Lemma 5.6 asserts then the crucial statement that one can take the product of two partitions of unity. If the first is a partition in n pieces and the second in m pieces, then the product is a partition in nm pieces. This shows that if one restricted to the second level this construction would not be stable by products.

Let A be an $\mathbb{S}$-algebra. We use the $\mathbb{S}$-module structure of A to define partitions of objects of $C^\infty(A)$. First we introduce some notations. Given a finite set F, we define, for $j \in F$, specific elements δ_j and an element Σ_F of $\operatorname{Hom}_{\Gamma^{\mathrm{op}}}(F_+, 1_+)$ as follows

$$\delta_j(i) := 1 \iff j = i, \quad \Sigma_F(i) := 1\,, \quad \forall i \in F.$$

Definition 5.5. *Let A be an $\mathbb{S}$-algebra and f an object of $C^\infty(A)$. A* partition *of f is a finite collection of morphisms $f_j \to f$, $j = 1, \ldots, n$, in $C^\infty(A)$ such that there exists $\xi \in A(n_+)$ with $(A(\delta_j)(\xi))^\infty = f_j$, $\forall j$, and $(A(\Sigma)\xi)^\infty = f$.*

The following multiplicativity property holds

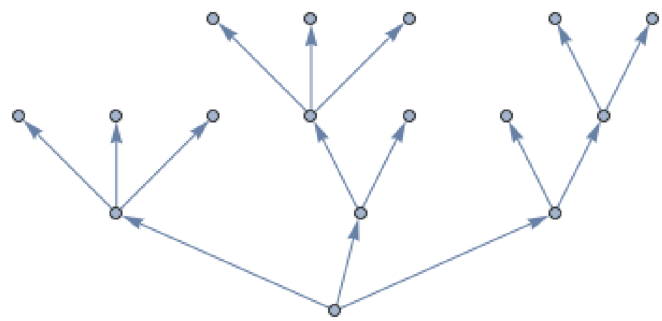

Figure 1. Rooted tree.

Lemma 5.6. *Let A be an $\mathbb{S}$-algebra. Let $(f_i)_{i\in F}$ and $(g_j)_{j\in G}$ be partitions of objects u, v of $C^\infty(A)$. Then the family $(f_i g_j)_{(i,j)\in F\times G}$ is a partition of uv.*

Proof. Let $\xi \in A(F_+)$ (resp. $\eta \in A(G_+)$) such that $A(\delta_i)(\xi) = a_i$, $a_i^\infty = f_i$, $\forall i \in F$, and $A(\Sigma)\xi = s$, $s^\infty = u$ (resp. $A(\delta_j)(\eta) = b_j$, $b_j^\infty = g_j$, $\forall j \in G$, and $A(\Sigma)\eta = t$, $t^\infty = v$). Consider the diagram (3) in the case $X = F_+$, $Y = G_+$, $X' = 1_+$, $Y' = 1_+$:

$$\begin{tikzcd} A(F_+) \wedge A(G_+) \arrow[r, "m_A"] \arrow[d, "A(\alpha)\wedge A(\beta)"'] & A(F_+ \wedge G_+) \arrow[d, "A(\alpha\wedge\beta)"] \\ A(1_+) \wedge A(1_+) \arrow[r, "m_A"'] & A(1_+ \wedge 1_+ = 1_+) \end{tikzcd} \tag{21}$$

Taking $\alpha = \delta_i$, $\beta = \delta_j$, one obtains $\alpha \wedge \beta = \delta_{(i,j)}$ and this gives the equality, with $\zeta := m_A(\xi \wedge \eta)$

$$A(\delta_{(i,j)})(\zeta) = a_i b_j.$$

Taking $\alpha = \Sigma_F$, $\beta = \Sigma_G$ one obtains $\alpha \wedge \beta = \Sigma_{F\times G}$ and this gives the equality $A(\Sigma_{F\times G})\zeta = st$. Finally, one derives from Lemma 5.4 that $(a_i b_j)^\infty = f_i g_j$ and $(st)^\infty = uv$. ☐

5.3 Rooted trees and multi-partitions

Next we construct, for an arbitrary $\mathbb{S}$-algebra A, a Grothendieck topology $\mathcal{J}(A)$ on $C^\infty(A)$. For a commutative unital ring R, we shall show that the corresponding Grothendieck topology on $A = HR$ coincides with the Zariski topology of $\operatorname{Spec} R$.
Iterating the process of partitioning an object of $C^\infty(A)$ is best labelled by a combinatorial datum called a rooted tree. We start by recalling the terminology pertaining to the notion of rooted tree. A rooted tree (T, r) is a connected graph T which is a tree and is pointed by the choice of a vertex r called the root. We shall consider a rooted tree (T, r) as a small category with objects the vertices of the tree and morphisms generated by the identity morphisms and the oriented edges of the graph, for the orientation of edges as shown in Figure 1. The *height* of a rooted tree is the maximal length of paths. In the following only trees of finite height are considered. Each internal vertex v has successors which form a subset $S(v)$ of the set of vertices. The vertices with no successor are called external vertices and their collection is denoted $\mathcal{E}(T)$.

Definition 5.7. *Let A be an $\mathbb{S}$-algebra, and (T, r) a rooted tree. A contravariant functor $F : (T, r) \longrightarrow C^\infty(A)$ is called a* multi-partition *of $F(r)$ if for each internal vertex v the collection*

$(F(u))_{u\in S(v)}$ *forms a partition of* $F(v)$ *in the sense of Definition 5.5.*

5.4 The Grothendieck topology on $C^\infty(A)$

Given a small category $\mathfrak{C}$, an object c of $\mathfrak{C}$ and a set X of morphisms with codomain c, we denote by $\mathrm{Sieve}(X)$ the sieve generated by X (see [22]). We introduce the notion of covering in the category $C^\infty(A)$ given by the function K which associates to each object c the collection $K(c)$ of families X of morphisms with codomain c for which there exists a multi-partition $F : (T,r) \longrightarrow C^\infty(A)$ with $F(r) = c$, such that for all external vertices e of T

$$F(r \to e) \in \mathrm{Sieve}(X). \tag{22}$$

One has

$$\mathrm{Sieve}(X) = \cup_{x\in X}\mathrm{Sieve}(x), \qquad \mathrm{Sieve}(x) = \{y^\infty x \mid y \in A(1_+)\}.$$

Lemma 5.8. *Let A be an $\mathbb{S}$-algebra. The function $c \mapsto K(c)$ defines a basis for a Grothendieck topology $\mathcal{J}(A)$ on $C^\infty(A)$.*

Proof. We first check that the function K that associates to an object c of $C^\infty(A)$ the collection $K(c)$ of families of morphisms of (22) fulfills the three conditions of Definition III. 2 of [22]. The first condition states that the family with a single element $c \to c$ belongs to $K(c)$. This follows taking the trivial partition of c. The second condition is the stability by pullback, which follows if we show that for $X \in K(c)$ and $d \to c$ a morphism, the family $dX := (xd)_{x\in X}$ belongs to $K(d)$. Indeed, one has $cd = d$. Let $F : (T,r) \longrightarrow C^\infty(A)$ be a multi-partition with $F(r) = c$ such that (22) holds. Lemma 5.6 shows that the functor $dF : (T,r) \longrightarrow C^\infty(A)$ defined as the pointwise product of F with d fulfills (22), with respect to dX which thus belongs to $K(d)$. Finally, the third condition states that if $X = \{f_i : c_i \to c \mid i \in I\}$ belongs to $K(c)$ and for each $i \in I$ one has a family $Y_i = \{g_{ij} : d_{ij} \to c_i \mid j \in I_i\}$, $Y_i \in K(c_i)$, then the family $Y := \{f_i \circ g_{ij} : d_{ij} \to c \mid i \in I, j \in I_i\}$ is in $K(c)$. By hypothesis there exists a multi-partition $F : (T,r) \longrightarrow C^\infty(A)$ with $F(r) = c$ such that (22) holds, and for each $i \in I$ a multi-partition $F_i : (T_i, r_i) \longrightarrow C^\infty(A)$ with $F_i(r_i) = c_i$ such that for all external vertices ϵ of T_i

$$F_i(r_i \to \epsilon) \in \mathrm{Sieve}(Y_i). \tag{23}$$

Let e be an external vertex of T. One has, by (22), $F(r \to e) \in \mathrm{Sieve}(X)$, so let $\psi(e) \in I$ be such that $F(r \to e) \in \mathrm{Sieve}(c_{\psi(e)})$. We thus obtain a map $\psi : \mathcal{E}(T) \to I$ from the set $\mathcal{E}(T)$ of external vertices of T to I and a map $\alpha : \mathcal{E}(T) \to A(1_+)$ such that

$$F(r \to e) = \alpha(e)^\infty c_{\psi(e)}.$$

We let then (T', r) be the rooted tree obtained by grafting the rooted tree $(T_{\psi(e)}, r_{\psi(e)})$ at each external vertex e of T. Then we define a multi-partition $F' : (T', r) \longrightarrow C^\infty(A)$ as follows. One has $F_i(r_i) = c_i$ for all $i \in I$ and hence for $i = \psi(e)$ one gets

$$c_{\psi(e)} = F_{\psi(e)}(r_{\psi(e)}), \quad F(r \to e) = \alpha(e)^\infty F_{\psi(e)}(r_{\psi(e)}),$$

which shows that the functors $\alpha(e)^\infty F_{\psi(e)}$ match with F and thus define a single functor $F' : (T', r) \longrightarrow C^\infty(A)$. It is a multi-partition since the condition of Definition 5.7 is checked locally for each internal vertex. Each external vertex e' of T' is an external vertex of a $(T_{\psi(e)}, r_{\psi(e)})$. By (23) one has $F'(r \to e') \in \mathrm{Sieve}(Y)$ thus one gets $Y \in K(c)$ as required. □

Definition 5.9. *Let A be an $\mathbb{S}$-algebra. We define the spectrum $\mathfrak{Spec}(A)$ as the Grothendieck site $(C^\infty(A), \mathcal{J}(A))$.*

We remark here that the information contained in the site is more precise than that of the associated topos: in §7.6 we shall give an example where this nuance plays a role.

The following Proposition describes the Grothendieck topology $\mathcal{J}(A)$ for $A = HR$, where R is a semiring.

Proposition 5.10. *Let R be a semiring and $A = HR$ the associated $\mathbb{S}$-algebra.*
(i) Let f be an object of $C^\infty(A)$, $f_j \to f$, $j = 1, \ldots, n$ a partition of f in the sense of Definition 5.5. Let $x_j, x \in R$ such that $x_j^\infty = f_j$ for $j = 1, \ldots, n$ and $x^\infty = f$. Then there exist elements $a_j \in R$, $j = 1, \ldots, n$ such that $\sum_{j=1}^n a_j x_j = x^k$, for some integer $k \geq 1$.
(ii) Let $F : (T, r) \longrightarrow C^\infty(A)$ be a multi-partition of $F(r)$, let $x, x_e \in R$ such that $x^\infty = F(r)$ and $x_e^\infty = F(e)$, for all $e \in \mathcal{E}(T)$. Then there exists a finite collection of elements $a_e \in R$ such that $\sum a_e x_e = x^k$ for some integer $k \geq 1$.
(iii) A family X of morphisms $x^\infty \to c^\infty$ with codomain c^∞ is a covering for the Grothendieck topology $\mathcal{J}(A)$ if and only if there exists a map $a : X \to R$ with finite support such that $\sum a(x)x$ is equal to an integer power of c.

Proof. (i) Let $\xi \in HR(n_+)$ with $(HR(\delta_j)(\xi))^\infty = f_j$, $\forall j$, and $(HR(\Sigma)\xi)^\infty = f$. By construction $HR(n_+) = R^n$ and thus one gets elements $\xi_j \in R$, $j = 1, \ldots, n$, such that $\xi_j^\infty = f_j = x_j^\infty$ for all j and $(\sum_1^n \xi_j)^\infty = x^\infty$. Thus, each x_j divides a power of ξ_j and we let m and $b_j \in R$ such that $\xi_j^m = b_j x_j$ for all j. Let $\xi = \sum_1^n \xi_j$. Then, there exist elements $c_j \in R$ such that

$$\xi^{mn} = (\sum_1^n \xi_j)^{mn} = \sum_1^n c_j \xi_j^m.$$

One has $\sum_1^n c_j \xi_j^m = \sum_1^n c_j b_j x_j$ and since $\xi^\infty = x^\infty$, ξ^{mn} divides a power of x, *i.e.* $x^k = a\xi^{mn}$ for some $a \in R$. Thus one gets $x^k = \sum_1^n a c_j b_j x_j = \sum a_j x_j$ as required.
(ii) Let us choose for each vertex w of T an element $x(w) \in R$ such that $F(w) = x(w)^\infty$ and that $x(e) = x_e$ for external vertices, while $x(v) = x$. Consider a vertex w such that all its successors (*i.e.* vertices in $S(w)$) are external. Then by hypothesis the $(F(u))_{u \in S(w)}$ forms a partition of $F(w)$ and hence by (i) one can find a power $x(w)^k$ of the form $\sum_{S(w)} a(u)x(u)$. One has $F(w) = (x(w)^k)^\infty$. Thus at the expense of raising the $x(w)$ to some power one can assume that for each vertex t of T one has elements $a(t, u) \in R$, $u \in S(t)$, such that

$$x(t) = \sum_{u \in S(t)} a(t, u) x(u).$$

It follows then that there exists elements $a_e \in R$ such that $\sum a_e x_e = x^k$ for some power of x.
(iii) Let $f = c^\infty$, $X \in K(f)$ and $F : (T, r) \longrightarrow C^\infty(A)$ with $F(r) = f$ such that (22) holds. This shows that for each $e \in \mathcal{E}(T)$ there exists an element $x(e) \in X$ and a $b(e) \in R$ such that $F(e) = (b(e)x(e))^\infty$. Thus by (ii) one can find elements $a(e) \in R$ for $e \in \mathcal{E}(T)$ and $k > 0$ with

$$c^k = \sum_{e \in \mathcal{E}(T)} a(e) b(e) x(e)$$

so that X satisfies the condition of the Lemma. Conversely assume that there exists a map $a : X \to R$ with finite support such that $\sum a(x)x$ is equal to a power c^k of c. Consider the partition of c^∞ given by the $(a(x)xc)^\infty$. One can view it as a multi-partition of height 1. For each external vertex one has $(a(x)xc)^\infty \in \mathrm{Sieve}(X)$ since $x^\infty \in X$. It follows that $X \in K(f)$. $\square$

5.5 Functoriality of $\mathfrak{Spec}(A)$

Let A, B be $\mathbb{S}$-algebras and $\phi : A \to B$ a morphism of $\mathbb{S}$-algebras. We show that ϕ determines a morphism of sites (in the sense of Grothendieck)

$$\tilde{\phi} : (C^\infty(A), \mathcal{J}(A)) \longrightarrow (C^\infty(B), \mathcal{J}(B)).$$

One first defines a covariant functor which assigns to an object $c = f^\infty$ of $C^\infty(A)$ the object $\tilde{\phi}(c) := \phi(f)^\infty$ of $C^\infty(B)$. This definition is meaningful in view of the implication

$$f^n \in gA(1_+) \implies \phi(f)^n \in \phi(g)B(1_+).$$

Moreover, this implication shows that $\tilde{\phi}$ defines a covariant functor $C^\infty(A) \longrightarrow C^\infty(B)$. This functor preserves finite products since $\phi(ab) = \phi(a)\phi(b)$, $\forall a, b \in A(1_+)$. It also preserves the terminal object 1 as well as the equalizers (since they are all trivial). Thus $\tilde{\phi}$ preserves finite limits. Let us show that it also preserves covers. This means that if S is a covering sieve of the object c of $C^\infty(A)$, then the sieve generated by the morphisms $\tilde{\phi}(u)$, for $u \in S$, is a covering sieve of the object $\tilde{\phi}(c)$ of $C^\infty(B)$. By definition of the Grothendieck topology $\mathcal{J}(A)$, the sieve S contains a family X of morphisms with codomain c for which there exists a multi-partition $F : (T, r) \longrightarrow C^\infty(A)$ with $F(r) = c$ such that for all external vertices e of T (22) holds *i.e.* $F(r \to e) \in \mathrm{Sieve}(X)$. Thus S in fact contains all the morphisms $F(r \to e)$. The composite $G := \tilde{\phi} \circ F$ is a multi-partition $G : (T, r) \longrightarrow C^\infty(B)$ and thus the sieve generated by the morphisms $\tilde{\phi}(u)$ for $u \in S$ contains all the morphisms $G(r \to e)$ and is hence a covering sieve of the object $\tilde{\phi}(c)$ of $C^\infty(B)$. We now apply Theorem VII, 10, 2 of [22] and obtain

Theorem 5.11. *Let A, B be $\mathbb{S}$-algebras and $\phi : A \to B$ a morphism of $\mathbb{S}$-algebras. Then composition with ϕ defines a morphism of sites $\tilde{\phi} : (C^\infty(A), \mathcal{J}(A)) \longrightarrow (C^\infty(B), \mathcal{J}(B))$ and an associated geometric morphism of toposes $f : \mathfrak{Spec}(B) \longrightarrow \mathfrak{Spec}(A)$. The direct image functor f_* sends a sheaf $\mathcal{F}$ on $\mathfrak{Spec}(B)$ to the composite $f_*(\mathcal{F}) = \mathcal{F} \circ \tilde{\phi}$.*

6. Examples of $\mathfrak{Spec}(A)$

6.1 The case of rings

Next proposition can be deduced directly from Proposition 5.10 but we give an independent and more geometric proof based on the identifications of categories in Proposition 5.3.

Proposition 6.1. *Let R be a commutative unital ring. The site $(C^\infty(HR), \mathcal{J}(HR))$ is isomorphic to the (Zariski) site associated to the topological space $\mathrm{Spec}\,(R)$ and the category of open sets of the form $D(f)$, $f \in R$.*

Proof. Let $A = HR$. Then it follows from Proposition 5.3 that the category $\mathcal{C} := C^\infty(HR)$ is isomorphic to the category of open sets of the form $D(f) \subseteq \mathrm{Spec}\,(R)$, $f \in R$. Let $f_j \to f$, $j = 1, \ldots, n$ be a partition of an object $f \in C^\infty(A)$ as in Definition 5.5. Let $\xi \in A(n_+)$, with $(A(\delta_j)(\xi))^\infty = f_j$, $\forall j$, and $(A(\Sigma)\xi)^\infty = f$. One has $A(n_+) = R^n$ and $\xi = (\xi_j)_{j=1,\ldots,n}$, moreover $(A(\delta_j)(\xi)) = \xi_j$ and $A(\Sigma)\xi = \sum \xi_j$. The equality $(A(\delta_j)(\xi))^\infty = f_j$ means $D(f_j) = D(\xi_j)$, while $(A(\Sigma)\xi)^\infty = f$ means $D(f) = D(\sum \xi_j)$. Thus if such a partition exists one has $D(f) = \cup D(f_j)$ (the case $f = 0$ is taken care by the empty partition $HR(0_+) \to HR(1_+)$). Hence the existence of a multi-partition $F : (T, r) \longrightarrow C^\infty(A)$ of $f = F(r)$ implies

$$D(f) = \cup_{e \in \mathcal{E}(T)} D(F(e)).$$

This argument shows that a family $X \in K(c)$ is a covering of $D(c)$ in the usual sense since each open set $D(F(e))$ of the ordinary covering associated to the multi-partition is contained in one of the form $D(x)$, for some $x \in X$. Conversely, if a family X of objects of $C^\infty(HR)$ is a covering in the usual (Zariski) sense, *i.e.* if $\cup_{x\in X} D(x) = D(c)$, then the ideal generated by X contains a power c^n of c and there exists a finite decomposition $c^n = \sum x\, a(x)$, with $a(x) \in R$ which provides a partition c_j, $j = 1, \dots, n$ of c, in the sense of Definition 5.5. Moreover, each c_j belongs to the sieve of X since it is of the form $x\, a(x)$ for some $x \in X$. Thus the usual covering X belongs to $K(c)$ (and the involved rooted tree is of height 1). $\square$

6.2 The case of monoids

Let M be an object of $\mathfrak{Mo}$ *i.e.* a multiplicative, commutative monoid with 0 and 1. Following our conventions [3] a prime ideal $I \subsetneq M$ of M contains 0. The algebraic spectrum $\operatorname{Spec} \mathbb{F}_M$ is the topological space whose points are the prime ideals $I \subsetneq M$ of M whose complement is stable under multiplication [7]. This spectrum always contains the prime ideal $\mathfrak{p}_M = (M^\times)^c$. The topology of $\operatorname{Spec} \mathbb{F}_M$ is defined by its closed sets which are either empty or associated to an ideal $J \subset M$ by

$$V(J) := \{\mathfrak{p} \in \operatorname{Spec} \mathbb{F}_M \mid J \subseteq \mathfrak{p}\}.$$

For any ideal $J \subset M$ one lets $\sqrt{J} := \{a \in M \mid \exists n,\ a^n \in J\}$. In the next part we use the notations of Proposition 5.1.

Proposition 6.2. *Let M be an object of $\mathfrak{Mo}$ and $A = \mathbb{S}M$ be the associated $\mathbb{S}$-algebra.*
(i) For any pair of elements $b, c \in M = \mathbb{S}M(1_+)$ one has

$$\exists F_b \to F_c \iff V(b) \subseteq V(c)$$

where $V(b) := \{\mathfrak{p} \in \operatorname{Spec} \mathbb{F}_M \mid b \in \mathfrak{p}\}$.
(ii) The map $D(b) \mapsto F_b$ defines an isomorphism of the category of open sets of $\operatorname{Spec} \mathbb{F}_M$ of the form $D(b)$, $b \in M$, with the category $C^\infty(\mathbb{S}M)$.
(iii) The site $(C^\infty(\mathbb{S}M), \mathcal{J}(\mathbb{S}M))$ is isomorphic to the site associated to the space $\operatorname{Spec}(\mathbb{F}_M)$, with topology defined by the category of open sets of the form $D(f)$.

Proof. (i) By applying Theorem 1.1 in [14], one has

$$\sqrt{bM} \supseteq \sqrt{cM} \iff V(b) \subseteq V(c).$$

Also, one has

$$\sqrt{bM} \supseteq \sqrt{cM} \iff \exists n \in \mathbb{N},\ c^n \in bM.$$

Hence, by Proposition 5.1 (iii), one gets

$$V(b) \subseteq V(c) \iff \exists F_b \to F_c.$$

(ii) follows from (i) and the uniqueness statement of Proposition 5.1 (iii), together with the equality $D(b) := V(b)^c$ which defines the open sets $D(b)$.
(iii) Let $b \in M$ and $D(b) := \{\mathfrak{p} \in \operatorname{Spec} \mathbb{F}_M \mid b \notin \mathfrak{p}\}$ the associated open set. Assume $D(b) \neq \varnothing$. Let $\mathfrak{p}_b := \cup_{\mathfrak{p}\in D(b)} \mathfrak{p}$: it is the largest prime ideal not containing b. Let then

$$(D(f_j))_{j\in I}, \quad \cup D(f_j) = D(b)$$

be an open covering of $D(b)$ of the topological space $\operatorname{Spec}(\mathbb{F}_M)$. There exists $j \in I$ such that $\mathfrak{p}_b \in D(f_j)$. One then has $f_j \notin \mathfrak{p}_b$ and it follows from the definition of $\mathfrak{p}_b$ that $f_j \notin \mathfrak{p}$, $\forall \mathfrak{p} \in D(b)$. This shows that $D(f_j) = D(b)$, thus the Grothendieck topology associated to the basis of open sets of the form $D(f)$ is the chaotic topology in which every presheaf is a sheaf. Let us now show that the same holds for the Grothendieck topology $\mathcal{J}(\mathbb{S}M)$. Let $f \in M$. Consider a partition of f in the sense of Definition 5.5 *i.e.* a finite collection of morphisms $f_j \to f$, $j = 1, \ldots, n$, in $C^\infty(A)$ such that there exists $\xi \in A(n_+)$, with $(A(\delta_j)(\xi))^\infty = f_j$, $\forall j$, and $(A(\Sigma)\xi)^\infty = f$. By construction of $A(n_+) = M \wedge n_+$, one has $\xi = m \wedge k$, for some $m \in M$ and $k \in n_+$. Thus $A(\delta_j)(\xi) = m$ for $j = k$ and $A(\delta_j)(\xi) = *$ otherwise. Moreover $A(\Sigma)\xi = m$. Thus such a partition of f consists of a single element equal to f. This implies that the same holds for a multi-partition. Hence the Grothendieck topology $\mathcal{J}(\mathbb{S}M)$ is the chaotic topology and agrees with that of $\mathrm{Spec}\,(\mathbb{F}_M)$. □

The proof of (iii) in the above proposition shows in particular that the topos $\mathfrak{Spec}(\mathbb{S}M)$ is of presheaf type.

6.3 The tropical case

Let $I \subset \mathbb{R}$ be an open bounded interval and $R = C_{\mathrm{Conv}}(I)$ be the semiring of continuous, piecewise affine with integral slope, convex functions $f : I \to \mathbb{R}$. In the following we consider only functions with finitely many slopes and moreover we add in R the constant function with value $-\infty$, which plays the role of the zero element of R. We consider elements of $R = C_{\mathrm{Conv}}(I)$ as maps $f : I \to \mathbb{R}_{\max}$ where $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ is the tropical semifield. The two operations in R are the pointwise sup for addition, and the pointwise sum for multiplication.
We let $A := HR$. Our first task is to determine $C^\infty(A)$. Given $f \in R$ we let $f'' := (\frac{d}{dx})^2 f$ be its second derivative understood as a distribution. It is by construction a finite sum of Dirac masses with positive integral coefficients

$$f''(x) = \sum n_j\, \delta_{x_j}(x) = \sum_{y \in Z(f)} n_f(y)\delta_y(x). \tag{24}$$

This equality defines uniquely the finite subset (tropical zeros) $Z(f) \subseteq I$ and the positive integer valued function $n_f : I \to \mathbb{N}$ that vanishes outside $Z(f)$.

Lemma 6.3. *Let $f, g \in R$ be two non-zero*[2] *elements.*
(i) The function g divides f in R if and only if $n_g \le n_f$.
(ii) One has

$$\exists k \in \mathbb{N},\ f^k \in gA(1_+) \iff Z(g) \subseteq Z(f). \tag{25}$$

Proof. (i) Since the product in R is given by the pointwise sum, the map $f \mapsto n_f$ transforms product into sums. It follows that if g divides f one has $n_g \le n_f$. Conversely if $n_g \le n_f$, let h be a double primitive of the distribution $\sum(n_f(y) - n_g(y))\delta_y$. The choice of the double primitive is unique up to an affine function, (invertible in R), thus this choice only affects the equality $f = gh$ up to an invertible element. It follows that g divides f in R.
(ii) One has $n_{f^k} = k\, n_f$, $Z(f^k) = Z(f)$. If $f^k \in gA(1_+)$ then $n_g \le k\, n_f$ follows from (i) so that $Z(g) \subseteq Z(f)$. Conversely, if $Z(g) \subseteq Z(f)$, there exists a finite $k \in \mathbb{N}$ such that $n_g \le k\, n_f$ and (i) shows that $f^k \in gA(1_+)$. □

Proposition 6.4. *Consider the (tropical) semiring $R = C_{\mathrm{Conv}}(I)$.*
(i) For any pair of non-zero elements $b, c \in R = HR(1_+)$ the following equivalence holds

$$\exists F_b \to F_c \iff Z(b) \subseteq Z(c).$$

[2] *i.e.* elements not equal to the constant function $-\infty$

(ii) The map $Z(b)^c \mapsto F_b$ defines an isomorphism between the category $\mathcal{F}$ of open subsets of I which are either empty or whose complement is finite, and the category $C^\infty(HR)$.
(iii) The coverings of an object Ω of the site $(C^\infty(HR), \mathcal{J}(HR))$ are the families of objects $(\Omega_j)_{j\in K}$ of $\mathcal{F}$ which are coverings of Ω in the usual topology, and such that each connected component $C \subseteq \Omega$ is covered by a single element $\Omega_{j(C)}$ of the family.
(iv) The category of points of the topos $\mathfrak{Spec}(HR)$ is canonically isomorphic to the category $\mathrm{Conv}(I)$ *of convex subsets of the interval I, with morphisms given by reverse inclusion.*

Proof. (i) The equivalence follows from (iii) of Proposition 5.1 combined with (ii) of Lemma 6.3.
(ii) For the zero element of R, *i.e.* the function constant equal to $-\infty$, one defines $Z(0) = I$. By Definition 5.2, the category $C^\infty(HR)$ is the opposite of the category formed by the flat functors F_b. Since every finite subset of I appears as a $Z(f)$ for some $f \in R$ using (24), the required isomorphism follows from (i).
(iii) In HR the empty partition $HR(0_+) \to HR(1_+)$ is a partition of 0. Let f be a non-zero element of $R = HR(1_+)$ and C be a connected component of the open set $Z(f)^c$. Let $(f_j)_{j\in K}$ be a finite family of elements of R such that $\vee f_j = f$. The function f restricted to C is affine and there exists an index $j \in K$ such that f_j agrees with f on a non-empty interval inside C. Then, since f_j is convex and $f_j \le f$ one has $f_j(x) = f(x)$, $\forall x \in C$. Indeed, one can assume that $f(x) = 0$, $\forall x \in C$ and use the convexity of f_j to show that its upper-graph is bounded below by the segment $\{(x,0), x \in C\}$. This shows that a partition, in the sense of Definition 5.5, of the object f^∞ of $C^\infty(HR)$ contains an element f_j with $Z(f_j) \cap C = \emptyset$. By following the relevant branch of the corresponding rooted tree, one sees that for any multi-partition $F : (T, r) \longrightarrow C^\infty(HR)$ of f^∞, there exists an external vertex e of T such that $Z(F(e)) \cap C = \emptyset$. Hence any covering of the open set $Z(f)^c$ for the Grothendieck topology $\mathcal{J}(HR)$ fulfills the condition of (iii). Conversely, let us show that if an ordinary covering $(\Omega_j)_{j\in K}$ of an object Ω fulfills the condition (iii) one can find an $f \in R$ with $f^\infty = \Omega$ and elements $f_j \in R$, with $f_j^\infty = \Omega_j$, such that $f = \vee f_j$ and hence that $(\Omega_j)_{j\in K}$ is a cover of Ω for the Grothendieck topology $\mathcal{J}(HR)$. For each connected component C of Ω, let $\Omega_{j(C)}$ cover C and $h_{j(C)}$ with $h^\infty_{j(C)} = \Omega_{j(C)}$ be such that it is constant equal to 0 on C. Let $h \in R$ with $h^\infty = \Omega$ and for each connected component C of Ω let L_C be the affine function which agrees with h on C. Then, by convexity, one has that $(h - L_c)(x)$ is strictly positive outside C and bounded below by the distance $d(x, C)$. This shows that, for n large enough, one obtains

$$nh = \vee_C (h_{j(C)} + nL_C).$$

This provides the required equality for $f := nh$ and $f_{j(C)} = h_{j(C)} + nL_C$, while the remaining f_j's do not matter.
(iv) A point of the topos $\mathfrak{Spec}(HR)$ associated to the site $(C^\infty(HR), \mathcal{J}(HR))$ is given by a flat continuous functor $F : C^\infty(HR) \longrightarrow \mathfrak{Sets}$. The uniqueness of morphisms in the category $C^\infty(HR)$ and the flatness entail that F only takes two possible values, the empty set and the singleton. Moreover the class $\mathcal{N}$ of objects Ω of $C^\infty(HR)$ for which $F(\Omega) \neq \emptyset$ is hereditary and stable under intersection:

$$\Omega \in \mathcal{N},\ \Omega \subseteq \Omega' \Longrightarrow \Omega' \in \mathcal{N},\ \Omega, \Omega' \in \mathcal{N} \Longrightarrow \Omega \cap \Omega' \in \mathcal{N}.$$

Let then $Z(\mathcal{N}) := \{x \in I \mid \{x\}^c \in \mathcal{N}\}$. One has $Z(\mathcal{N}) \neq \emptyset$ if $\mathcal{N} \neq \{I\}$, moreover one has

$$\Omega \in \mathcal{N} \iff \Omega^c \subseteq Z(\mathcal{N}).$$

Indeed, if $\Omega \in \mathcal{N}$ then any element x of its finite complement in I is the complement of a larger $\Omega' \in \mathcal{N}$ so that $x \in Z(\mathcal{N})$. Conversely, if the finite set $\Omega^c \subseteq Z(\mathcal{N})$ then Ω is a finite intersection

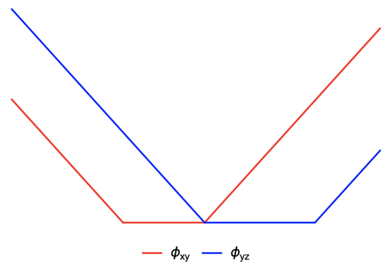

Figure 2. $\phi_{xy} \vee \phi_{yz} = \phi_{yy}$

of elements of $\mathcal{N}$ and hence is in $\mathcal{N}$.
Let $E = Z(\mathcal{N})^c$. Next we show that if the flat functor F is continuous then E is convex. Assume, on the contrary, that there exist three elements $x < y < z$ of the interval I such that $x, z \in E$ and $y \notin E$. One has $y \in Z(\mathcal{N})$ and thus the open set $\Omega := \{y\}^c \in \mathcal{N}$. Then we can construct a covering $(\Omega_j)_{j=1,2}$ of Ω for the Grothendieck topology $\mathcal{J}(HR)$ such that $\Omega_j \notin \mathcal{N}$ for $j = 1, 2$ thus contradicting the continuity of the flat functor F.
For $a, b \in I$ and $a \leq b$, let ϕ_{ab} be the element of $R = C_{\rm Conv}(I)$ which is the function identically 0 on the interval $[a, b]$ and has slope -1 for $x < a$, and slope 1 for $x > b$. By construction: $Z(\phi_{ab}) = \{a, b\}$. Let $x < y < z \in I$ be as above, then $\phi_{xy} \vee \phi_{yz} = \phi_{yy}$ (see Figure 2). This equality shows that the open sets $\Omega_1 := \{x, y\}^c$ and $\Omega_2 := \{y, z\}^c$ form a covering of $\Omega := \{y\}^c$ for the Grothendieck topology $\mathcal{J}(HR)$. One has $x \in E = Z(\mathcal{N})^c$ and hence $\Omega_1 \notin \mathcal{N}$, similarly $z \in E = Z(\mathcal{N})^c$ and hence $\Omega_2 \notin \mathcal{N}$. But since $y \notin E$, $\Omega := \{y\}^c \in \mathcal{N}$. Thus the flat functor F fails to be continuous. This argument shows that for any continuous flat functor F the subset $E = Z(\mathcal{N})^c \subseteq I$ is convex. Conversely, given a convex subset $E \subseteq I$ the associated flat functor defined by

$$F(\Omega) = \begin{cases} \{*\} & \text{if } E \subseteq \Omega \\ \emptyset & \text{otherwise} \end{cases}$$

is continuous since $E \subseteq \Omega$ implies that there exists a connected component C of Ω which contains E. Finally, it is easy to check that the existence of morphisms of functors is governed by the reverse inclusion of convex sets thus one gets the required result. $\square$

6.4 The case of semirings

The tropical case developed in §6.3 together with Proposition 3.4.1 of [13] suggest that, for an arbitrary semiring R, one can compare the spectrum of the $\mathbb{S}$-algebra HR with the prime spectrum of R.

Theorem 6.5. *Let R be a semiring.*
(i) The points of the topos $\mathfrak{Spec}(HR)$ coincide with the prime spectrum, i.e. the set of prime ideals of R.
(ii) The site $(C^\infty(HR), \mathcal{J}(HR))$ is isomorphic to the Zariski spectrum Spec (R) *endowed with the basis of open sets of the form $D(f)$, for $f \in R$.*

Proof. Let $F = C^\infty(HR) \longrightarrow \mathfrak{Sets}$ be the flat continuous functor associated to a point of the topos $\mathfrak{Spec}(HR)$. The uniqueness of morphisms in the category $C^\infty(HR)$ and the flatness entail that the functor F only takes two possible values: the empty set and the singleton. Moreover, the subset $M \subset R$ of elements $f \in R$ such that $F(f^\infty) \neq \varnothing$ is hereditary and stable under products

$$f \in M \,\&\, \exists n \in \mathbb{N},\ f^n \in gR \Longrightarrow g \in M, \quad f,g \in M \Longrightarrow fg \in M.$$

Let $J = M^c$, let $f,g \in J$ and assume that $h = f + g \notin J$. Proposition 5.10 shows that the pair $\{(fh)^\infty \to h^\infty, (gh)^\infty \to h^\infty\}$ forms a partition of h^∞ (since $h^2 = hf + hg$). Thus, if the functor F is continuous, one gets $fh \in M$ or $gh \in M$. Assume $fh \in M$, then as M is hereditary one has $f \in M$, but this contradicts $f \in J$. Thus it follows that J is stable under addition. It is an ideal since M is hereditary and furthermore it is prime since its complement M is stable under multiplication. Conversely, let $J \subset R$ be a prime ideal, and $F_J : C^\infty(HR) \longrightarrow \mathfrak{Sets}$ be the functor

$$F_J(f^\infty) = \begin{cases} \{*\} & \text{if } f \notin J \\ \varnothing & \text{otherwise.} \end{cases}$$

Since J is a prime ideal one derives that F_J is flat. It remains to show that F_J is continuous. By Proposition 5.10, a family X of morphisms with codomain f is a covering for the Grothendieck topology $\mathcal{J}(A)$ if and only if there exists a map $a : X \to R$ with finite support such that $\sum a(x)x = f^k$ for some $k > 0$. Assuming that $f \in M$, one needs to show that $x \in M$ for some $x \in X$, but this follows since J is an ideal. Thus the points of the topos $\mathfrak{Spec}(HR)$ correspond canonically to prime ideals $\mathfrak{p} \in \mathrm{Spec}(R)$ *i.e.* elements of the prime spectrum $\mathrm{Spec}(R)$.
(ii) We first extend Proposition 5.3 in the semiring context. Proposition 7.28 of [15], states that for any ideal $I \subset R$ of a (commutative) semiring R the intersection of prime ideals containing I is

$$\sqrt{I} \coloneqq \{x \in R \mid \exists n \in \mathbb{N},\ x^n \in I\}.$$

Setting $V(b) \coloneqq \{\mathfrak{p} \in \mathrm{Spec}\, R \mid b \in \mathfrak{p}\}$, where $\mathrm{Spec}\, R \coloneqq \mathrm{Prim} R$ is the set of prime ideals of R,

$$\sqrt{bR} = \cap_{\mathfrak{p} \in V(b)} \mathfrak{p}, \ \sqrt{bR} \supseteq \sqrt{cR} \iff V(b) \subseteq V(c), \ \sqrt{bR} \supseteq \sqrt{cR} \iff \exists n \in \mathbb{N},\ c^n \in bR$$

By applying Proposition 5.1 (iii) one gets $V(b) \subseteq V(c) \iff \exists F_b \to F_c$. A similar proof as in Proposition 5.3 shows that the map $D(b) \mapsto F_b$ is an isomorphism between the category of open sets of $\mathrm{Spec}(R)$ of the form $D(b) = V(b)^c$, $b \in R$, and the category $C^\infty(HR)$. The open sets $D(b) = V(b)^c$, $b \in R$ form a basis for the Zariski topology and are stable under finite intersections since $D(ab) = D(a) \cap D(b)$. Note that $D(0) = \varnothing$. It remains to show that the covering families (say $\{D(b_j)\}$) of an open set $D(b)$ for the Grothendieck topology $\mathcal{J}(HR)$ are the same as for the Zariski topology. By Proposition 5.10, $\{D(b_j)\}$ is a covering of $D(b)$ for $\mathcal{J}(HR)$ if and only if there exist $a_j \in R$ and $n \in \mathbb{N}$ with $b^n = \sum a_j b_j$. If this holds and $\mathfrak{p} \in D(b)$ is a prime ideal, then $\mathfrak{p} \in D(b_j)$ for some j since otherwise all $b_j \in \mathfrak{p}$ and hence $b^n \in \mathfrak{p}$. Conversely, let $\{D(b_j)\}$ be a covering of $D(b)$ for the Zariski topology, and I be the ideal generated by the b_j. Then any prime ideal $\mathfrak{p}$ containing I is in the intersection $\cap V(b_j) \subseteq V(b)$. Thus the intersection of prime ideals containing I contains b and Proposition 7.28 of [15] then shows that there exist $a_j \in R$ and $n \in \mathbb{N}$ with $b^n = \sum a_j b_j$. Thus, we conclude that the open coverings are the same for the Grothendieck topology $\mathcal{J}(HR)$ as for the Zariski topology. $\square$

7. The structure sheaf on $\mathfrak{Spec}(A)$

In this section we define the structure sheaf on the spectrum of an arbitrary $\mathbb{S}$-algebra A and show in §7.3 that the construction agrees with the classical one for rings. The construction of the presheaf of $\mathbb{S}$-algebras on $\mathfrak{Spec}(A)$ is based on the process of localization of $\mathbb{S}$-algebras as explained in §7.1. The new feature here is that it is in general false that the structure presheaf is automatically a sheaf (see Remark 7.13). An important step then is, as shown in §7.2, that the associated sheaf is a sheaf of $\mathbb{S}$-algebras. The functoriality of this construction is proved in §7.4. The new behavior is described in the case of the quotient by a subgroup and is explored in §7.5. Finally §7.6 investigates the spectrum of the $\mathbb{S}$-algebra $|H\mathbb{Q}|_1$ in the context of the Arakelov compactification of $\operatorname{Spec}\mathbb{Z}$.

7.1 Localization of $\mathbb{S}$-algebras

Let A be an $\mathbb{S}$-algebra and $M \subseteq A(1_+)$ a multiplicatively closed subset of the monoid $A(1_+)$. We denote by $M - \mathfrak{Sets}_*$ the category of pointed sets endowed with an action of the monoid M, we call M-sets its objects, morphisms are M-equivariant map of pointed sets. The next two lemmas show how to obtain the localization $M^{-1}A$ as an $\mathbb{S}$-algebra.

Lemma 7.1. *Let M be a multiplicative, commutative monoid.*
(i) Let X be an M-set. The following defines an equivalence relation on $M \times X$ whose quotient is denoted $M^{-1}X$

$$(s,x) \sim (t,y) \iff \exists a \in M \mid atx = asy. \tag{26}$$

(ii) Let $f : X \to Y$ be a morphism in $M - \mathfrak{Sets}_*$. Then there exists a unique induced map $\tilde f : M^{-1}X \to M^{-1}Y$ by setting $\tilde f(s,x) = (s, f(x))$ for all $s \in M, x \in X$.
(iii) The association $X \mapsto M^{-1}X$, $f \mapsto \tilde f$ defines a functor $M^{-1} : M - \mathfrak{Sets}_* \longrightarrow \mathfrak{Sets}_*$.

Proof. (i) The relation is symmetric and reflexive by construction. We show that it is transitive. Assume that $(s,x) \sim (t,y)$ and $(t,y) \sim (u,z)$. Then for some $a,b \in M$, one has $atx = asy$ and $buy = btz$. By substitution, one derives $abtux = abusy = abtsz$, so that $(s,x) \sim (u,z)$.
(ii) One checks, using the equivariance of f, that $(s,x) \sim (t,y)$ implies $(s,f(x)) \sim (t,f(y))$.
(iii) follows from (ii). □

Lemma 7.2. *Let A be an $\mathbb{S}$-algebra and M a multiplicatively closed subset of the monoid $A(1_+)$.*
(i) The composite $M^{-1} \circ A$ of the functor M^{-1} with A (viewed as a functor $A : \Gamma^{\mathrm{op}} \longrightarrow M - \mathfrak{Sets}_*$), defines an $\mathbb{S}$-module $M^{-1}A$.
(ii) The structure of $\mathbb{S}$-algebra of A induces a structure of $\mathbb{S}$-algebra on the $\mathbb{S}$-module $M^{-1}A$.
(iii) Let $N \subseteq M$ be a multiplicatively closed subset. Then the natural inclusion $i : N \subseteq M$ induces the morphism of $\mathbb{S}$-algebras

$$N^{-1}A \to M^{-1}A,\ (n,x) \to (i(n),x).$$

Proof. (i) By (1), the product $A(1_+) \wedge A(k_+) \to A(k_+)$ restricted to $M \wedge A(k_+)$ turns A into a functor $\Gamma^{\mathrm{op}} \longrightarrow M - \mathfrak{Sets}_*$. The statement then follows from Lemma 7.1 (iii).
(ii) Define the product $M^{-1}A(X) \wedge M^{-1}A(Y) \to M^{-1}A(X \wedge Y)$ by

$$(s,x) \wedge (t,y) \mapsto (st, m_A(x \wedge y)). \tag{27}$$

One checks, using $m_A(ux \wedge vy) = uv\, m_A(x \wedge y)$, $\forall u,v \in M$, that the right hand side only depends upon the classes of (s,x) and (t,y). The naturality, commutativity and associativity of the product follow from the same properties for A and the commutativity of the monoid M.
(iii) By construction, the map $(n,x) \mapsto (i(n),x)$ is compatible with the equivalence relation (26), since $N \subseteq M$ and in agreement with the structure of $\mathbb{S}$-algebras. □

Next proposition shows that the process of localization of $\mathbb{S}$-algebras coincides with the classical localization for semirings and monoids.

Proposition 7.3. *(i) Let R be a semiring and M a multiplicatively closed subset of R. Then the map $(m,(a_j)) \mapsto ((m,a_j))$ defines an isomorphism of $\mathbb{S}$-algebras $\rho : M^{-1}HR \xrightarrow{\sim} HM^{-1}R$.*
(ii) Let N be a pointed commutative monoid and M a multiplicatively closed subset of N. Then $M^{-1}\mathbb{S}N \simeq \mathbb{S}M^{-1}N$.

Proof. (i) The map ρ is well defined since for $(m,(a_j)) \sim (n,(b_j))$ there exists $u \in M$ such that $una_j = umb_j$ for all j, and this implies $(m,a_j) \sim (n,b_j)$. Conversely, if $(m,a_j) \sim (n,b_j)$ for all j, there exists for each j an element $u_j \in M$ such that $u_j n a_j = u_j m b_j$. By taking $u = \prod u_j$ one obtains that $(m,(a_j)) \sim (n,(b_j))$. This shows that ρ is injective. It is surjective since for any finite family (n_j,a_j) one derives by "reduction to the same denominator", with $n = \prod n_j$, that for each j, $(n_j,a_j) \sim (n,b_j)$, where $b_j = a_j \prod_{i\neq j} n_i$. Finally, one verifies that ρ is a morphism of functors $\rho : M^{-1}HR \longrightarrow HM^{-1}R$ and is compatible with the product (27).
(ii) The statement follows because both $M^{-1}\mathbb{S}N(k_+)$ and $\mathbb{S}M^{-1}N(k_+)$ are given by the smash product $M^{-1}N \wedge k_+$. □

7.2 The sheaf $\mathcal{O}^{++}$ of $\mathbb{S}$-algebras

Let A be an $\mathbb{S}$-algebra. We start by constructing a presheaf $\mathcal{O}$ of $\mathbb{S}$-algebras for the small category $C^\infty(A)$. Let $f \in A(1_+)$ and $M(f^\infty)$ the multiplicative monoid

$$M(f^\infty) := \{g \in A(1_+) \mid \exists n \in \mathbb{N}, n > 0,\ f^n \in gA(1_+)\}. \tag{28}$$

Using the notations of Lemma 7.2, we define an $\mathbb{S}$-algebra by localization

$$\mathcal{O}(f^\infty) := M(f^\infty)^{-1}A. \tag{29}$$

Next Lemma shows that $\mathcal{O}$ is a presheaf of $\mathbb{S}$-algebras *i.e.* a contravariant functor from the category $C^\infty(A)$ to the category of $\mathbb{S}$-algebras.

Lemma 7.4. *(i) Let $\phi \in Hom_{C^\infty(A)}(a^\infty, b^\infty)$. The inclusion $M(b^\infty) \subseteq M(a^\infty)$ induces a morphism of $\mathbb{S}$-algebras $\mathcal{O}(\phi) : \mathcal{O}(b^\infty) \to \mathcal{O}(a^\infty)$.*
(ii) The maps $f^\infty \mapsto \mathcal{O}(f^\infty)$ and $\phi \mapsto \mathcal{O}(\phi)$ given in (i), define a contravariant functor $\mathcal{O} : C^\infty(A) \longrightarrow \mathfrak{S}$ from the category $C^\infty(A)$ to the category of $\mathbb{S}$-algebras.

Proof. (i) By Proposition 5.1 (iii) one has

$$g \in M(f^\infty) \iff \mathrm{Hom}_{C^\infty(A)}(f^\infty, g^\infty) \neq \varnothing.$$

Thus, the existence of $\phi \in \mathrm{Hom}_{C^\infty(A)}(a^\infty, b^\infty)$ implies $M(b^\infty) \subseteq M(a^\infty)$. One then applies Lemma 7.2 (iii) to conclude.
(ii) follows from the transitivity of the inclusions $M(c^\infty) \subseteq M(b^\infty) \subseteq M(a^\infty)$. □

The presheaf $\mathcal{O}$ just constructed is not always a sheaf as shown in Remark 7.13, thus one needs to pass to the associated sheaf. In general, as explained in [22], Section III. 5, passing from a presheaf P to the associated sheaf $\mathfrak{a}P$ requires two applications of the operation $P \mapsto P^+$ from presheaves to presheaves (the first application makes the presheaf separated and the second makes it a sheaf). As a preliminary step we show that the operation $P \mapsto P^+$ transforms presheaves of $\mathbb{S}$-algebras into presheaves of $\mathbb{S}$-algebras. This amounts to prove that the matching families for a given sieve still form an $\mathbb{S}$-algebra. We recall for completeness the definition of matching families ([22], p. 121).

Definition 7.5. *Given a site, a sieve R which is a cover of an object C and a presheaf P, a matching family for R is a function which assigns to every element $f : D \to C$ of R an element $x_f \in P(D)$ so that*

$$P(g)(x_f) = x_{fg}\,,\quad \forall g : E \to D. \tag{30}$$

One denotes by $\mathrm{Match}(R,P)$ the matching families for the sieve R and the presheaf P.

Let $(A_\alpha)_{\alpha\in I}$ be a family of $\mathbb{S}$-algebras: we define the product $\mathbb{S}$-module $A := \prod A_\alpha$ as follows

$$A(k_+) := \prod A_\alpha(k_+),\ \ A(\phi)((a_\alpha)_{\alpha\in I}) := (A_\alpha(\phi)(a_\alpha))_{\alpha\in I}.$$

The product $m_A : A(X)\wedge A(Y) \to A(X\wedge Y)$ given by

$$m_A((a_\alpha)_{\alpha\in I},(b_\alpha)_{\alpha\in I}) := (m_{A_\alpha}(a_\alpha,b_\alpha))_{\alpha\in I},$$

endows A with a structure of $\mathbb{S}$-algebra.

Lemma 7.6. *Let $\mathcal{C}$ be a small category, C an object of $\mathcal{C}$, R a sieve on C and P a presheaf of $\mathbb{S}$-algebras. Let P_k be the presheaf of sets obtained by evaluation on the object k_+ of Γ^{op}.*
The map $k_+ \to \mathrm{Match}(R,P_k)$ defines an $\mathbb{S}$-subalgebra of the product $\mathbb{S}$-algebra: $A = \prod_{f\in R} P(\mathrm{Dom}(f))$.

Proof. Let us first show that $k_+ \to \mathrm{Match}(R,P_k)$ is a subfunctor of the functor $k_+ \to A(k_+)$. By construction, the matching families $(x_f)_{f\in R} \in \mathrm{Match}(R,P_k)$ form a subset of $A(k_+)$. Let $\phi \in \mathrm{Hom}_{\Gamma^{\mathrm{op}}}(k_+,\ell_+)$ and $(x_f)_{f\in R} \in \mathrm{Match}(R,P_k)$. We show that $A(\phi)(x_f)_{f\in R} \in \mathrm{Match}(R,P_\ell)$. Let $g : E \to D$ be a morphism in $\mathcal{C}$. Since $P(g) : P(D) \to P(E)$ is a morphism of $\mathbb{S}$-algebras, it is a natural transformation of functors $\Gamma^{\mathrm{op}} \longrightarrow \mathfrak{Sets}$ and this gives the commutative diagram

$$\begin{tikzcd} P_k(D) \arrow[r, "P(D)(\phi)"] \arrow[d, "P_k(g)"'] & P_\ell(D) \arrow[d, "P_\ell(g)"] \\ P_k(E) \arrow[r, "P(E)(\phi)"] & P_\ell(E) \end{tikzcd} \tag{31}$$

One has $x_f \in P_k(D)$, $A(\phi)(x_f) = P(D)(\phi)(x_f)$, and thus, using (31) and (30) one obtains

$$P_\ell(g)A(\phi)(x_f) = P_\ell(g)P(D)(\phi)(x_f) = P(E)(\phi)P_k(g)x_f = P(E)(\phi)x_{fg} = A(\phi)(x_{fg}).$$

Let us show that $k_+ \to \mathrm{Match}(R,P_k)$ is stable under the product $m_A : A(k_+)\wedge A(\ell_+) \to A(k_+\wedge\ell_+)$. Since $P(g) : P(D) \to P(E)$ is a morphism of $\mathbb{S}$-algebras one has a commutative diagram

$$\begin{tikzcd} P_k(D)\wedge P_\ell(D) \arrow[r, "m_A"] \arrow[d, "P_k(g)\wedge P_\ell(g)"'] & P_{k\ell}(D) \arrow[d, "P_{k\ell}(g)"] \\ P_k(E)\wedge P_\ell(E) \arrow[r, "m_A"'] & P_{k\ell}(E) \end{tikzcd} \tag{32}$$

Let then $(x_f)_{f\in R} \in \mathrm{Match}(R,P_k)$ and $(y_f)_{f\in R} \in \mathrm{Match}(R,P_\ell)$. To show that $m_A(x_f\wedge y_f)_{f\in R} \in \mathrm{Match}(R,P_{k\ell})$ one uses

$$P_{k\ell}(g)(m_A(x_f\wedge y_f)) = m_A(P_k(g)x_f\wedge P_\ell(g)y_f) = m_A(x_{fg}\wedge y_{fg}).$$

This proves that the map $k_+ \to \mathrm{Match}(R,P_k)$ defines an $\mathbb{S}$-subalgebra of A. ∎

With the notations of Lemma 7.6, one has by construction

$$P_k^+(C) := \varinjlim_{R \in J(C)} \mathrm{Match}(R, P_k).$$

The associated sheaf is obtained by iterating twice the transformation $P \mapsto P^+$ on presheaves. One derives the following

Lemma 7.7. *Let $(\mathcal{C}, J)$ be a Grothendieck site, and P a presheaf of $\mathbb{S}$-algebras. Then the associated sheaf $k_+ \to P_k^{++}$ is a sheaf of $\mathbb{S}$-algebras.*

Proof. It suffices to show that the transformation $\mathcal{O} \mapsto \mathcal{O}^+$ maps presheaves of $\mathbb{S}$-algebras to presheaves of $\mathbb{S}$-algebras. By Lemma 7.6, $P^+(C)$ is a pointwise filtered colimit of $\mathbb{S}$-algebras and is thus an $\mathbb{S}$-algebra [17] (3.7). □

Definition 7.8. *Let A be an $\mathbb{S}$-algebra. The* structure presheaf $\mathcal{O}_A$ *(resp.* structure sheaf $\mathcal{O}_A^{++}$*) on the site $(C^\infty(A), \mathcal{J}(A))$ is the presheaf $\mathcal{O}$ of Lemma 7.4 (resp. the sheaf $\mathcal{O}^{++}$ of $\mathbb{S}$-algebras associated to $\mathcal{O}$ on $\mathfrak{Spec}(A)$).*

7.3 The case of rings

Next we prove that the general construction of the structure sheaf $\mathcal{O}_A^{++}$ on $\mathfrak{Spec}(A)$ as presented in §7.2 agrees with the classical construction when $A = HR$ for a ring R.

Proposition 7.9. *Let R be a ring and $A = HR$ be the associated $\mathbb{S}$-algebra. Then the structure presheaf $\mathcal{O}_A$ on $\mathfrak{Spec}(A)$ is a sheaf and is canonically isomorphic to the sheaf of $\mathbb{S}$-algebras $H\mathcal{O}_R$ on* $\mathrm{Spec}\, R$.

Proof. It is well known ([16], Proposition II. 2.2) that for an element $f \in R$ the ring $\mathcal{O}_R(D(f))$, for the classical structure sheaf $\mathcal{O}_R$ on the prime spectrum $\mathrm{Spec}\, R$, is isomorphic to the localized ring R_f. This latter ring is the same as the localization $M(f^\infty)^{-1}R$, since the elements involved in $M(f^\infty)$ as in (28) all divide a power of f. By applying Proposition 7.3, one derives $HR_f = M(f^\infty)^{-1}HR$ and one derives, by *op.cit.* , that the presheaf $D(f) \mapsto R_f$ is already a sheaf. Thus the presheaf on $\mathfrak{Spec}(A)$ given by $f^\infty \mapsto HR_f$ is a sheaf, and thus so is the presheaf $f^\infty \mapsto M(f^\infty)^{-1}HR$. This argument proves that this presheaf coincides with $\mathcal{O}_A$ and that, in the above case, there is no need to sheafify the presheaf of Lemma 7.4. □

7.4 Functoriality of the structure sheaf $\mathcal{O}_A^{++}$

Let $\phi : (C, J) \to (D, K)$ be a morphism of sites as in §5.5. Let $f : \mathrm{Sh}(D, K) \longrightarrow \mathrm{Sh}(C, J)$ be the associated geometric morphism and f_* the direct image functor. We denote by $\mathfrak{a}$ the sheafification functor. Let $\mathcal{F}$ be a presheaf on D, then one has a natural morphism of sheaves

$$\gamma : \mathfrak{a} f_* \mathcal{F} \to f_* \mathfrak{a} \mathcal{F} \tag{33}$$

induced by the universal property of $\mathfrak{a} f_* \mathcal{F}$. Indeed, any map ψ of presheaves from $f_*\mathcal{F}$ to a sheaf $\mathcal{L}$ factors uniquely through the canonical map $\eta : f_*\mathcal{F} \to \mathfrak{a} f_* \mathcal{F}$ (see [22] Lemma III. 5.3), so that ψ induces a morphism of sheaves $\tilde{\psi} : \mathfrak{a} f_* \mathcal{F} \to \mathcal{L}$ such that $\psi = \tilde{\psi} \circ \eta$. One applies this to the morphism of presheaves $\psi : f_*\mathcal{F} \to f_* \mathfrak{a}\mathcal{F}$ obtained by functoriality of f_* at the level of presheaves (since it is given by composition with $\phi : C \longrightarrow D$) from the canonical morphism of presheaves $\mathcal{F} \to \mathfrak{a}\mathcal{F}$. Thus the γ of (33) is $\gamma = \tilde{\psi}$.

Let now $\mathcal{E}$ be a presheaf on C and $\rho : \mathcal{E} \to f_*\mathcal{F}$ a morphism of presheaves. By functoriality of the sheafification functor $\mathfrak{a}$ one obtains a morphism of sheaves $\mathfrak{a}(\rho) : \mathfrak{a}(\mathcal{E}) \to \mathfrak{a}(f_*\mathcal{F})$. We let

$$\alpha(\rho) := \gamma \circ \mathfrak{a}(\rho) : \mathfrak{a}(\mathcal{E}) \to f_*\mathfrak{a}(\mathcal{F}) \tag{34}$$

be the morphism of sheaves obtained by composition with γ of (33).

Theorem 7.10. *Let A, B be $\mathbb{S}$-algebras and $\phi : A \to B$ a morphism of $\mathbb{S}$-algebras. Let $\tilde{\phi} : (C^\infty(A), \mathcal{J}(A)) \longrightarrow (C^\infty(B), \mathcal{J}(B))$ be the associated morphism of sites as in Theorem 5.11. The localization of ϕ defines a morphisms of presheaves $\rho : \mathcal{O}_A \to \tilde{\phi}_*(\mathcal{O}_B)$*

$$\rho : M(f^\infty)^{-1}A \to M(\phi(f)^\infty)^{-1}B. \tag{35}$$

Moreover, the morphism of sheaves $\alpha(\rho) = \gamma \circ \mathfrak{a}(\rho)$ associated to ρ by (34), is a morphism from the structure sheaf $\mathcal{O}_A^{++}$ to the direct image of the structure sheaf $\mathcal{O}_B^{++}$ by the geometric morphism associated to the morphism of sites $\tilde{\phi}$.

Proof. The map which to a pair $(m, a) \in M(f^\infty) \times A$ assigns the pair $(\phi(m), \phi(a))$ that belongs to $M(\phi(f)^\infty) \times B$, defines, as in Lemma 7.2 (iii), a morphism of $\mathbb{S}$-algebras. These morphisms are compatible with the restriction morphisms of Lemma 7.4, thus they define a morphism of presheaves of $\mathbb{S}$-algebras as in (35). Lemma 7.6 implies that the sheafification functor $\mathfrak{a}$ as well as the universal map from a presheaf to the associated sheaf are compatible with the $\mathbb{S}$-algebra structures. It follows that the morphism of sheaves $\alpha(\rho)$ is itself compatible with the $\mathbb{S}$-algebra structures. $\square$

7.5 Quotient by a subgroup

Let A be an $\mathbb{S}$-algebra and $G \subset A(1_+)$ be a multiplicative subgroup of the monoid $A(1_+)$. By applying the compatibility property (17) one sees that the functor $A : \Gamma^{\rm op} \longrightarrow \mathfrak{Sets}_*$ is G-equivariant and induces, by composition with taking the quotient by G, a functor $A/G : \Gamma^{\rm op} \longrightarrow \mathfrak{Sets}_*$ which assigns to an object X of $\Gamma^{\rm op}$ the set of G-orbits $A(X)/G$ and to a morphism its action on the G-orbits. Moreover, the product $m_A : A(X) \wedge A(Y) \to A(X \wedge Y)$ fulfills $m_A(u\xi \wedge v\eta) = uv\ m_A(\xi \wedge \eta)$ and hence induces a product which turns A/G into an $\mathbb{S}$-algebra. We let $q_G : A \to A/G$ be the natural transformation given by the canonical map to the quotient by G. It is by construction a morphism of $\mathbb{S}$-algebras.

Proposition 7.11. *Let A be an $\mathbb{S}$-algebra, $G \subset A(1_+)$ a subgroup and $q_G : A \to A/G$ the canonical morphism to the quotient by G.*
(i) The geometric morphism $\mathfrak{Spec}(A/G) \to \mathfrak{Spec}(A)$ induced by q_G is an isomorphism of sites.
(ii) The morphism ρ of presheaves as in (35) is given by the localization of the morphism q_G.

Proof. (i) We first show that the morphism of sites $\tilde{q}_G : (C^\infty(A), \mathcal{J}(A)) \longrightarrow (C^\infty(A/G), \mathcal{J}(A/G))$ is an isomorphism. First, the morphism $q_G : A \to A/G$ induces a morphism of monoids $q_G : A(1_+) \to A/G(1_+)$ whose effect on $M = A(1_+)$ is to divide M by the subgroup G. We claim that the quotient map $q : M \to M/G$ induces an isomorphism $\tilde{q}_G : C^\infty(A) \xrightarrow{\sim} C^\infty(A/G)$. To prove this it is enough to show that for any $f \in M$ and $g \in G$ one has the equality $f^\infty = (fg)^\infty$ in $C^\infty(A)$. This follows from Proposition 5.1, since f divides fg and fg divides f. Next, let us show that $\tilde{q}_G : C^\infty(A) \to C^\infty(A/G)$ is an isomorphism for the Grothendieck topologies. Recall (Definition 5.5) that a partition of an object f in $C^\infty(A)$ is a collection of morphisms $f_j \to f$, $j = 1, \ldots, n$, such that there exists $\xi \in A(n_+)$ with $(A(\delta_j)(\xi))^\infty = f_j$, $\forall j$, and $(A(\Sigma)\xi)^\infty = f$. This notion is unchanged if one replaces A by A/G. Thus the notion of multi-partition is also

unchanged, therefore the morphism of sites $\tilde{q}_G : (C^\infty(A), \mathcal{J}(A)) \longrightarrow (C^\infty(A/G), \mathcal{J}(A/G))$ is an isomorphism.
(ii) follows from the commutation of the operations of localization with respect to a multiplicative subset $M \subseteq A(1_+)$ containing G, and of quotient by G, *i.e.* the equality

$$(M/G)^{-1}(A/G) = (M^{-1}A)/G. \tag{36}$$

□

Next corollary displays the role of the general theory of $\mathbb{S}$-algebras in the context of the adele class space of a global field (compare (i) with example §3.5, example 11, in [3]).

Corollary 7.12. *Let K be a global field, $\mathbb{A}_K$ the ring of adeles of K. Let $A = H\mathbb{A}_K/K^\times$ be the $\mathbb{S}$-algebra associated to the subgroup $G = K^\times \subset H\mathbb{A}_K(1_+) = \mathbb{A}_K$.*
(i) The map of sets $\nu : Hom_{\mathfrak{S}}(\mathbb{S}[T], A) \to A(1_+)$ of Corollary 2.3 defines a canonical bijection

$$\nu : Hom_{\mathfrak{S}}(\mathbb{S}[T], H\mathbb{A}_K/K^\times) \xrightarrow{\sim} \mathbb{A}_K/K^\times.$$

(ii) The spectrum $\mathfrak{Spec}(H\mathbb{A}_K/K^\times)$ is canonically isomorphic, as a site, to the spectrum of the ring of adeles $\mathbb{A}_K$.

Proof. (i) follows directly from Corollary 2.3 and the equality $H\mathbb{A}_K(1_+) = \mathbb{A}_K$.
(ii) follows from Proposition 7.11 and Proposition 7.9. □

Remark 7.13. *It is false, in general and in the above context, that the presheaf given by (36) on $C^\infty(A/G)$ is a sheaf. This shows, in particular, that one needs to pass to the associated sheaf. In the following we describe some examples where this issue happens.*
Let $R = R_1 \times R_2$ be the product of two commutative rings, G a commutative group and $\iota_j : G \to R_j$, for $j = 1, 2$, two morphisms to the group of units. To be more specific, we consider two places v_j of $\mathbb{Q}$, we let $R_j = \mathbb{Q}_{v_j}$ and $G = \mathbb{Q}^\times$ with the canonical inclusions in the local fields obtained as completions (much of this argument applies in a more general setup). The spectrum of HR/G is, by Propositions 6.1 and 7.11 isomorphic to the prime spectrum of R and hence consists of the two point set $\{v_1, v_2\}$ endowed with the discrete topology. The structure sheaf of HR is obtained by applying H to the structure sheaf of R, which has stalk R_j at v_j. The canonical presheaf on $\mathfrak{Spec}(HR/G)$ assigns to the open set $\{v_1, v_2\}$ the $\mathbb{S}$-algebra HR/G, but when one passes to the associated sheaf one obtains a non-trivial quotient. One can see this already at the level 1_+, i.e. by evaluation on 1_+. For the canonical presheaf $\mathcal{O}$ one obtains instead

$$\mathcal{O}(\{v_1, v_2\})(1_+) = (R_1 \times R_2)/G, \ \mathcal{O}(\{v_1\})(1_+) = R_1/G, \ \mathcal{O}(\{v_2\})(1_+) = R_2/G,$$

so that for the associated sheaf $\mathfrak{a}\mathcal{O}$ one has

$$\mathfrak{a}\mathcal{O}(\{v_1, v_2\})(1_+) = R_1/G \times R_2/G.$$

This proves that the canonical morphism $\mathcal{O} \to \mathfrak{a}\mathcal{O}$ to the associated sheaf is in general not injective.

In view of the above Remark 7.13 the canonical presheaf on $\mathfrak{Spec}(HR/G)$ is in general not a sheaf. However, next statement guarantees that it is a sheaf, provided the ring R has no zero divisors.

Proposition 7.14. *Let R be a commutative ring with no zero divisors, and $G \subset R$ a subgroup of the multiplicative group. Then the canonical presheaf on $\mathfrak{Spec}(HR/G)$ is a sheaf.*

Proof. It follows from Propositions 6.1 and 7.11 that $\mathfrak{Spec}(HR/G)$ is isomorphic to the prime spectrum of R. It is thus an irreducible topological space, *i.e.* any pair of non-empty open sets has a non-empty intersection. Let $\mathcal{O}_R$ be the structure sheaf of R. Since R has no zero divisors the restriction maps $\mathcal{O}_R(U) \to \mathcal{O}_R(V)$ are injective and the action of G by multiplication is free except for its action on the fixed point 0. These two properties continue to hold for finite powers $\mathcal{O}_R^n$ and are hence fulfilled by the structure sheaf of HR. The result is then ensured by the next general lemma. ☐

Lemma 7.15. *Let X be an irreducible topological space, G a group and $\mathcal{F}$ a sheaf of pointed G-sets on X such that*

(i) *The action of G on the complement of the base point is free on $\mathcal{F}(U)$ for any non-empty open set U.*

(ii) *The restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ are injective.*

Then the presheaf $U \mapsto \mathcal{F}(U)/G$ is a sheaf.

Proof. Let $(U_j)_{j\in I}$ be an open cover of the open set $U \subseteq X$ and consider the equalizer diagram for the sheaf $\mathcal{F}$

$$\mathcal{F}(U) \xrightarrow{\epsilon} \prod_{j\in I} \mathcal{F}(U_j) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j). \tag{37}$$

Let $i_0 \in I$, with $U_{i_0} \neq \emptyset$. The restriction map $\mathcal{F}(U) \to \mathcal{F}(U_{i_0})$ is injective by (ii), and G-equivariant, thus it remains injective on the orbit space *i.e.* for the induced map $\mathcal{F}(U)/G \to \mathcal{F}(U_{i_0})/G$. Let $\xi = (\xi_j)_{j\in I} \in \prod_{j\in I} \mathcal{F}(U_j)$ be such that the restrictions of ξ_i and ξ_j are equal in $\mathcal{F}(U_i \cap U_j)/G$ for all $i,j \in I$. We can assume that all ξ_j are distinct from the base point $*$, since if one of them is $*$ the same holds for all. Since the action of G is free on the complement of the base point, there exists unique elements $g(i,j) \in G$ such that

$$g(i,j)\xi_j = \xi_i \text{ on } U_i \cap U_j\,, \quad \forall i,j \in I.$$

One has $g(i,i) = 1$, $g(i,j)g(j,k) = g(i,k)$ for all $i,j,k \in I$. Thus $g(i,j) = g(i,i_0)g(j,i_0)^{-1}$ and with $\eta_j := g(j,i_0)^{-1}\xi_j$ one has

$$\eta_j = \eta_i \text{ on } U_i \cap U_j\,, \quad \forall i,j \in I,$$

so that since (37) is an equalizer there exists a section $\eta \in \mathcal{F}(U)$ which restricts to η_j on U_j for all $j \in I$. This shows that the class of η in $\mathcal{F}(U)/G$ restricts to the class of ξ_j in $\mathcal{F}(U_j)/G$ for all $j \in I$. Hence the diagram (37) remains an equalizer after passing to the sets of G-orbits. ☐

7.6 The spectrum of $|H\mathbb{Q}|_1$

The definition of the structure sheaf of $\mathbb{S}$-algebras associated to the Arakelov compactification $\overline{\operatorname{Spec}\mathbb{Z}}$ given in [5], involves the $\mathbb{S}$-subalgebra of $H\mathbb{Q}$ defined by

$$|H\mathbb{Q}|_1(n_+) := \{(q_j)_{j=1,\dots,n} \in H\mathbb{Q}(n_+) \mid \sum |q_j| \le 1\},$$

where $|q|$ is the archimedean norm of $q \in \mathbb{Q}$. Next proposition shows that $\mathfrak{Spec}(|H\mathbb{Q}|_1)$ and its structure sheaf behave in a similar way as the spectrum of the local ring associated to a non-archimedean place of $\mathbb{Q}$. On the other hand one finds a substantial nuance when testing the Grothendieck topology.

Proposition 7.16. (*i*) *The spectrum* $\mathfrak{Spec}(|H\mathbb{Q}|_1)$ *is the site given by the small category* $0 \to u \to 1$ *endowed with the Grothendieck topology for which the single morphism* $u \to 1$ *is a covering of* 1.
(*ii*) *The structure presheaf* $\mathcal{O}$ *fulfills* $\mathcal{O}(1) = |H\mathbb{Q}|_1$, $\mathcal{O}(u) = H\mathbb{Q}$, *and the restriction map is given by the inclusion* $|H\mathbb{Q}|_1 \subset H\mathbb{Q}$.

Proof. (*i*) Let $A = |H\mathbb{Q}|_1$. One has $A(1_+) = \{q \in \mathbb{Q} \mid |q| \le 1\}$. Thus for $r, s \in A(1_+)$, one has

$$r \in sA(1_+) \iff |r| \le |s|.$$

This proves that the elements $u = r^\infty$ for $0 < |r| < 1$ are all equal in the category $C^\infty(A)$. This category has three objects, u, 0^∞ and 1^∞. The morphisms are $0^\infty \to u \to 1^\infty$. The partition of unity $\frac{1}{2} + \frac{1}{2} = 1$ shows that the single morphism $u \to 1$ is a covering of 1.
(*ii*) The localization of the $\mathbb{S}$-algebra $A = |H\mathbb{Q}|_1$, with respect to the monoid M of non-zero elements of $A(1_+)$, is $H\mathbb{Q}$ since for any n-tuple $(q_j)_{j=1,\ldots,n} \in H\mathbb{Q}(n_+)$ one can find a non-zero multiple $(qq_j)_{j=1,\ldots,n} \in H\mathbb{Q}(n_+)$ such that $\sum |qq_j| \le 1$. □

In the above proposition one sees that u corresponds to the generic point and the stalk at u is $H\mathbb{Q}$ as expected. On the other hand the closed point disappears due to the prescribed Grothendieck topology. This example exhibits the need to keep all the information from the site and presheaf instead of passing directly to the associated topos and sheaf.

8. Relation with Töen-Vaquié [24]

In their paper [24], B. Töen and M. Vaquié have developed a general theory of algebraic geometry which applies to any symmetric monoidal closed category $\mathcal{C}$ which is complete and cocomplete. Their theory gives the expected result, *i.e.* usual algebraic geometry, when applied to the category of abelian groups. It also gives the naive $\mathbb{F}_1$ geometric theory of monoids when applied to the category of pointed sets as suggested by [18]. The category $\mathbb{S}-\mathrm{Mod}$ of Γ-sets (equivalently of $\mathbb{S}$-modules) is an example of a symmetric monoidal closed category which is complete and cocomplete. The closed structure of the category $\mathbb{S}-\mathrm{Mod}$ is defined by setting

$$\underline{\mathrm{Hom}}_{\mathbb{S}}(M,N) = \{k_+ \mapsto \mathrm{Hom}_{\mathbb{S}}(M, N(k_+ \wedge -))\}, \tag{38}$$

where $\wedge$ is the smash product of pointed sets. This formula uniquely defines the smash product of Γ-sets by applying the adjunction

$$\underline{\mathrm{Hom}}_{\mathbb{S}}(M_1 \wedge M_2, N) = \underline{\mathrm{Hom}}_{\mathbb{S}}(M_1, \underline{\mathrm{Hom}}_{\mathbb{S}}(M_2, N)).$$

Thus it would seem perfectly natural to apply the general theory of [24] to the category $\mathbb{S}-\mathrm{Mod}$ rather than developing the new theory presented in this paper. However we shall prove in this section that even though the category $\mathfrak{Ring}$ of rings is a full subcategory of the category of $\mathbb{S}$-algebras, the theory developed in [24], when applied to the category $\mathbb{S}-\mathrm{Mod}$, does not agree with ordinary algebraic geometry when restricted to $\mathfrak{Ring}$. We shall first give the concrete case where this happens and in the final part of this section we shall explain the conceptual reason for this failure.

Lemma 8.1. *Let* K *be a field* $K \neq \mathbb{F}_2$, *consider the ring* $R = K^2$, *and the* $\mathbb{S}$*-algebra* $A = HR$. *Let* $p_j : R \to K$, $j = 1, 2$ *be the two projections. Then the family with two elements*

$$(Hp_j)_{j=1,2}, \quad Hp_j : HR \to HK$$

fails to be a Zariski covering of HR *in the sense of [24], Definition 2.10.*

Proof. Let $G = K^\times$ be the multiplicative group of K, viewed as a diagonal subgroup of $R = K^2$, using the diagonal embedding $g \mapsto (g,g) \in K^2$. Consider the $\mathbb{S}$-algebra $A' := HR/G$ with the canonical morphism $\rho : A \to A'$. One has $HR = HK \times HK$ and the morphism $Hp_j : HR \to HK$ is simply the canonical projection to one of the factors. We view A' as an A-module using the morphism ρ. We first compute the tensor product $X_j := A' \otimes_A B_j$, where A acts on $B_j = HK$ through the morphism Hp_j. By [24] (p. 454), one has a natural isomorphism of B_j-modules of the form

$$A' \coprod_A B_j \to A' \otimes_A B_j,$$

where the colimit $A' \coprod_A B_j$ is computed in the category $\mathfrak{S}$ of $\mathbb{S}$-algebras. Next we show that $A' \coprod_A B_j$ is given by

$$\begin{tikzcd} & HR \arrow[dl, "\rho"'] \arrow[dr, "Hp_j"] & \\ HR/G \arrow[dr, "\pi_j"'] & & HK \arrow[dl, "\rho_j"] \\ & HK/K^* & \end{tikzcd} \tag{39}$$

where $\rho_j : HK \to HK/K^*$ is the quotient morphism and $\pi_j : HR/G \to HK/K^*$ is the projection. Consider an object X of $\mathfrak{S}$ and morphisms $\alpha : A' \to X$ and $\beta : B_j \to X$ in $\mathfrak{S}$ such that the following diagram commutes

$$\begin{tikzcd} & A \arrow[dl, "\rho"'] \arrow[dr, "Hp_j"] & \\ A' \arrow[dr, "\alpha"'] & & B_j \arrow[dl, "\beta"] \\ & X & \end{tikzcd} \tag{40}$$

We show that, for fixed j, there exists a unique morphism $\gamma : HK/K^\times \to X$ such that the following equalities hold

$$\alpha = \gamma \circ \pi_j, \quad \beta = \gamma \circ \rho_j.$$

Let us take $j = 1$ for simplicity. By construction $\beta : HK \to X$ is a morphism of $\mathbb{S}$-algebras. For any integer $n > 0$ and any $\xi_1, \xi_2 \in HK(n_+)$ one has $(\xi_1, \xi_2) \in HR(n_+)$, and

$$\alpha(\rho((\xi_1,\xi_2))) = \alpha((\xi_1,\xi_2)G) = \beta(\xi_1).$$

This shows that $\beta(\xi_1)$ is unchanged if one multiplies ξ_1 by any $u \in K^\times$ and that there exists a unique morphism $\gamma : HK/K^\times \to X$ in $\mathfrak{S}$ such that $\beta = \gamma \circ \rho_1$. Then one derives $\alpha = \gamma \circ \pi_1$. This shows that the colimit $A' \coprod_A B_j$ is given by the diagram (39). Thus the functor

$$\mathcal{L} :\ A - \mathrm{Mod} \longrightarrow \prod B_j - \mathrm{Mod}, \quad Y \to \prod Y \otimes_A B_j,$$

when evaluated on the A-module A', transforms A' into the product of two copies of $HK/K^\times$, viewed as modules over $B_j = HK$. Take $u \in K^\times$, $u \neq 1$ and $v = (1,u) \in R = A(1_+)$. The endomorphism V of the A-module A' given by multiplication by $\rho(v) \in A'(1_+)$ is well defined and non trivial since $\rho(v) \neq 1$. But the action of V on each $A' \otimes_A B_j = HK/K^\times$ is the identity and thus the functor $\mathcal{L}$ is not faithful. This shows that the covering is not faithfully flat and hence is not a Zariski cover. $\square$

Lemma 8.1 shows that Töen-Vaquié 's theory for the symmetric monoidal closed category $\mathbb{S}-\mathrm{Mod}$ cannot restrict to the usual algebraic geometry on the full subcategory $\mathfrak{Ring}$ of $\mathfrak{S}$. Indeed, if it would, the spectrum of a field K_j would be a single point $\{p_j\}$ and the spectrum of the product $R = K_1 \times K_2$ of two fields would be a pair of points $\{p_1, p_2\}$ with maps corresponding to the two projections $p_j : R \to K_j$ forming a Zariski cover. On the other hand, Lemma 8.1 shows that one does not obtain a Zariski cover in the sense of [24]. The conceptual reason for this failure is that the notion of cover as in [24] involves all A-modules. Even though ordinary R-modules, for a ring R, form a full subcategory of the category of HR-modules, there are new HR-modules in the wider context of $\mathbb{S}$-algebras: their existence constrains the notion of cover and excludes the simplest covers.

9. Appendix : Gluing two categories using adjoint functors

We briefly recall, for convenience of the reader, the process of glueing together two categories using a pair of adjoint functors, as explained to us by P. Cartier (see [3] for more details).
Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories connected by a pair of adjoint functors $\beta : \mathcal{C} \to \mathcal{C}'$ and $\beta^* : \mathcal{C}' \to \mathcal{C}$. By definition one has a canonical identification

$$\mathrm{Hom}_{\mathcal{C}'}(\beta(H), R) \overset{\Phi}{\cong} \mathrm{Hom}_{\mathcal{C}}(H, \beta^*(R)) \qquad \forall\ H \in \mathrm{Obj}(\mathcal{C}),\ R \in \mathrm{Obj}(\mathcal{C}'). \tag{41}$$

The naturality of Φ is expressed by the commutativity of the following diagram where the vertical arrows are given by composition, $\forall f \in \mathrm{Hom}_{\mathcal{C}}(G, H)$ and $\forall h \in \mathrm{Hom}_{\mathcal{C}'}(R, S)$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}'}(\beta(H), R) \arrow[r, "\Phi"] \arrow[d, "\mathrm{Hom}(\beta(f),h)"'] & \mathrm{Hom}_{\mathcal{C}}(H, \beta^*(R)) \arrow[d, "\mathrm{Hom}(f,\beta^*(h))"] \\
\mathrm{Hom}_{\mathcal{C}'}(\beta(G), S) \arrow[r, "\Phi"'] & \mathrm{Hom}_{\mathcal{C}}(G, \beta^*(S))
\end{tikzcd} \tag{42}$$

We define the category $\mathcal{C}'' = \mathcal{C} \cup_{\beta,\beta^*} \mathcal{C}'$ by gluing $\mathcal{C}$ and $\mathcal{C}'$. The collection of objects of $\mathcal{C}''$ is obtained as the *disjoint union* of the collection of objects of $\mathcal{C}$ and $\mathcal{C}'$. For $R \in \mathrm{Obj}(\mathcal{C}')$ and $H \in \mathrm{Obj}(\mathcal{C})$, one sets $\mathrm{Hom}_{\mathcal{C}''}(R, H) = \varnothing$. On the other hand, one defines

$$\mathrm{Hom}_{\mathcal{C}''}(H, R) = \mathrm{Hom}_{\mathcal{C}'}(\beta(H), R) \cong \mathrm{Hom}_{\mathcal{C}}(H, \beta^*(R)). \tag{43}$$

The morphisms between objects contained in a same category are unchanged. The composition of morphisms in $\mathcal{C}''$ is defined as follows. For $\phi \in \mathrm{Hom}_{\mathcal{C}''}(H, R)$ and $\psi \in \mathrm{Hom}_{\mathcal{C}}(H', H)$, one defines $\phi \circ \psi \in \mathrm{Hom}_{\mathcal{C}''}(H', R)$ as the composite

$$\phi \circ \beta(\psi) \in \mathrm{Hom}_{\mathcal{C}'}(\beta(H'), R) = \mathrm{Hom}_{\mathcal{C}''}(H', R). \tag{44}$$

Similarly, for $\theta \in \mathrm{Hom}_{\mathcal{C}'}(R, R')$ one defines $\theta \circ \phi \in \mathrm{Hom}_{\mathcal{C}''}(H, R')$ as the composite

$$\theta \circ \phi \in \mathrm{Hom}_{\mathcal{C}'}(\beta(H), R') = \mathrm{Hom}_{\mathcal{C}''}(H, R'). \tag{45}$$

We recall from [3] the following fact

Proposition 9.1. *$\mathcal{C}'' = \mathcal{C} \cup_{\beta,\beta^*} \mathcal{C}'$ is a category which contains $\mathcal{C}$ and $\mathcal{C}'$ as full subcategories. Moreover, for any object H of $\mathcal{C}$ and R of $\mathcal{C}'$, one has*

$$Hom_{\mathcal{C}''}(H, R) = Hom_{\mathcal{C}'}(\beta(H), R) \cong Hom_{\mathcal{C}}(H, \beta^*(R)).$$

One defines specific morphisms α_H and α'_R as follows

$$\alpha_H = \mathrm{id}_{\beta(H)} \in \mathrm{Hom}_{\mathcal{C}'}(\beta(H), \beta(H)) = \mathrm{Hom}_{\mathcal{C}''}(H, \beta(H)) \tag{46}$$

$$\alpha'_R = \Phi^{-1}(\mathrm{id}_{\beta^*(R)}) \in \Phi^{-1}(\mathrm{Hom}_{\mathcal{C}}(\beta^*(R), \beta^*(R))) = \mathrm{Hom}_{\mathcal{C}''}(\beta^*(R), R). \tag{47}$$

By construction one gets

$$\mathrm{Hom}_{\mathcal{C}''}(H, R) = \{g \circ \alpha_H \mid g \in \mathrm{Hom}_{\mathcal{C}'}(\beta(H), R)\}, \tag{48}$$

and for any morphism $\rho \in \mathrm{Hom}_{\mathcal{C}}(H, K)$ the following equation holds

$$\alpha_K \circ \rho = \beta(\rho) \circ \alpha_H. \tag{49}$$

Similarly, it also turns out that

$$\mathrm{Hom}_{\mathcal{C}''}(H, R) = \{\alpha'_R \circ f \mid f \in \mathrm{Hom}_{\mathcal{C}}(H, \beta^*(R))\} \tag{50}$$

and the associated equalities hold

$$\alpha'_S \circ \beta^*(\rho) = \rho \circ \alpha'_R \qquad \forall \rho \in \mathrm{Hom}_{\mathcal{C}'}(R, S) \tag{51}$$

$$g \circ \alpha_H = \alpha'_R \circ \Phi(g) \qquad \forall g \in \mathrm{Hom}_{\mathcal{C}'}(\beta(H), R). \tag{52}$$

In [3] we applied the above process to construct the category $\mathfrak{MR} = \mathfrak{Ring} \cup_{\beta,\beta^*} \mathfrak{Mo}$ in order to refine the notion of "variety over $\mathbb{F}_1$" introduced by Soulé in [23] to that of $\mathbb{F}_1$-scheme and prove that Chevalley groups are $\mathbb{F}_{1^2}$-schemes. An $\mathbb{F}_1$-scheme is defined in terms of a covariant functor $\mathcal{X} : \mathfrak{MR} \to \mathfrak{Sets}$ required to fulfill three additional conditions :

- The restriction $\mathcal{X}_{\mathfrak{Ring}}$ of $\mathcal{X}$ to the full subcategory $\mathfrak{Ring}$ is a $\mathbb{Z}$-scheme.
- The restriction $\mathcal{X}_{\mathfrak{Mo}}$ of $\mathcal{X}$ to $\mathfrak{Mo}$ is an $\mathfrak{Mo}$-scheme.
- The map $\mathcal{X}(\alpha'_R) : \mathcal{X}_{\mathfrak{Mo}} \circ \beta^*(R) \to \mathcal{X}_{\mathfrak{Ring}}(R)$ is a bijection (of sets), when R is a field.

The first condition asserts that when $\mathcal{X}$ is restricted to the full subcategory $\mathfrak{Ring}$ one obtains a scheme in the usual sense (see [10]). The second condition involves the notion of $\mathfrak{Mo}$-scheme developed in [3] as the functorial version of the notion of scheme [7] over the category $\mathfrak{Mo}$. The requirement of locality of the functor is automatic and the only condition on the functor $\mathcal{X}_{\mathfrak{Mo}} : \mathfrak{Mo} \longrightarrow \mathfrak{Sets}$ is that it is locally affine. The third condition involves the evaluation of the functor $\mathcal{X}$ on the specific morphism α'_R of (47). Morphisms of $\mathbb{F}_1$-schemes are natural transformations of the corresponding functors. The conceptual reason why Chevalley groups are $\mathbb{F}_{1^2}$-schemes is the validity of the Bruhat decomposition of a Chevalley group over any field (see [3] where the subtle nuance between $\mathbb{F}_1$ and $\mathbb{F}_{1^2}$ is addressed).

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