Prolate operator and Riemann Zeta

Alain Connes[a,b,1] **and Henri Moscovici**[c,2]

[a]A. Connes: Collège de France, 3 rue d'Ulm, 75005 Paris, France; [b]I.H.E.S., 35 Rte de Chartres, 91440 Bures-sur-Yvette, France; [c]Department of Mathematics, The Ohio State University, Columbus, OH 43210, USA

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We describe a remarkable new property of the selfadjoint extension of the prolate spheroidal operator introduced in 1998 by the first author. The restriction of this operator to the interval whose characteristic function commutes with it is well known, has discrete positive spectrum and is well understood. What we have discovered is that the restriction of the prolate differential operator to the complement of the finite interval admits (besides a replica of the above positive spectrum) negative eigenvalues whose ultraviolet behavior reproduce that of the squares of zeros of the Riemann zeta function. Moreover we show that their corresponding eigenfunctions belong to the Sonin space. This feature fits with the proof (by the first author and C. Consani) of Weil's positivity at the archimedean place, which uses the compression of the scaling action to the Sonin space. Furthermore we construct an isospectral family of Dirac operators whose spectra have the same ultraviolet behavior as the zeros of the Riemann zeta function.

Zeta | Prolate | Dirac | Spectrum

The prolate spheroidal wave functions play a key role in (1–3) in relation with the Riemann zeta function. In all these applications they appear as eigenfunctions of the angle operator between two orthogonal projections in the Hilbert space $L^2(\mathbb{R})^{\rm ev}$ of even square integrable function on $\mathbb{R}$. These projections depend on a parameter $\lambda > 0$, the projection P_λ is given by the multiplication with the characteristic function of the interval $[-\lambda, \lambda] \subset \mathbb{R}$. The projection $\widehat{P_\lambda}$ is its conjugate by the Fourier transform $\mathbb{F}_{e_\mathbb{R}}$ which is the unitary operator in $L^2(\mathbb{R})^{\rm ev}$ defined by

$$\mathbb{F}_{e_\mathbb{R}}(\xi)(y) = \int \xi(x) \exp(-2\pi i x y) dx.$$

In all the above applications of prolate spheroidal wave functions the miraculous existence, discovered* by the Bell Labs group (6–8), of a differential operator W_λ commuting with the angle operator, plays only an auxiliary role. In the present paper we uncover another "miracle": a careful study of the natural selfadjoint extension of W_λ introduced in (9, Lemma 6) (see also (10, §3.3)) to $L^2(\mathbb{R})$ shows that it still has discrete spectrum and that its negative eigenvalues reproduce the ultraviolet behavior of the squares of zeros of the Riemann zeta function. In a similar way the positive spectrum corresponds, in the ultraviolet regime, to the trivial zeros. This coincidence holds for two values $\lambda = 1$ and $\lambda = \sqrt{2}$. The conceptual reason for this coincidence is the link between the operator

$$(W_\lambda \xi)(x) = -\partial_x(\lambda^2 - x^2)\partial_x \xi(x) + (2\pi\lambda)^2 x^2 \xi(x) \qquad [1]$$

and the square of the scaling operator $S := x\partial_x$. In (2) the compression of $f(S)$ to Sonin's space (which consists of functions $f \in L^2(\mathbb{R})$ such that $P_\lambda(f) = 0 = P_\lambda(\mathbb{F}_{e_\mathbb{R}} f)$), was shown to be (for $\lambda = 1$) the root of Weil's positivity at the archimedean place on test functions supported in the interval $[2^{-1/2}, 2^{1/2}]$ but, since Sonin's space is not preserved by scaling, one could not restrict scaling to this space. It turns out that W_λ commutes with the orthogonal projection on Sonin's space. Thus one can restrict W_λ to Sonin's space and the ultraviolet spectral similarity with the squares of non-trivial zeros of zeta suggests that one has spectrally captured the contribution of the archimedean place to the mysterious zeta spectrum. In fact using the Darboux process we construct an isospectral family of Dirac square-root operators of W_λ depending on a deformation parameter, whose spectrum has the same ultraviolet behavior as the zeros of the Riemann zeta function.

Our paper is organized as follows: In Section 1 we show that there exists a unique selfadjoint extension $W_{\rm sa}$ of the symmetric operator $W_{\rm min}$ defined on Schwartz space $\mathcal{S}(\mathbb{R})$ by Eq. (1). Moreover $W_{\rm sa}$ commutes with Fourier transform and has discrete spectrum unbounded in both directions. In Section 2 we show that the eigenvectors for negative eigenvalues of $W_{\rm sa}$ belong to Sonin's space. In Section 3 we compute the semiclassical approximation to the number of negative eigenvalues of $W_{\rm sa}$ whose absolute value is less than E^2. In Section 4, we use the Darboux method combined with solutions of a Riccati equation to construct an isospectral family of Dirac operators $\not{D}$ whose squares are direct sums of two copies of $W_{\rm sa}$; in particular their spectrum is discrete and contained in $\mathbb{R} \cup i\mathbb{R}$, with imaginary eigenvalues symmetric under complex conjugation. In Section 5 we specialize to the case $\lambda = \sqrt{2}$ and show that function $N(E)$, counting the eigenvalues of positive imaginary part less than E of the operator $2\not{D}$, fulfills the same asymptotic law as the Riemann formula

$$N(E) \sim \frac{E}{2\pi}\left(\log\left(\frac{E}{2\pi}\right) - 1\right) + O(\log E). \qquad [2]$$

We also show numerical evidence for the ultraviolet spectral similarity between the eigenvalues of $2\not{D}$ and the zeros of the Riemann zeta function. Section 6 contains more speculative final remarks, in particular on a natural two-dimensional black hole geometry intrinsically related to the operator $2\not{D}$.

Significance Statement

We show that the eigenvalues of the selfadjoint extension (introduced by the first author in 1998) of the prolate spheroidal operator reproduce the ultraviolet behavior of the squares of zeros of the Riemann zeta function and we construct an isospectral family of Dirac operators whose spectra have the same ultraviolet behavior as those zeros.

*As pointed out in (4) this discovery can be traced back to the work of Bateman in 1907 (5)

[1]alain@connes.org

1. The selfadjoint prolate wave operator

The prolate spheroidal operator Eq. (1) is an operator of Sturm-Liouville type,

$$(W_\lambda\xi)(x) = \partial_x\big(p(x)\partial_x\xi(x)\big) + q(x)\xi(x), \qquad x \in \mathbb{R}$$
$$\text{where} \qquad p(x) = x^2 - \lambda^2, \quad q(x) = (2\pi\lambda)^2x^2, \qquad [3]$$

but having two interior singular points it is not directly treatable by the usual Sturm-Liouville theory. However its restrictions to each of the intervals $(-\infty, -\lambda)$, $(-\lambda, \lambda)$ and (λ, ∞) are standard, in fact quasi-regular, Sturm-Liouville operators.

Henceforth W_λ will be simply denoted W whenever λ is a general parameter. To begin with, we regard W as an unbounded operator on $L^2(\mathbb{R})$ with core the Schwartz space $\mathcal{S}(\mathbb{R})$. As such, W is real, symmetric and invariant under the parity exchange $x \mapsto -x$. These features are inherited by its closure in the graph norm $W_{\min}$, as well as by $W_{\max} = W_{\min}{}^*$, the latter having domain

$$\mathrm{Dom}(W_{\max}) = \{\xi \in L^2(\mathbb{R}) \mid W\xi \in L^2(\mathbb{R})\}, \qquad [4]$$

with $W\xi$ viewed as a tempered distribution. In addition W has the remarkable property of commuting with the Fourier transform

$$\mathbb{F}_{e_\mathbb{R}}(f)(y) := \int_{-\infty}^{\infty} f(x)\exp(-2\pi ixy)dx. \qquad [5]$$

Since both the Schwartz space $\mathcal{S}(\mathbb{R})$ and its dual are globally invariant under the Fourier transform, the domains $\mathrm{Dom}W_{\min}$ and $\mathrm{Dom}W_{\max}$ are invariant too, therefore both $W_{\min}$ and $W_{\max}$ commute with $\mathbb{F}_{e_\mathbb{R}}$.

Lemma 1.1 *The deficiency indices of* $W_{\min}$ *are* $(4, 4)$.

Any $\xi \in \mathrm{Dom}(W_{\max})$ satisfying $W\xi = \pm i\xi$ is a piecewise real analytic function and is uniquely specified by six parameters in the complement of the two regular singular points $\pm\lambda$. The known form of the solutions (*cf.* (11)) together with the fact that $W\xi \in L^2(\mathbb{R})$ imply that the logarithmic singularities of ξ on the left and the right of $\pm\lambda$ have to match. This reduces the number of parameters to 4. Conversely, since all 4 singular points are LC (limit circle case), any solution of $W\xi = \pm i\xi$ belongs to $\mathrm{Dom}(W_{\max})$, hence $\dim \mathrm{Ker}(W_{\max} \pm iI) = 4$.

Lemma 1.2 *Let* $\xi \in \mathrm{Dom}W_{\max}$ *and denote* $a = \pm\lambda$. *The distribution* $p(x)\partial_x\xi$ *coincides with a continuous function* f *in a neighborhood of* a *and the evaluation map* $L(\xi) := f(a)$ *defines a non-zero continuous linear form on* $\mathrm{Dom}W_{\max}$ *which vanishes on the closed subspace* $\mathrm{Dom}W_{\min}$.

Let ψ be the distribution $\psi = p(x)\partial_x\xi(x)$. Since $\xi \in \mathrm{Dom}W_{\max}$ the derivative of ψ in the sense of distributions belongs locally to L^2 and hence locally to L^1. It follows that ψ coincides with the integral of an L^1 function and hence with a continuous function f. The evaluation $L(\xi) := f(a)$ is by construction continuous in the graph norm of $\mathrm{Dom}W_{\max}$. For $\xi \in \mathcal{S}(\mathbb{R})$ the distribution $\psi = p(x)\partial_x\xi(x)$ is a function vanishing at $x = a$ and thus $L(\xi) = 0$. By the density of $\mathcal{S}(\mathbb{R})$ in $\mathrm{Dom}W_{\min}$ for the graph norm, it follows that L vanishes on the closed subspace $\mathrm{Dom}W_{\min}$.

Let P_λ be the cutoff projection associated to the interval $[-\lambda, \lambda]$, *i.e.* the multiplication operator by the characteristic function $1_{[-\lambda,\lambda]}$, and let $\widehat{P}_\lambda = \mathbb{F}_{e_\mathbb{R}}P_\lambda\mathbb{F}_{e_\mathbb{R}}^{-1}$ denote its conjugate by the Fourier transform.

Lemma 1.3 *If* $\xi \in \mathrm{Dom}W_{\min}$ *then* $P_\lambda\xi \in \mathrm{Dom}W_{\max}$ *and* $WP_\lambda\xi = P_\lambda W\xi$. *The same holds with respect to* $\widehat{P}_\lambda$.

Let $f \in C^\infty(V)$, where V is a neighborhood of the interval $[-\lambda, \lambda]$. Then $P_\lambda f \in \mathrm{Dom}W_{\max}$ and viewing $W(P_\lambda f)$ as distribution one gets, for any $\phi \in \mathcal{S}(\mathbb{R})$

$$\langle W(P_\lambda f), \phi\rangle = \int_{-\lambda}^{\lambda} f(x)(W\phi)(x)dx =$$
$$\int_{-\lambda}^{\lambda} -f(x)\partial_x(\lambda^2 - x^2)\partial_x\phi(x)dx + \int_{-\lambda}^{\lambda} f(x)(2\pi\lambda)^2x^2\phi(x)dx.$$

Using twice integration by parts, together with the fact that $(\lambda^2 - x^2)\phi'(x)$ and $(\lambda^2 - x^2)f'(x)$ vanish on the boundary, one obtains

$$\langle W(P_\lambda f), \phi\rangle =$$
$$\int_{-\lambda}^{\lambda} f'(x)((\lambda^2 - x^2)\phi')(x)dx + \int_{-\lambda}^{\lambda} f(x)(2\pi\lambda)^2x^2\phi(x)dx =$$
$$-\int_{-\lambda}^{\lambda}(\partial_x((\lambda^2 - x^2)f'(x)))\phi(x)dx + \int_{-\lambda}^{\lambda} f(x)(2\pi\lambda)^2x^2\phi(x)dx$$
$$= \int_{-\lambda}^{\lambda} Wf(x)\phi(x)dx,$$

which shows that $W(P_\lambda f) = P_\lambda Wf$. In particular the same is true for any $f \in \mathcal{S}(\mathbb{R})$, and by the density of $\mathcal{S}(\mathbb{R})$ in $\mathrm{Dom}W_{\min}$ for the graph norm it follows that

$$\xi \in \mathrm{Dom}W_{\min} \implies P_\lambda\xi \in \mathrm{Dom}W_{\max} \text{ and } W_{\max}P_\lambda\xi = P_\lambda W\xi.$$

The claim now follows from the fact that W commutes with $\mathbb{F}_{e_\mathbb{R}}$.

The selfadjoint extensions of $W_{\min}$ are parametrized by self-orthogonal subspaces of $\mathcal{E} := \mathrm{Dom}(W_{\max})/\mathrm{Dom}(W_{\min})$ with respect to the anti-symmetric sesquilinear form given by the pairing

$$\Omega(\xi, \eta) := \frac{1}{i}\Big(\langle W_{\max}\xi \mid \eta\rangle - \langle\xi \mid W_{\max}\eta\rangle\Big), \ \xi, \eta \in \mathrm{Dom}(W_{\max}) \qquad [6]$$

which descends to a non-degenerate form on $\mathcal{E}$.

The Ω-pairing can be expressed in terms of boundary values as usual. One starts with the Lagrange identity

$$\frac{d}{dx}[\xi, \eta] = \xi\, W\eta - \eta\, W\xi, \qquad [7]$$

where $\xi, \eta \in C^1(\mathbb{R}) \cap \mathrm{Dom}W_{\max}$, and

$$[\xi, \eta] := -p\left(\xi\frac{d\eta}{dx} - \eta\frac{d\xi}{dx}\right), \qquad p(x) = x^2 - \lambda^2, \qquad [8]$$

is the (generalized) Wronskian. By integrating it on compact subintervals $[a, b] \subset \mathbb{R} \setminus \{\pm\lambda\}$ one obtains Green's formula

$$\int_a^b (W(\xi)\bar{\eta} - \xi W(\bar{\eta}))(x)dx = [\xi, \bar{\eta}]\big|_a^b \qquad [9]$$
$$:= \lim_{x\to b}[\xi, \bar{\eta}](x) - \lim_{x\to a}[\xi, \bar{\eta}](x).$$

Passage to the lateral limits towards the endpoints of the three subintervals partitioning $\mathbb{R} \setminus \{\pm\lambda\}$ extends this identity to the whole real line, allowing to express Ω in terms of Lagrange brackets as follows:

$$i\Omega(\xi,\eta) = [\xi,\bar\eta]|_{-\infty}^{-\lambda} + [\xi,\bar\eta]|_{-\lambda}^{\lambda} + [\xi,\bar\eta]|_{\lambda}^{\infty} \qquad [10]$$

for all pairs $\xi,\eta \in \mathrm{Dom}W_{\max}$.

Since W is invariant under parity exchange, it preserves the orthogonal decomposition $L^2(\mathbb{R}) = L^2_+(\mathbb{R}) \oplus L^2_-(\mathbb{R})$ into even, resp. odd functions, which in turn induces corresponding splittings $W = W^+ \oplus W^-$, $\Omega = \Omega_+ . \oplus \Omega_-$ and $\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-$. Note also that $\mathcal{E}_\pm$ are invariant under Fourier transform.

The following auxiliary lemma is a straightforward computation.

Lemma 1.4 *(i) Let* $f(x) = \frac{1}{2}\log((\lambda^2 - x^2)^{-2})$ *viewed as a tempered distribution. Then the Fourier transform* $\mathbb{F}_{e_\mathbb{R}} f$ *is a distribution which coincides outside* 0 *with the function*

$$\widetilde{f}(y) = \frac{\cos(2\pi\lambda y)}{|y|}.$$

(ii) Let 1_I *be the characteristic function of the interval* $I = [-\lambda,\lambda]$ *then*

$$\mathbb{F}_{e_\mathbb{R}} 1_I(y) = \frac{\sin(2\pi\lambda y)}{\pi y}.$$

We now proceed to construct a basis of $\mathcal{E}$. First, for $\mathcal{E}_+$ we pick an even function $\alpha_+ \in C_c^\infty(\mathbb{R})$ such that $\alpha_+(x) = \log|\lambda^2 - x^2|$ for $x \in [\frac{3}{4}\lambda, \frac{5}{4}\lambda]$ and with support in $(\frac{1}{2}\lambda, \frac{3}{2}\lambda)$. Then we take $\beta_+(x) = 1_I$, the characteristic function of the interval $I = [-\lambda,\lambda]$, which belongs to $P_\lambda\mathcal{S}(\mathbb{R})$ and hence to $\mathrm{Dom}W_{\max}$. Next for $\mathcal{E}_-$ we let $\alpha_-(x) := x\alpha_+(x)$ and $\beta_-(x) := x\beta_+(x)$.

Lemma 1.5 *The quadruplet* $\{\alpha_\pm, \beta_\pm, \widehat{\alpha}_\pm, \widehat{\beta}_\pm\}$ *forms a basis of* $\mathcal{E}_\pm$.

One checks using the expression Eq. (10) of the Ω-pairing together with Lemma 1.4 that the matrix representation of Ω_+ with respect to the given quadruplet has a single nonzero entry in each row and column.

In the odd case the calculation is similar.

The Ω-pairings with the above basis elements yield boundary conditions of Sturm-Liouville type. Denoting, for $\xi \in \mathrm{Dom}(W^\pm_{\max})$,

$$\begin{aligned} \mathbb{L}_{\alpha_\pm}(\xi) &:= i\Omega_\pm(\xi,\alpha_+), & \mathbb{L}_{\widehat{\alpha}_\pm}(\xi) &:= i\Omega_\pm(\xi,\widehat{\alpha}_+), \\ \mathbb{L}_{\beta_\pm}(\xi) &:= i\Omega_\pm(\xi,\beta_+), & \mathbb{L}_{\widehat{\beta}_\pm}(\xi) &= i\Omega_\pm(\xi,\widehat{\beta}_-), \end{aligned} \qquad [11]$$

the minimal domains are characterized in these terms as being the intersection

$$\mathrm{Dom}(W^\pm_{\min}) = \mathrm{Ker}\,\mathbb{L}_{\alpha_\pm} \cap \mathrm{Ker}\,\mathbb{L}_{\beta_\pm} \cap \mathrm{Ker}\,\mathbb{L}_{\widehat{\alpha}_\pm} \cap \mathrm{Ker}\,\mathbb{L}_{\widehat{\beta}_\pm} \qquad [12]$$

and the induced functionals on $\mathcal{E}_\pm = \mathrm{Dom}(W^+_{\max})/\mathrm{Dom}(W^\pm_{\min})$ form a basis of $\mathcal{E}^*_\pm$.

By straightforward calculation, using the fact that one can always restrict the computation to $\mathbb{R}^+$, one obtains explicit expressions for the boundary functionals. Up to a nonzero constant factor they are as follows. In the even case,

$$\begin{aligned} \mathbb{L}_{\alpha_+}(\xi) &= \lim_{x\nearrow\lambda}((x-\lambda)\log(\lambda-x)\partial_x\xi(x) - \xi(x)) \\ &\quad - \lim_{x\searrow\lambda}((x-\lambda)\log(x-\lambda)\partial_x\xi(x) - \xi(x)); \\ \mathbb{L}_{\beta_+}(\xi) &:= \lim_{x\nearrow\lambda}((\lambda-x)\partial_x\xi(x)) = \lim_{x\searrow\lambda}((\lambda-x)\partial_x\xi(x)) \end{aligned} \qquad [13]$$

$$\begin{aligned} \mathbb{L}_{\widehat{\alpha}_+}(\xi) &:= \frac{2}{\pi}\lim_{x\to\infty}\Big(x\cos(2\pi\lambda x)\partial_x\xi(x) + \Big(2\pi\lambda x\sin(2\pi\lambda x) + \cos(2\pi\lambda x)\Big)\xi(x)\Big); \\ \mathbb{L}_{\widehat{\beta}_+}(\xi) &:= -\frac{2}{\pi}\lim_{x\to\infty}\Big(x\sin(2\pi\lambda x)\partial_x\xi(x) - \Big(2\pi\lambda x\cos(2\pi\lambda x) - \sin(2\pi\lambda x)\Big)\xi(x)\Big). \end{aligned} \qquad [14]$$

We note that the existence of the limit defining $\mathbb{L}_{\beta_+}(\xi)$, *i.e.* the equality of the lateral limits, is ensured by Lemma 1.2.

Similar formulas define the functionals $\mathbb{L}_{\alpha_-}, \mathbb{L}_{\beta_-}, \mathbb{L}_{\widehat{\alpha}_-}, \mathbb{L}_{\widehat{\beta}_-}$ in the odd case.

Since both $\mathrm{Dom}(W_{\min})$ and $\mathrm{Dom}(W_{\max})$, as well as the symplectic form Ω, are globally invariant under the Fourier transform, the quotient inherits induced transformations $f^\pm_{e_\mathbb{R}} : \mathcal{E}_\pm \to \mathcal{E}_\pm$ which relates the boundary functionals as follows:

$$\mathbb{L}_{\widehat{\beta}_\pm} = \mathbb{L}_{\beta_\pm} \circ f_{e_\mathbb{R}} \quad \text{and} \quad \mathbb{L}_{\widehat{\alpha}_\pm} = \mathbb{L}_{\alpha_\pm} \circ f_{e_\mathbb{R}}. \qquad [15]$$

This association gives rise to two distinguished self-orthogonal subspaces, namely

$$\mathcal{L}_\beta = \bigcap_\pm \mathrm{Ker}\mathbb{L}_{\beta_\pm} \cap \bigcap_\pm \mathrm{Ker}\mathbb{L}_{\widehat{\beta}_\pm} \; \mathcal{L}_\alpha = \bigcap_\pm \mathrm{Ker}\mathbb{L}_{\alpha_\pm} \cap \bigcap_\pm \mathrm{Ker}\mathbb{L}_{\widehat{\alpha}_\pm} \qquad [16]$$

Definition. We denote by W_{sa} the restriction of the operator $W_{\max}$ to the subspace $\mathcal{L}_\beta = \bigcap_\pm \mathrm{Ker}\mathbb{L}_{\beta_\pm} \cap \bigcap_\pm \mathrm{Ker}\mathbb{L}_{\widehat{\beta}_\pm}$. Explicitly, its domain $\mathrm{Dom}W_{\mathrm{sa}}$ consists of the elements $\xi \in \mathrm{Dom}(W_{\max})$ satisfying the following boundary conditions:

$$\lim_{x\to\pm\lambda}(\lambda^2 - x^2)\partial_x\xi(x) = 0, \qquad [17]$$

and at $\pm\infty$, writing $\xi = \xi^+ + \xi^-$ with $\xi^\pm \in \mathrm{Dom}(W^\pm_{\max})$,

$$\lim_{x\to\pm\infty}\Big(x\sin(2\pi\lambda x)\partial_x\xi^+(x) - \Big(2\pi\lambda x\cos(2\pi\lambda x) - \sin(2\pi\lambda x)\Big)\xi^+(x)\Big) = 0, \qquad [18]$$

$$\lim_{x\to\pm\infty}\Big(x\cos(2\pi\lambda x)\partial_x\xi^-(x) + \Big(2\pi\lambda x\sin(2\pi\lambda x) + \cos(2\pi\lambda x)\Big)\xi^-(x)\Big) = 0. \qquad [19]$$

We are now in a position to establish the main result of this section.

Theorem 1.6 *(i)* W_{sa} *is selfadjoint and commutes with the Fourier transform.*
(ii) W_{sa} *commutes with the projections* P_λ *and* $\widehat{P}_\lambda$.
(iii) W_{sa} *is the only selfadjoint extension of* $W_{\min}$ *commuting with* P_λ *and* $\widehat{P}_\lambda$. *(iv) The spectrum of* W_{sa} *is discrete and unbounded on both sides; its negative eigenvalues are simple, while the positive eigenvalues (with possibly finitely many exceptions) have multiplicity* 2.

(*i*) $W_{\rm sa}$ is selfadjoint by construction, and its domain $\mathcal{L}_\beta$ is invariant under the Fourier transform also by construction.

(*ii*) Since $\mathrm{Dom}W_{\min}$ is given by Eq. (12). every element of $\mathcal{L}_\beta$ is a linear combination of an element $\xi \in \mathrm{Dom}W_{\min}$ and the 4 vectors $\beta_\pm, \widehat{\beta}_\pm$ of Lemma 1.1. Each $\beta_\pm$ is of the form $P_\lambda f_\pm$ with $f_\pm$ smooth with compact support and thus one has, using Lemma 1.3,

$$P_\lambda \beta_\pm = \beta_\pm \in \mathcal{S}, \quad W_{\rm sa} P_\lambda \beta_\pm = W_{\rm sa} P_\lambda f_\pm = P_\lambda W f_\pm,$$

which shows that $W_{\rm sa} P_\lambda \beta_\pm = P_\lambda W_{\rm sa} P_\lambda \beta_\pm = P_\lambda W_{\rm sa} \beta_\pm$ giving the required commutation for the $\beta_\pm$.

(*iii*) The domain of a selfadjoint extension of $W_{\min}$ commuting with P_λ and $\widehat{P}_\lambda$ must be contained in $\mathrm{Dom}W_{\max}$ and also contain both $P_\lambda \mathcal{S}(\mathbb{R})$ and $\widehat{P}_\lambda \mathcal{S}(\mathbb{R})$. Thus it must contain $\mathcal{L}_\beta$, and cannot be larger due to selfadjointness.

(*iv*) The operator $P_\lambda W_{\rm sa}$ is selfadjoint and positive on $(-\lambda, \lambda)$ and has simple spectrum. It coincides with the selfadjoint operator extending $W\big|[-\lambda,\lambda]$ which corresponds to the boundary condition Eq. (17) and has simple spectrum consisting of strictly positive eigenvalues (*cf.* (12)). To handle the operator $W'_{\rm sa} = (I - P_\lambda)W_{\rm sa}$ in $\mathcal{H} = (I - P_\lambda)L^2(\mathbb{R})$ we use the orthogonal decomposition $L^2(\mathbb{R}) = L^2_+(\mathbb{R}) \oplus L^2_-(\mathbb{R})$ into even, resp. odd functions, which induces the orthogonal decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ and the splitting $W'_{\rm sa} = {W'_{\rm sa}}^+ \oplus {W'_{\rm sa}}^-$. Under the canonical isomorphisms $\mathcal{H}^\pm \sim L^2(\lambda, \infty)$ (obtained by restriction) the operators ${W'_{\rm sa}}^\pm$ are the selfadjoint extensions of the restriction of W with boundary condition at λ given by Eq. (17) and at infinity by Eq. (18) and Eq. (19) respectively. Thus they are covered by standard results in Sturm-Liouville theory (*cf.* (12–15)). Indeed the endpoints are LC (*limit circle case*) in the Weyl classification (see *e.g.* (14, 15) for relevant definitions and properties), which can be easily checked by using explicit bases of formal solutions for $W\xi - \mu\xi = 0$, $\mu \in \mathbb{C}$, around each singular point (*cf. e.g.* (11, §2)). The endpoint λ is LCNO (*non-oscillatory*), while $+\infty$ is LCO (*oscillatory*) since the prolate spheroidal wave functions (which provide principal solutions around λ) have infinitely many zeros in the neighborhood of $+\infty$ (*cf.* (6–8)). The spectrum of $W'^\pm$ is discrete (*cf.* (13, §§19.2-3)) but not bounded on either side, since ∞ is an LCO endpoint (see (15, Theorem 10.7.1)). Moreover, their spectrum has multiplicity 1 (see (14, Theorem 10.7)). The operators $W'^\pm$ inherit the positive eigenvalues corresponding to the Fourier transforms of the (prolate spheroidal) eigenfunctions of $P_\lambda W_{\rm sa}$ which are eigenvectors for the same positive eigenvalues. Thus the corresponding positive spectrum of the original operator $W_{\rm sa}$ is of multiplicity 2. Note finally that the spectra of the operators ${W'_{\rm sa}}^\pm$ are disjoint because the boundary conditions at ∞ are exclusive of each other.

Corollary 1.7 *If ϕ is an eigenfunction of ${W_{\rm sa}}^\pm$ then*

(i) *ϕ is regular on $[\lambda, \lambda+\epsilon)$ and on $(\lambda-\epsilon, \lambda]$ for some $\epsilon > 0$, with a possible discontinuity at λ;*

(ii) *the leading term of the asymptotic expansion of ϕ at ∞ is proportional to $\frac{\sin(2\pi\lambda x)}{x}$ if ϕ is even and to $\frac{\cos(2\pi\lambda x)}{x}$ if ϕ is odd.*

This follows from the above characterization Eq. (17), Eq. (18), Eq. (19) of the domain of $W_{\rm sa}$ combined with the known bases of formal solutions for the equation $W\xi = \mu\xi$, $\mu \in \mathbb{R}$, around $\pm\lambda$ and $\pm\infty$ (*cf.* (11)).

2. Sonin space and negative eigenvalues

We translate the requirement that the Fourier transform $\mathbb{F}_{e_\mathbb{R}} f$ of an $f \in \mathrm{Dom}W_{\max}$ has no logarithmic singularity at the singular points into a condition on the asymptotic behavior of f at ∞. For simplicity we only deal with even functions, and for notational convenience take $\lambda = 1$.

We can then find the asymptotic expansion at ∞ using the boundary condition that the leading term there is $\frac{\sin(2\pi\lambda y)}{y}$. We take for simplicity $\lambda = 1$ and use (11) to get for the tentative eigenvector for eigenvalue μ the expansion at ∞

$$\xi_\mu(x) \sim \frac{\sin(2\pi x)}{x} + \frac{(\mu - 4\pi^2)\cos(2\pi x)}{4\pi x^2} + \frac{-\mu^2 + 8\pi^2\mu + 2\mu - 16\pi^4 + 8\pi^2}{32\pi^2 x^3}\sin(2\pi x) + O(x^{-4}).$$

In fact as shown in Proposition 14 of (11), the coefficients of this expansion are directly related to the coefficients of the expansion of the finite solution at λ and taking for simplicity $\lambda = 1$, if the latter is of the form

$$f_\mu(x) = \sum U_n(\mu)(x-1)^n, \quad U_0(\mu) = 1, \; U_1(\mu) = \frac{\mu - 4\pi^2}{2}$$
$$U_2(\mu) = \frac{\mu^2 - 8\pi^2\mu - 2\mu + 16\pi^4 - 8\pi^2}{16}, \; \ldots$$

then the asymptotic series at infinity which governs the solution which has leading term in $\exp(-2\pi i x)/x$ is equal to $v(x)\exp(-2\pi i x)/x$ where

$$v(x) \sim \sum n! U_n(\mu)(2\pi i x)^{-n}$$

When one applies the Borel summation to this series the first step is to replace it by its Borel transform which is, up to normalization,

$$B(y) := \sum U_n(\mu) y^n$$

and is related to $v(x)$ by $\int_0^\infty t^n \exp(-zt)\,dt = z^{-n-1}\Gamma(n+1)$ *i.e.* the Laplace transform

$$\frac{v(x)}{2\pi i x} = \int_0^\infty \exp(-2\pi i x t) B(t) dt$$

Lemma 2.1 *For any $\mu \in \mathbb{R}$ the asymptotic expansion of the unique solution ξ_μ which at ∞ is asymptotically $\sim -\frac{\sin(2\pi x)}{\pi x}$ is Borel summable and is equal to the Fourier transform of the unique even solution ϕ_μ which is zero on $[-1,1]$ and agrees with $f_\mu(x)$ for $x > 1$.*

One has the equality

$$\frac{v(x)}{2\pi i x} = \int_0^\infty \exp(-2\pi i x t) B(t) dt =$$

$$\int_0^\infty \exp(-2\pi i x t) f_\mu(t+1) dt = \int_1^\infty \exp(-2\pi i x(y-1)) f_\mu(y) dy$$

Thus one gets

$$v(x)\exp(-2\pi i x)/(2\pi i x) = \int_1^\infty \exp(-2\pi i x y) f_\mu(y) dy$$

The function ϕ_μ is even and vanishes on $[-1,1]$ so

$$\int_{-\infty}^\infty \exp(-2\pi i x y)\phi_\mu(y) dy =$$

$$= \int_1^\infty \exp(-2\pi ixy) f_\mu(y) dy + \overline{\int_1^\infty \exp(-2\pi ixy) f_\mu(y) dy} =$$

$$= v(x)\exp(-2\pi ix)/(2\pi ix) + \overline{v(x)\exp(-2\pi ix)/(2\pi ix)}$$

Now these two terms are asymptotic solutions since μ is real and $v(x)\exp(-2\pi ix)/(2\pi ix)$ is an asymptotic solution. Moreover the leading behavior at ∞ is in

$$\exp(-2\pi ix)/(2\pi ix) - \exp(2\pi ix)/(2\pi ix) = -\frac{\sin(2\pi x)}{\pi x}$$

Thus it follows that the Fourier transform $\mathbb{F}_{e_\mathbb{R}} \phi_\mu = \xi_\mu$.

Corollary 2.2 *With the above notation, assume μ is a negative eigenvalue. Then ϕ_μ belongs to the Sonin space.*

The Sonin space coincides with the orthogonal of the eigenspaces of $W_{\rm sa}$ associated to the classical prolate functions and their Fourier transforms. We should note that at this point we do not claim (although this is supported by numerical evidence) that all eigenvalues of the restriction of $W_{\rm sa}$ to Sonin's space are negative, however there could be only finitely many exceptions (see §B).

3. Semiclassical approximation and counting function

In this section we use the semiclassical estimate for the function counting the number of eigenvalues and investigate the negative eigenvalues of the operator $W_{\rm sa}$. We consider the classical Hamiltonian [†]

$$H_\lambda(p,q) = (p^2 - \lambda^2)(q^2 - \lambda^2) \qquad [20]$$

and use it as a semiclassical approximation of $W_{\rm sa}$ via the formal relation

$$W_\lambda \sim -4\pi^2 H_\lambda + 4\pi^2 \lambda^4 \qquad [21]$$

using the correspondence $q \to x$ and $p \to \frac{1}{2\pi i}\partial_x$ associated to the choice of the Fourier transform $\mathbb{F}_{e_\mathbb{R}}$. Sonin's space corresponds to the conditions $p^2 - \lambda^2 \geq 0$ and $q^2 - \lambda^2 \geq 0$ and the region of interest for the counting of eigenvalues is thus

$$\Omega_\lambda(E) := \{(q,p) \mid q \geq \lambda, p \geq \lambda, H_\lambda(p,q) \leq \left(\frac{E}{2\pi}\right)^2 + \lambda^4\}$$

The area of $\Omega_\lambda(E)$ is given, with $a = \left(\frac{E}{2\pi}\right)^2 + \lambda^4$, by the convergent integral

$$I_\lambda(a) = \int_\lambda^\infty \left(\frac{\sqrt{a + \lambda^2 x^2 - \lambda^4}}{\sqrt{x^2 - \lambda^2}} - \lambda \right) dx$$

One obtains, by a change of variables, the equality

$$I_\lambda(a) = \lambda^2 I_1(a\,\lambda^{-4}) \qquad [22]$$

We recall that the elliptic integrals $E(m)$ and $K(m)$ are defined by

$$E(m) := \int_0^{\pi/2} \sqrt{1 - m\,\sin^2\theta}\, d\theta,$$

$$K(m) := \int_0^{\pi/2} \frac{1}{\sqrt{1 - m\,\sin^2\theta}}\, d\theta$$

† using the standard position-momentum (q, p) notation for the phase space

Lemma 3.1 *The integral $I(a) = I_1(a)$ is given by the sum of elliptic integrals*

$$I(a) = aK(1-a) - E(1-a) + 1 \qquad [23]$$

This follows from a straightforward computation. We thus get

Proposition 3.2 *The semiclassical approximation to the number of negative eigenvalues ξ of $W_{\rm sa}$ with $-\xi \leq E^2$ on even functions is the same as on odd functions and is equal to $2\sigma(E,\lambda)$ where*

$$\sigma(E,\lambda) \sim \frac{E}{2\pi}\left(\log\left(\frac{E}{2\pi}\right) - 1 + \log(4) - 2\log(\lambda)\right) + \lambda^2 + o(1) \qquad [24]$$

The semiclassical approximation corresponds, for the restriction to even functions (or to odd functions), to twice the area of $\Omega_\lambda(E)$ and hence to $I_\lambda(a) = \lambda^2 I(a\,\lambda^{-4})$, for $a = \left(\frac{E}{2\pi}\right)^2 + \lambda^4$. One has the asymptotic expansion for $a \to \infty$

$$I(a) \sim \frac{1}{2}\sqrt{a}\,(\log(a) - 2 + 4\log(2)) + 1 + \qquad [25]$$

$$+ \frac{1}{8}\sqrt{\frac{1}{a}}(-\log(a) - 4\log(2)) + O\left(\frac{1}{a}\right)$$

so that

$$I_\lambda(a) \sim \frac{1}{2}\sqrt{a}\,(\log(a) - 2 + 4\log(2) - 4\log\lambda) + \lambda^2 + o(1) \qquad [26]$$

We then use the expansions

$$\sqrt{a} = \frac{E}{2\pi} + O(1/E), \quad \log(a) = 2\log\left(\frac{E}{2\pi}\right) + O(1/E^2)$$

and obtain Eq. (24).

4. Dirac operators

The results of Section 3 show that for suitable values of λ the negative spectrum of $W_{\rm sa}$, or equivalently of $W'_{\rm sa} = (I - P_\lambda)W_{\rm sa}$, has the same ultraviolet behavior as the squares of zeros of the Riemann zeta function. Since $W_{\rm sa}$ is a differential operator of second order we liken it to the Klein-Gordon operator and construct the analogue of the Dirac operator. This can be done by means of the Darboux process (see (16), (17), (4)), which allows to factorize $W_{\rm sa}$ as a product of two first order differential operators. It suffices to treat the case of ${W'_{\rm sa}}^+$, which under the canonical isomorphism $\mathcal{H}^+ \cong L^2(\lambda,\infty)$ is identified with the selfadjoint operator given by the restriction of W to (λ,∞) subject to the boundary conditions Eq. (17) and Eq. (18).

Lemma 4.1 *Let $p(x)$ and $q(x)$ be as in* Eq. (3), $\nabla := p^{1/4}\partial_x p^{1/4}$. *Then the following is a Riccati equation ($x \in (\lambda,\infty)$)*

$$p^{1/2}(x)\partial w(x) + w(x)^2 = -q(x) + \left(\frac{p''(x)}{4} - \frac{p'(x)^2}{16p(x)}\right) \qquad [27]$$

Any solution w of this equation gives rise to a factorization

$$W|(\lambda,\infty) = (\nabla + w)(\nabla - w). \qquad [28]$$

Let f be a smooth function on $\mathbb{R}$ and consider the differential operators $T_1 := f\partial_x f$ and $T_2 := \partial_x f^4 \partial_x$. Let us show that $T_1^2 - T_2$ is an operator of order zero: one has

$$T_1^2 = f\partial_x f^2 \partial_x f = -f' f^2 \partial_x f + \partial_x f^3 \partial_x f$$

$$-f' f^2 \partial_x f = -f'^2 f^2 - f' f^3 \partial_x, \quad \partial_x f^3 \partial_x f = \partial_x f^4 \partial_x + \partial_x f^3 f'$$

so that $T_1^2 - T_2$ is the multiplication by $2f'^2 f^2 + f^3 f''$. Applying this for $f(x) = p(x)^{1/4}$ and the conclusion follows using Eq. (27).

The solutions of the equation Eq. (27) can be found by the standard reduction to a Bernoulli equation and are as follows.

Lemma 4.2 *Let u_1, u_2 be two linearly independent real valued solutions of the equation $(Wu)(x) = 0$, $x \in (\lambda, \infty)$.*
(i) For $z \in \mathbb{C}$ the solution $u = u_1 + zu_2$ has no zero in (λ, ∞) if $z \notin \mathbb{R}$ and has infinitely many zeros otherwise.
(ii) All solutions of the Riccati equation Eq. (27) *are of the form*

$$w_z = \frac{\nabla u}{u} \qquad [29]$$

with $u = u_1 + zu_2$ and $z \in \mathbb{C} \setminus \mathbb{R}$.
(iii) The map $z \mapsto w_z$ from $\mathbb{C} \setminus \mathbb{R}$ to the space of solutions of Eq. (27) *is a homeomorphism.*

Proposition 4.3 *Let w be a solution of the Riccati equation* Eq. (27) *and let $\not{D}_w$ be the operator on $\mathcal{H}^+ \oplus \tilde{\mathcal{H}}^+$ (with $\tilde{\mathcal{H}}^+$ another copy of $\mathcal{H}^+$) defined by*

$$\not{D}_w = \begin{pmatrix} 0 & \nabla + w \\ \nabla - w & 0 \end{pmatrix} \qquad [30]$$

with domain $\mathrm{Dom}\not{D}_w :=$

$$\left\{ \begin{pmatrix} \xi \\ \tilde{\xi} \end{pmatrix} ; \xi \in \mathrm{Dom}{W'_{\mathrm{sa}}}^+, (\nabla + w)(\tilde{\xi}) \in \mathrm{Dom}{W'_{\mathrm{sa}}}^+ \right\} \qquad [31]$$

(i) Then

$$\not{D}_w^2 = \begin{pmatrix} {W'_{\mathrm{sa}}}^+ & 0 \\ 0 & {W'_{\mathrm{sa}}}^+ + 2\nabla w \end{pmatrix} \qquad [32]$$

with the diagonal terms isospectral.
(ii) The spectrum of $\not{D}_w$ is $\{\pm\sqrt{\mu} \mid \mu \in \mathrm{Spec}{W'_{\mathrm{sa}}}^+\}$, independently of w.

One uses the factorization Eq. (28), which in particular implies that $\mathrm{Dom}\not{D}_w$ so defined is contained in the domain of the closure of $\not{D}_w^2$. Let (ξ_μ) be an orthonormal basis of eigenfunctions for ${W'_{\mathrm{sa}}}^+$ (indexed by the eigenvalues μ). Using the vectors $\tilde{\xi}_\mu = (\nabla - w)\xi_\mu \in \tilde{\mathcal{H}}^+$ the operator $\not{D}_w$ splits as a direct sum of two by two matrices of the form $\begin{pmatrix} 0 & \mu \\ 1 & 0 \end{pmatrix}$ whose eigenvalues are the $\pm\sqrt{\mu}$. We choose the inner product on $\tilde{\mathcal{H}}^+$ for which the $\mu^{-1/2}\tilde{\xi}_\mu$ form an orthonormal basis.

5. Ultraviolet behavior of spectrum of Dirac

In this section we take $\lambda = \sqrt{2}$, and consider the operator $2\not{D}$ where $\not{D} = \not{D}_w$ is as defined in Proposition 4.3.

Theorem 5.1 *The operator $2\not{D}$ has discrete simple spectrum contained in $\mathbb{R} \cup i\mathbb{R}$. Its imaginary eigenvalues are symmetric under complex conjugation and the counting function $N(E)$ counting those of positive imaginary part less than E fulfills*

$$N(E) \sim \frac{E}{2\pi}\left(\log\left(\frac{E}{2\pi}\right) - 1\right) - \frac{\log E}{2\pi} + O(1) \qquad [33]$$

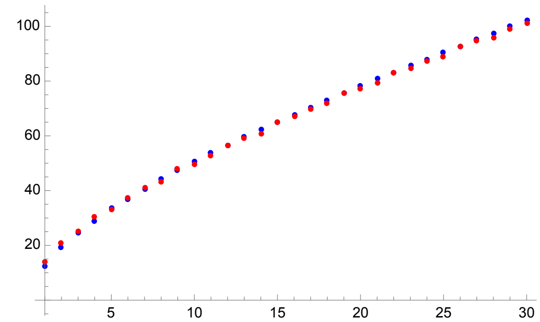

Fig. 1. Plot of the imaginary part of the n-th eigenvalue (in blue) of $2\not{D}$and of the n-th zero of zeta (red). When the red dot hides the blue dot the two values are too close to each other to be distinguished.

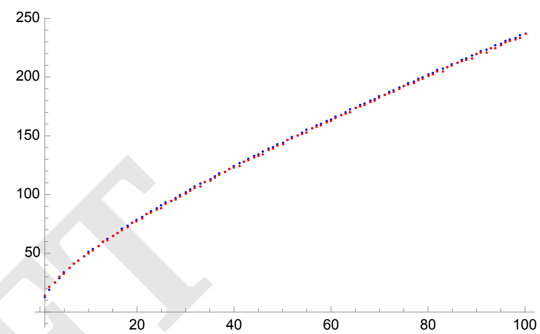

Fig. 2. Comparison for the first 100 eigenvalues

By Proposition 4.3 the spectrum of $2\not{D}$ consists of the complex numbers of the form $\xi = \pm 2\sqrt{\mu}$ where μ varies in the spectrum of ${W'_{\mathrm{sa}}}^+$. The latter is real and to estimate the number of negative eigenvalues of ${W'_{\mathrm{sa}}}^+$ less, in absolute value, than $(E/2)^2$ we apply Heywood's formula (1.4) of (18), which was subsequently extended to a larger class of potentials (see (19, (1.4)) and references therein, also (20, Theorem 1) for a slightly stronger version). A Liouville transformation shows that ${W'_{\mathrm{sa}}}^+$ is unitarily equivalent to the Sturm-Liouville operator S on $(0, \infty)$ where $S(\phi)(y) := \partial_y^2 \phi(y) - q(y)\phi(y)$ and

$$q(y) = -(4\pi)^2 \cosh(y)^2 - \frac{1}{4}\left(\coth^2(y) - 2\right) \qquad [34]$$

A delicate computation which is a refinement of the semiclassical estimate of Proposition 3.2, gives the additional logarithmic contribution $-\frac{\log E}{2\pi}$ of Eq. (33). One can use the computer to obtain the first eigenvalues of $2\not{D}$ (of positive imaginary part) and compare them with the zeros of the Riemann zeta function as shown in Figures 1,2.

6. Final remarks

We gather in this final section a number of more speculative remarks.

A. Geometric meaning of Theorem 5.1. The operator $2\not{D}$ of Theorem 5.1 together with the action by multiplication of smooth functions on the interval $[\sqrt{2}, \infty)$ would define a standard spectral triple if $2\not{D}$ were selfadjoint (or skew adjoint) but its spectrum contains both real and imaginary pieces. Still, the key property that the resolvent is compact is fulfilled and moreover since the leading term $\left(2\sqrt{x^2 - 2}\right)\partial_x$ of $2\not{D}$ is equivalent to $2x\partial_x$ for $x \to \infty$ the algebra of functions having bounded commutator with $2\not{D}$ contains smooth functions which are Lipchitz for the scale invariant metric dx/x. The classical metric associated to $2\not{D}$ is

$$ds^2 = -\frac{1}{4}dx^2/(x^2-2) = \frac{1}{\alpha(x)}dx^2, \quad \alpha(x) = -4(x^2-2)$$

This ds^2 changes sign when crossing the boundary $x = \sqrt{2}$ and this suggests, in order to handle all even functions on $\mathbb{R}$ and also to take into account the real and imaginary eigenvalues of the square of $2\not{D}$, to look for a two dimensional metric with signature $(-1, 1)$ of the form

$$ds^2 = -\alpha(x)dt^2 + \frac{1}{\alpha(x)}dx^2$$

This geometry corresponds to a black hole in two space-time dimensions with horizon at $x = \pm\sqrt{2}$. It fulfils the 2-dimensional analogue of Einstein's equation with a cosmological constant $= 8$ and no source (21).

B. Positive eigenvalues and trivial zeros of Zeta. The operator ${W'_{\rm sa}}^{+}$ admits as positive eigenvalues the eigenvalues $\chi(2n)$, $n \in \mathbb{N}$, of the restriction of $W_{\rm sa}$ to even functions in the interval $[-\lambda, \lambda]$. Moreover by Heywood's formula (1.5) of (18), one can show that the counting function for the positive eigenvalues of ${W'_{\rm sa}}^{+}$ differs only by an $O(1)$ from the counting function for the $\chi(2n)$. This implies that there are at most finitely many positive eigenvalues of ${W'_{\rm sa}}^{+}$ besides the $\chi(2n)$. In fact we conjecture that there are none. The well understood asymptotic form of the eigenvalues $\chi(n)$ ((22) Theorem 3.11) implies that, independently of the value of λ,

$$\chi(2n) = \left(2n + \frac{1}{2}\right)^2 + O(1), \quad n \to \infty.$$

This behavior is the same as that of the squares of the trivial zeros of the Riemann zeta function with the same shift of $\frac{1}{2}$ as for the critical line.

C. Spectral truncation. In order to eliminate the real eigenvalues of $2\not{D}$ coming from the positive eigenvalues of $W_{\rm sa}$ one can effect a spectral truncation (23), the algebra of functions acting by multiplication is then replaced by the operator system obtained by compression on Sonin's space.

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1. A Connes, Trace formula in noncommutative geometry and the zeros of the riemann zeta function. *Sel. Math. (New Series)* **5**, 29–106 (1999).
2. A Connes, C Consani, Weil positivity and trace formula, the archimedean place. *Sel. Math. (New Series)* **27**, xx–xx (2021).
3. A Connes, C Consani, Spectral triples and ζ-cycles. *Sel. Math. (New Series)* **27**, xx–xx (2021).
4. W Casper, FA Grünbaum, M Yakimov, I Zurrián, Reflective prolate spheroidal operators and the kp-kdv equations. *Commun. Pure Appl. Math.* **xx**, xx–xx (xxxx).
5. H Bateman, On the inversion of a definite integral. *Proc. Lond. Math. Soc.* **4**, 461–498 (1907).
6. D Slepian, H Pollack, Prolate spheroidal wave functions, fourier analysis and uncertainty. *Bell Syst. technical J.* **43**, 3009–3057 (1964).
7. D Slepian, Some asymptotic expansions for prolate spheroidal wave functions. *J. Math. Phys.* **44**, 99–140 (1965).
8. D Slepian, Some comments on fourier analysis, uncertainty and modeling. *Siam Rev.* **23**, 379–393 (1983).
9. A Connes, Formules explicites, formules de trace et réalisation spectrale des zéros de la fonction zêta. *N/A* **N/A** (1998).
10. A Connes, M Marcolli, *Noncommutative Geometry, Quantum Fields, and Motives*, "American Mathematical Society Colloquium Publications". (American Mathematical Society) Vol. 55, (2008).
11. F Richard-Jung, JP Ramis, J Thomann, F Fauvet, New characterization for the eigenvalues of the prolate spheroidal wave equation. *Stud. Appl. Math.* **138**, 3–42 (2017).
12. V Katsnelson, Self-adjoint boundary conditions for the prolate spheroid differential operator XXmissing booktitle & seriesXX"Indefinite inner product spaces, Schur analysis, and differential equations". (Springer/Birkhäuser, Cham), pp. 357–386 (2018).
13. MA Naimark, *Linear Differential Operators, Part II.* (Ungar Publishing Company, New York), (1968).
14. J Weidmann, *Spectral theory of ordinary differential operators*, "Lecture Notes in Mathematics". (Springer, Berlin) Vol. 1258, (1987).
15. A Zettl, *Sturm-Liouville Theory*, "Mathematical Surveys and Monographs". (American Mathematical Society) Vol. 121, (2005).
16. PA Deift, Applications of a commutation formula. *Duke Math. J.* **45**, 267–310 (1968).
17. FA Grünbaum, Commuting integral and differential operators and the master symmetries of the korteweg–de vries equation. *Inverse Probl.* **37** (2021).
18. P Heywood, On the asymptotic distribution of eigenvalues. *Proc. Lond. Math. Soc.* **4**, 456–470 (1954).
19. M Nursultanov, G Rozenblum, Eigenvalue asymptotics for the sturm-liouville operator with potential having a strong local negative singularity. *Opuscula Math.* **37**, 109 (2017).
20. FV Atkinson, CT Fulton, Asymptotic formulae for eigenvalues of limit circle problems on a half line. *Annali di Matematica Pura ed Appl.* **135**, 363–398 (1983).
21. RB Mann, A Shiekh, L Tarasov, Classical and quantum properties of two-dimensional black holes. *Nucl. Phys. B341* **N/A**, 134–154 (1990).
22. A Osipov, V Rokhlin, H Xiao, *Prolate spheroidal wave functions of order zero. Mathematical tools for bandlimited approximation*, "Applied Mathematical Sciences". (Springer, New York) Vol. 187, (2013).
23. A Connes, WD van Suijlekom, Spectral truncations in noncommutative geometry and operator systems. *Commun. Math. Phys.* **383**, 2021–2067 (2021).