Rankin-Cohen Brackets and the Hopf Algebra of Transverse Geometry

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Dedicated to Pierre Cartier

Abstract

We settle in this paper a question left open in our paper "Modular Hecke algebras and their Hopf symmetry", by showing how to extend the Rankin-Cohen brackets from modular forms to modular Hecke algebras. More generally, our procedure yields such brackets on any associative algebra endowed with an action of the Hopf algebra of transverse geometry in codimension one, such that the derivation corresponding to the Schwarzian derivative is inner. Moreover, we show in full generality that these Rankin-Cohen brackets give rise to associative deformations.

Introduction

In [1] H. Cohen constructed a collection of bilinear operations on functions on the complex plane, which are covariant with respect to the 'slash' action of $\mathrm{PSL}(2,\mathbb{R})$ and therefore give rise to bi-differential operators on the graded algebra of modular forms. If f and g denote two modular forms of

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weight k and ℓ, the nth *Rankin-Cohen bracket* of f and g is given (using the normalization in [11]) by the formula

$$[f,g]_n := \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+\ell-1}{r} f^{(r)} g^{(s)} . \tag{0.1}$$

Don Zagier later investigated the abstract algebraic structure defined by such bilinear operations [11].

We noticed in [8] that the formulas for the perturbations by inner of the action of the Hopf algebra $\mathcal{H}_1$ of [5] on the modular Hecke algebras were similar to the formulas occuring in Zagier's definition [11] of 'canonical' Rankin-Cohen algebras. This suggested that there should be a close relationship between Rankin-Cohen brackets and actions of the Hopf algebra $\mathcal{H}_1$. We shall show here that this is indeed the case, by exhibiting canonical bilinear operators that generalize the Rankin-Cohen brackets, for any associative algebra $\mathcal{A}$ endowed with an action of the Hopf algebra $\mathcal{H}_1$ of [5] for which the 'Schwarzian' derivation δ_2' of [8] is inner,

$$\delta_2'(a) = \Omega\, a - a\, \Omega\,, \qquad \forall a \in \mathcal{A}\,. \tag{0.2}$$

Such an action of the Hopf algebra $\mathcal{H}_1$ defines the noncommutative analogue of a one-dimensional "projective structure", while Ω plays the role of the quadratic differential.
We had given in [8] the first term in the deformation as (twice) the fundamental class $[F]$ in the cyclic cohomology $PHC^{\mathrm{ev}}_{\mathrm{Hopf}}(\mathcal{H}_1)$, i.e. the class of the cyclic 2-cocycle

$$F := X \otimes Y - Y \otimes X - \delta_1\, Y \otimes Y\,, \tag{0.3}$$

which in the foliation context represents the transverse fundamental class. Since the antipode $S(X)$ is given by

$$S(X) = -X + \delta_1\, Y\,, \tag{0.4}$$

one has

$$-F \;=\; S(X) \otimes Y + Y \otimes X\,,$$

and one could reasonably expect the higher brackets to be increasingly more complicated expressions involving $S(X)$, X, Y and Ω, beginning with the first bracket

$$RC_1(a,b) := S(X)(a)\, 2Y(b) \,+ 2Y(a)\, X(b)\,, \qquad a, b \in \mathcal{A}\,. \tag{0.5}$$

We shall obtain in this paper the canonical formulas for the higher brackets. As an illustration, the second bracket RC_2 is given by

$$\begin{aligned} RC_2(a,b) := \quad & S(X)^2(a)\, Y(2\,Y+1)(b) \,+\, S(X)\,(2\,Y+1)(a)\, X(2\,Y+1)(b) \\ & + Y(2\,Y+1)(a)\, X^2(b) \,-\, Y(a)\,\Omega\, Y(2Y+1)(b) \\ & - Y(2Y+1)(a)\,\Omega\, Y(b) \end{aligned} \tag{0.6}$$

The "quadratic differential" Ω and its higher derivatives $X^j(\Omega)$ always intervene between polynomial expressions $P(S(X),Y)(a)$ and $Q(X,Y)(b)$. We shall also prove (Theorem 8) that when applied to the modular Hecke algebras they yield a family of associative formal deformations, which in particular incorporate the 'tangent groupoid' deformation. This will be achieved by showing that the canonical formula commutes with the crossed product construction under conditions which are realized for the action of Hecke operators on modular forms, and then relying on the results of [2]. The commutation with crossed product will in fact uniquely dictate the general formula. We shall show that the general formula is invariant under inner perturbations of the action of the Hopf algebra. In order to obtain the whole sequence of higher components we shall closely follow the method of [11] and [2].

Passing to the general algebraic context, in the last section we establish (Theorem 10) the associativity of the formal deformations corresponding to the Rankin-Cohen brackets for an arbitrary associative algebra endowed with an action of $\mathcal{H}_1$ such that the derivation corresponding to the Schwarzian derivative is inner.

Contents

1 Zero Quadratic Differential

For the clarity of the exposition, we shall first develop the higher Rankin-Cohen brackets for actions of $\mathcal{H}_1$ in the simplified case when the quadratic differential Ω is zero, i.e. when $\delta_2' = 0$.

We recall that as an algebra $\mathcal{H}_1$ coincides with the universal enveloping algebra of the Lie algebra with basis $\{X, Y, \delta_n \,;\, n \geq 1\}$ and brackets

$$[Y, X] = X \,,\ [Y, \delta_n] = n\, \delta_n \,,\ [X, \delta_n] = \delta_{n+1} \,,\ [\delta_k, \delta_\ell] = 0 \,, \quad n, k, \ell \geq 1 \,,$$

while the coproduct which confers it the Hopf algebra structure is determined by the identities

$$\begin{aligned} \Delta\, Y &= Y \otimes 1 + 1 \otimes Y \,, \quad \Delta\, \delta_1 = \delta_1 \otimes 1 + 1 \otimes \delta_1 \,, \\ \Delta\, X &= X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y \,, \end{aligned}$$

together with the property that $\Delta : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_1$ is an algebra homomorphism.

We let $\mathcal{H}_s$ be the quotient of $\mathcal{H}_1$ by the ideal generated by δ_2'. The basic lemma is to put the antipode $S(X^n)$ in normal order, i.e. with the $\delta' s$ first then the X then the $Y' s$.

Lemma 1. *One has the following identity in* $\mathcal{H}_{\mathrm{s}}$*,*

$$S(X^n) = \sum (-1)^{n-k} \binom{n}{k} \frac{\delta_1^k}{2^k}\, X^{n-k} (2Y + n - k)_k$$

where $(\alpha)_k := \alpha\, (\alpha + 1) .... (\alpha + k - 1)$.

Proof. For $n = 1$ one has $S(X) = -X + \delta_1\, Y$ and the r.h.s. of the formula gives $-X + \dfrac{\delta_1}{2}\, (2Y)$. Let us assume that it holds for n and check it for $n+1$ by multiplication on the left by $S(X) = -X + \delta_1\, Y$. One has $Y\, \dfrac{\delta_1^k}{2^k}\, X^{n-k} = \dfrac{\delta_1^k}{2^k}\, X^{n-k} (Y + n)$ so that the new terms corresponding to $\delta_1\, Y\, S(X^n)$ give

$$\sum (-1)^{n-k} \binom{n}{k} \frac{\delta_1^{k+1}}{2^{k+1}}\, X^{n-k} (2Y + 2n)(2Y + n - k)_k$$

One has $[X,\,\delta_1^k] = \,k\,\frac{\delta_1^{k+1}}{2}$ thus the new terms corresponding to $-X\,S(X^n)$ give

$$\sum(-1)^{n+1-k}\binom{n}{k}\,\frac{\delta_1^k}{2^k}\,X^{n+1-k}(2Y+n-k)_k$$

and

$$\sum(-1)^{n+1-k}\binom{n}{k}\,\frac{\delta_1^{k+1}}{2^{k+1}}\,X^{n-k}\,k\,(2Y+n-k)_k$$

The first of these two expressions can be written as

$$\sum(-1)^{n-k}\binom{n}{k+1}\,\frac{\delta_1^{k+1}}{2^{k+1}}\,X^{n-k}(2Y+n-k-1)_{k+1}$$

thus all the term are right multiples of

$$(-1)^{n-k}\,\frac{\delta_1^{k+1}}{2^{k+1}}\,X^{n-k}\,(2Y+n-k)_k$$

with coefficients

$$\binom{n}{k}\,(2Y+2n)\,-\,\binom{n}{k}\,k+\,\binom{n}{k+1}\,(2Y+n-k-1)$$

but this is the same as

$$\left(\binom{n}{k}\,+\,\binom{n}{k+1}\right)\,(2Y+n)\,+\,(n-k)\,\binom{n}{k}\,-\,(k+1)\,\binom{n}{k+1}$$
$$=\,\binom{n+1}{k+1}\,(2Y+n)$$

$\square$

The content of Lemma 1 can be written using the generating function

$$\Phi(X)(s) := \sum\frac{s^n\,X^n}{n!}\,\Gamma(2Y+n)^{-1} \tag{1.1}$$

as the equality,

$$\Phi(X-Z\,Y)(s) = e^{-\frac{s\,Z}{2}}\,\Phi(X)(s) \tag{1.2}$$

under the only assumption that the operators X, Y, Z fulfill,

$$[Y,X]=X\,,\quad [Y,Z]=Z\,,\quad [X,Z]=\frac{1}{2}Z^2\,, \tag{1.3}$$

Expanding the tensor product

$$\Phi(S(X))(s) \otimes \Phi(X)(s) = \sum s^n \, RC_n^{\mathcal{H}_s} \;\; (\Gamma(2Y+n) \otimes \Gamma(2Y+n))^{-1} \quad (1.4)$$

dictates the following formula for the higher Rankin-Cohen brackets for an action of $\mathcal{H}_s$ on an algebra $\mathcal{A}$,

$$RC_n(x,\, y) := \sum_{k=0}^{n} (\frac{S(X)^k}{k!}\,(2Y+k)_{n-k})(x)(\frac{X^{n-k}}{(n-k)!}\,(2Y+n-k)_k)(y) \quad (1.5)$$

The main preliminary result then is the following.

Lemma 2. *Let the Hopf algebra $\mathcal{H}_1$ act on an algebra $\mathcal{A}$ and $u \in \mathcal{A}$ be invertible and such that,*

$$X(u) = 0,\; Y(u) = 0,\; \delta_1(u^{-1}\delta_1(u)) = 0,\; \delta_2'(u) = 0\,.$$

1^0. *For all values of n,*

$$RC_n(x\,u,\; y) = \; RC_n(x,\; u\,y) \quad \forall x,\, y \in \mathcal{A}$$

2^0. *For all $x,\, y \in \mathcal{A}$,*

$$RC_n(u\,x,\; y) = \; u\, RC_n(x,\;\; y)\,, \quad RC_n(x,\; y\,u) = \; RC_n(x,\;\; y)\,u$$

3^0. *Let α be the inner automorphism implemented by u one has,*

$$RC_n(\alpha(x),\, \alpha(y)) = \; \alpha(RC_n(x,\, y)) \quad \forall x,\, y \in \mathcal{A}$$

Proof. For $x \in \mathcal{A}$ let L_x be the operator of left multiplication by x in $\mathcal{A}$. One has

$$X\, L_u = L_u(\, X - L_\nu\, Y) \quad (1.6)$$

where $\nu := -u^{-1}\delta_1(u) = \delta_1(u^{-1})u$. Moreover

$$Y(\nu) = \nu\,, \quad X(\nu) = \frac{1}{2}\nu^2\,.$$

so that the operators $X, Y, Z := L_\nu$ fulfill (1.3) and by (1.2)

$$\Phi(X - L_\nu\, Y)(s) = L_{e^{-\frac{s\nu}{2}}}\, \Phi(X)(s) \quad (1.7)$$

One has

$$\Delta(S(X)) = S(X) \otimes 1 + 1 \otimes S(X) + Y \otimes \delta_1$$

and using R_x for right multiplication by $x \in \mathcal{A}$,

$$S(X)\, R_u = R_u(\, S(X) - R_{\nu'}\, Y) \tag{1.8}$$

where $\nu' := u\delta_1(u^{-1})$. Moreover $\delta_1(\nu') = 0$ and

$$Y(\nu') = \nu'\,, \quad S(X)(\nu') = \frac{1}{2}(\nu')^2\,,$$

so that $S(X)$, Y, $R_{\nu'}$ fulfill (1.3) and by (1.2)

$$\Phi(S(X) - R_{\nu'}\, Y)(s) = R_{e^{-\frac{s\,\nu'}{2}}}\, \Phi(S(X))(s) \tag{1.9}$$

Since $u\,\nu = \nu'\,u$ we get 1^0 using (1.4) and the commutation of Y with L_u and R_u. Statement 2^0 follows from the commutation of R_u with X and Y, while 3^0 follows from 1^0 and 2^0. $\square$

Let us compute the above brackets for small values of n. We shall express them as elements

$$RC_n^{\mathcal{H}_s} \in \mathcal{H}_s \otimes \mathcal{H}_s\,.$$

For $n = 1$ we get

$$RC_1^{\mathcal{H}_s} \;=\; S(X) \otimes 2Y + 2Y \otimes X$$

which is -2 times the transverse fundamental class F (defined in (0.3)).
For $n = 2$ we get

$$RC_2^{\mathcal{H}_s} = \frac{1}{2}S(X)^2 \otimes (2\,Y)(2\,Y+1) \;+\; S(X)\,(2\,Y+1) \otimes X\,(2\,Y+1)$$
$$+\frac{1}{2}(2\,Y)(2\,Y+1) \;\otimes\; X^2\,.$$

For $n = 3$ we get

$$RC_3^{\mathcal{H}_s} = \frac{1}{6}S(X)^3 \otimes (2\,Y)(2\,Y+1)(2\,Y+2)$$
$$+\frac{1}{2}S(X)^2\,(2\,Y+2) \;\otimes\; X\,(2\,Y+1)(2\,Y+2)$$
$$+\frac{1}{2}S(X)\,(2\,Y+1)\,(2\,Y+2) \;\otimes\; X^2\,(2\,Y+2)$$
$$+\frac{1}{6}(2\,Y)(2\,Y+1)\,(2\,Y+2) \;\otimes\; X^3\,.$$

2 Arbitrary Quadratic Differential

Let us now pass to the general case where we assume that $\mathcal{H}_1$ acts on an algebra $\mathcal{A}$ and that the derivation δ'_2 is inner implemented by an element $\Omega \in \mathcal{A}$ so that,

$$\delta'_2(a) = \Omega\, a - a\, \Omega\,,\ \forall a \in \mathcal{A} \tag{2.1}$$

where we assume, owing to the commutativity of the δ_k,

$$\delta_k(\Omega) = \ 0\,,\ \forall k \in \mathbb{N} \tag{2.2}$$

It follows then that

$$\delta_k(X^j(\Omega)) = \ 0\,,\ \forall k\,,\ j \in \mathbb{N} \tag{2.3}$$

so that by (2.1), Ω commutes with all $X^j(\Omega)$ and the algebra $\mathcal{P} \subset \mathcal{A}$ generated by the $X^j(\Omega)$ is commutative, while both Y and X act as derivations on $\mathcal{P}$.

To understand how to obtain the general formulas we begin by computing in the above case ($\Omega = 0$) how the formulas for RC_n get modified by a perturbation of the action of the form

$$Y \to Y\,, \quad X \to X + \mu\, Y\,, \quad \delta_1 \to \delta_1 + ad(\mu) \tag{2.4}$$

where $Y(\mu) = \mu$ and $\delta_n(\mu) = 0$ for all n. The computation shows that RC_1 is unchanged, while $RC_2(a,b)$ gets modified by the following term,

$$\delta RC_2(a,b) = Y(a)\,\Omega\, Y(2Y+1)(b) + Y(2Y+1)(a)\,\Omega\, Y(b) \tag{2.5}$$

where $\Omega := X(\mu) + \frac{1}{2}\mu^2$. Note that the perturbed action fulfills (2.1) for that value of Ω.

This already indicates that in the general case the full formula for $RC_2(a,b)$ should be

$$\begin{aligned} RC_2(a,b) := \quad & S(X)^2(a)\, Y(2\,Y+1)(b) \,+\, S(X)\,(2\,Y+1)(a)\, X(2\,Y+1)(b) \\ & + Y(2\,Y+1)(a)\, X^2(b) \,-\, Y(a)\,\Omega\, Y(2Y+1)(b) \\ & - Y(2Y+1)(a)\,\Omega\, Y(b) \end{aligned}$$

so that the above perturbation then leaves RC_2 unaffected.

In order to obtain the general formulas, we consider the algebra $\mathcal{L}(\mathcal{A})$ of linear operators in $\mathcal{A}$. For $a \in \mathcal{A}$ we use the short hand notation

$$a := L_a \,, \quad a^o := R_a$$

for the operators of left and right multiplication by a when no confusion can arise.
We define by induction elements $B_n \in \mathcal{L}(\mathcal{A})$ by the equation

$$B_{n+1} := X\, B_n - n\, \Omega\, (Y - \frac{n-1}{2}) B_{n-1} \tag{2.6}$$

while $B_0 := 1$ and $B_1 := X$. The first values for the B_n are the following,

$$B_2 = X^2 - \Omega\, Y$$

$$B_3 = X^3 - \Omega\, X\, (3Y+1) - X(\Omega)\, Y$$

and

$$B_4 = X^4 - \Omega\, X^2\, (6Y+4) - X(\Omega)\, X\, (4Y+1) - X^2(\Omega)\, Y + 3\, \Omega^2\, Y\, (Y+1)$$

Let us define more generally for any two operators Z and Θ acting linearly in $\mathcal{A}$ and fulfilling

$$[Y, Z] = Z \,, \quad [Y, \Theta] = 2\Theta$$

the sequence of operators, $C_0 := 1$, $C_1 := Z$,

$$C_{n+1} := Z\, C_n - n\, \Theta\, (Y - \frac{n-1}{2}) C_{n-1} \tag{2.7}$$

and the series,

$$\Phi(Z, \Theta)(s) := \sum \frac{s^n\, C_n}{n!}\, \Gamma(2Y+n)^{-1}$$

Lemma 3. Φ *is the unique solution of the differential equation*

$$s(\frac{d}{ds})^2 \Phi - 2(Y-1)\, \frac{d}{ds}\Phi + Z\, \Phi - \frac{s}{2}\, \Theta\, \Phi = 0$$

which fulfills the further conditions,

$$\Phi(0) = \Gamma(2Y)^{-1} \,, \quad \frac{d}{ds}\Phi(0) = Z\, \Gamma(2Y+1)^{-1}$$

Proof. One has

$$\frac{d}{ds}\Phi = \sum \frac{s^n\, C_{n+1}}{n!}\,\Gamma(2Y+n+1)^{-1}$$
$$s(\frac{d}{ds})^2\Phi = \sum \frac{s^n\, C_{n+1}}{n!}\,n\,\Gamma(2Y+n+1)^{-1}$$
$$2(Y-1)\,\frac{d}{ds}\Phi = \sum \frac{s^n\, C_{n+1}}{n!}\,(2Y+2n)\,\Gamma(2Y+n+1)^{-1}$$
$$(n-(2Y+2n))\Gamma(2Y+n+1)^{-1} = -\Gamma(2Y+n)^{-1}$$

so that

$$s(\frac{d}{ds})^2\Phi - 2(Y-1)\,\frac{d}{ds}\Phi = -\sum \frac{s^n\, C_{n+1}}{n!}\,\Gamma(2Y+n)^{-1}$$

but by (2.6)

$$C_{n+1} = ZC_n - n\,\Theta\,(Y - \frac{n-1}{2})C_{n-1}$$

the first term gives -$Z\,\Phi$ while the second gives,

$$\Theta\sum \frac{s^n\, C_{n-1}}{n!}\,(n\,(Y + \frac{n-1}{2}))\Gamma(2Y+n)^{-1}$$

which equals

$$\frac{s\,\Theta}{2}\sum \frac{s^n\, C_n}{n!}\,(2Y+n)\Gamma(2Y+n+1)^{-1} = \frac{s\,\Theta}{2}\,\Phi$$

□

Let now μ be an operator in $\mathcal{A}$ such that

$$[\Theta,\,\mu] = 0\,, \quad [Y,\,\mu] = \mu\,, \quad [[Z,\,\mu],\,\mu] = 0 \tag{2.8}$$

we let

$$\Psi(s) := e^{\frac{s\,\mu}{2}}\,\Phi(Z,\,\Theta)(s)$$

Lemma 4. *$\Psi(s)$ satisfies the following differential equation*

$$s(\frac{d}{ds})^2\Psi - 2(Y-1)\,\frac{d}{ds}\Psi + (Z+\mu\,Y)\,\Psi - \frac{s}{2}\,(\Theta + [Z,\,\mu] + \frac{\mu^2}{2})\Psi = 0$$

Proof. One has

$$\frac{d}{ds}\Psi = \frac{\mu}{2}\Psi + e^{\frac{s\mu}{2}}\,\frac{d}{ds}\Phi$$

$$s(\frac{d}{ds})^2\Psi = s\,\frac{\mu^2}{4}\,\Psi + s\,\mu\,e^{\frac{s\mu}{2}}\,\frac{d}{ds}\Phi + e^{\frac{s\mu}{2}}\,s(\frac{d}{ds})^2\Phi$$

$$2(Y-1)\,\frac{d}{ds}\Psi = \mu\,Y\,\Psi + s\,\mu\,e^{\frac{s\mu}{2}}\,\frac{d}{ds}\Phi + e^{\frac{s\mu}{2}}\,2(Y-1)\,\frac{d}{ds}\Phi$$

where for the last equality we used $[Y,\,\mu] = \mu$ to get

$$[2Y,\,e^{\frac{s\mu}{2}}] = s\,\mu\,e^{\frac{s\mu}{2}}$$

thus,

$$(s(\frac{d}{ds})^2 - 2(Y-1)\,\frac{d}{ds})\Psi = s\,\frac{\mu^2}{4}\,\Psi - \mu\,Y\,\Psi + e^{\frac{s\mu}{2}}\,(s(\frac{d}{ds})^2 - 2(Y-1)\,\frac{d}{ds})\Phi$$

but

$$(s(\frac{d}{ds})^2 - 2(Y-1)\,\frac{d}{ds})\Phi = -Z\,\Phi + \frac{s}{2}\,\Theta\,\Phi$$

and by (2.8) one has,

$$[Z,\,e^{\frac{s\mu}{2}}] = \frac{s}{2}\,[Z,\,\mu]\,e^{\frac{s\mu}{2}}$$

so that

$$e^{\frac{s\mu}{2}}\,(-Z) = (-Z)\,e^{\frac{s\mu}{2}} + \frac{s}{2}\,[Z,\,\mu]\,e^{\frac{s\mu}{2}}$$

and since μ commutes with Θ,

$$(s(\frac{d}{ds})^2 - 2(Y-1)\,\frac{d}{ds})\Psi = s\,\frac{\mu^2}{4}\,\Psi - \mu\,Y\,\Psi - Z\,\Psi + \frac{s}{2}\,[Z,\,\mu]\,\Psi + \frac{s}{2}\,\Theta\,\Psi$$

□

Note also that

$$\Psi(0) = \Gamma(2Y)^{-1}\,, \quad \frac{d}{ds}\Psi(0) = (Z + \mu\,Y)\,\Gamma(2Y+1)^{-1}$$

It thus follows that the following holds.

Proposition 5. *Let μ fulfill conditions (2.8), then*

$$\Phi(Z + \mu\,Y,\ \Theta + [Z,\,\mu] + \frac{\mu^2}{2})(s) = e^{\frac{s\mu}{2}}\,\Phi(Z,\Theta)(s)$$

One has by construction

$$\Phi(X,\Omega)(s) = \sum \frac{s^n\, B_n}{n!}\, \Gamma(2Y+n)^{-1}$$

and similarly,

$$\Phi(S(X),\Omega^o)(s) = \sum \frac{s^n\, A_n}{n!}\, \Gamma(2Y+n)^{-1}$$

where the A_n are obtained by induction using,

$$A_{n+1} := S(X)\, A_n - n\, \Omega^o\, (Y - \frac{n-1}{2}) A_{n-1} \tag{2.9}$$

while $A_{-1} := 0$, $A_0 := 1$.
The first values of A_n are

$$A_1 = S(X)$$
$$A_2 = S(X)^2 - \Omega^o\, Y$$
$$A_3 = S(X)^3 - \Omega^o\, S(X)\,(3Y+1) + X(\Omega)^o\, Y$$

and

$$\begin{aligned} A_4 = {} & S(X)^4 - \Omega^o\, S(X)^2\,(6Y+4) + X(\Omega)^o\, S(X)\,(4Y+1) \\ & - X^2(\Omega)^o\, Y + 3\,(\Omega^o)^2\, Y\,(Y+1) \end{aligned}$$

The general formula for RC_n is obtained as in (1.4) by expanding the product

$$\Phi(S(X),\Omega^o)(s)(a)\, \Phi(X,\Omega)(s)(b)$$

which gives,

$$RC_n(a,b) := \sum_{k=0}^{n} \frac{A_k}{k!}\, (2Y+k)_{n-k}(a)\; \frac{B_{n-k}}{(n-k)!}\, (2Y+n-k)_k(b) \tag{2.10}$$

Lemma 6. *Let $\gamma \in Z^1(\mathcal{H}_1, \mathcal{A})$ be a 1-cocycle such that $\gamma(X) = \gamma(Y) = 0$, $\delta_k(\gamma(h)) = 0$ $\forall h \in \mathcal{H}_1$, $k \in \mathbb{N}$. Then the brackets RC_n are invariant under the inner perturbation of the action of $\mathcal{H}_1$ associated to γ.*

Proof. Let $\mu := \gamma(\delta_1)$. One has $\delta_k(\mu) = 0 \ \forall k \in \mathbb{N}$. The cocycle law

$$\gamma(h\,h') = \sum \gamma(h_{(1)})\, h_{(2)}(\gamma(h'))\,, \quad \forall h \in \mathcal{H}\,. \tag{2.11}$$

shows that $\gamma(\delta_2) = X(\mu) + \mu^2$ and $\gamma(\delta_2') = X(\mu) + \frac{1}{2}\mu^2$, which using $[\delta_1,\, \delta_2'] = 0$ gives $[\mu,\, X(\mu)] = 0$. We then get

$$[\Omega,\, \mu] = 0\,, \quad Y(\mu) = \mu\,, \quad [\mu,\, X(\mu)] = 0 \tag{2.12}$$

The effect of the perturbation on the generators is

$$\begin{aligned} &Y \to Y\,, \quad X \to X + L_\mu Y\,, \quad \delta_1 \to \delta_1 + L_\mu - R_\mu\,, \\ &\Omega \to \Omega + X(\mu) + \frac{\mu^2}{2} \end{aligned} \tag{2.13}$$

where L_μ is the operator of left multiplication by μ and R_μ is right multiplication by μ.
We first apply Proposition 5 to the operator L_μ of left multiplication by μ, and get, using (2.12) to check (2.8),

$$\Phi(X + L_\mu Y,\ \Omega + X(\mu) + \frac{\mu^2}{2})(s) = L_{e^{\frac{s\mu}{2}}}\, \Phi(X, \Omega)(s) \tag{2.14}$$

The effect of the perturbation on $S(X)$ is

$$S(X) \to S(X) - R_\mu Y, \tag{2.15}$$

where R_μ is right multiplication by μ.
One has

$$\Phi(S(X), \Omega^o)(s) = \sum \frac{s^n A_n}{n!}\, \Gamma(2Y + n)^{-1}$$

We now apply Proposition 5 to the operator $-R_\mu$ of right multiplication by $-\mu$, using $S(X)(\mu) = -X(\mu)$ and (2.12) to check (2.8) for the operators $S(X)$, Y, $\Omega^o = R_\Omega$, $-R_\mu$ and get,

$$\Phi(S(X) - R_\mu Y,\ (\Omega + X(\mu) + \frac{\mu^2}{2})^o)(s) = R_{e^{-\frac{s\mu}{2}}}\, \Phi(S(X), \Omega)(s) \tag{2.16}$$

combining (2.14) and (2.16) shows that the product

$$\Phi(S(X), \Omega^o)(s)(a)\, \Phi(X, \Omega)(s)(b)$$

is unaltered by the perturbation and gives the required invariance. ∎

Lemma 7. *Let the Hopf algebra* $\mathcal{H}_1$ *act on an algebra* $\mathcal{A}$ *with* δ_2' *inner as above and* $u \in \mathcal{A}$ *be invertible and such that with* $\mu = u^{-1}\delta_1(u)$,

$$X(u) = 0,\, Y(u) = 0,\, \delta_n(\mu) = 0,\, \forall\, n \in \mathbb{N}$$

1^0. *For all values of* n,

$$RC_n(x\,u,\, y) = RC_n(x,\, u\,y) \quad \forall x,\, y \in \mathcal{A}$$

2^0. *For all* $x,\, y \in \mathcal{A}$,

$$RC_n(u\,x,\, y) = u\,RC_n(x,\ y)\,, \quad RC_n(x,\, y\,u) = RC_n(x,\ y)\,u$$

3^0. *Let* α *be the inner automorphism implemented by* u *one has,*

$$RC_n(\alpha(x),\, \alpha(y)) = \alpha(RC_n(x,\, y)) \quad \forall x,\, y \in \mathcal{A}$$

Proof. One has $Y(\mu) = \mu$, let us show that

$$[X(\mu),\, \mu] = 0 \tag{2.17}$$

One has $\delta_2(u) = [X,\, \delta_1](u) = X(\delta_1(u)) = X(u\,\mu) = u\,(X(\mu) + \mu^2)$, so that,

$$\delta_2'(u) = u\,\rho\,, \quad \rho := X(\mu) + \frac{1}{2}\mu^2 \tag{2.18}$$

Since $\delta_1(X(\mu)) = -[X,\, \delta_1](\mu) = -\delta_2(\mu) = 0$, one has $\delta_1(\rho) = 0$. The commutation $[\delta_2',\, \delta_1]\,(u) = 0$ then entails

$$u\,\rho\,\mu = u\,\mu\,\rho\,, \quad [\mu,\, \rho] = 0$$

which implies (2.17). Then as in (1.6)

$$X\,L_u = L_u(\,X + L_\mu\,Y) \tag{2.19}$$

where $\mu := u^{-1}\delta_1(u) = -\delta_1(u^{-1})u$. Moreover,

$$\Omega\,L_u = L_u(\,\Omega + [X, L_\mu] + \frac{1}{2}L_\mu^2) \tag{2.20}$$

which one gets using

$$\delta_2'(u) = [\Omega,\, u]$$

Since $[\Omega,\, \mu] = \delta_2'(\mu) = 0$ the hypothesis (2.8) of Proposition 5 is fulfilled by $X,\, Y,\, \Omega,\, L_\mu$ and we then get

$$\Phi(X,\ \Omega)(s) L_u = \ L_u\, L_{e^{\frac{s\mu}{2}}}\, \Phi(X, \Omega)(s)$$

In a similar manner,

$$S(X)\, R_u = R_u(\, S(X) + R_{\mu'}\, Y) \tag{2.21}$$

where $\mu' := \delta_1(u)\, u^{-1}$. Moreover

$$\Omega^o\, R_u = R_u(\, \Omega^o + \ [S(X), R_{\mu'}] \ + \frac{1}{2}\, R_{\mu'}^2) \tag{2.22}$$

where Ω^o really stands for R_Ω. So by Proposition 5 we get

$$\Phi(S(X),\ \Omega^o)(s)) R_u = \ R_u\, R_{e^{\frac{s\mu'}{2}}}\, \Phi(S(X), \Omega)(s)\, .$$

Since $u\, \mu = \mu'\, u$ we conclude as in the proof of Lemma 2. □

We then obtain the following naturality property of the construction of the higher brackets.

Theorem 8. *When applied to any of the modular Hecke algebras $\mathcal{A}(\Gamma)$ the functor RC_* yields the reduced algebra of the crossed products of the algebra of modular forms endowed with the Rankin-Cohen brackets by the action of the group* $\mathrm{GL}(2, \mathbb{A}_f)^0$.

Proof. The algebra $\mathcal{A}(\Gamma)$ is the reduced algebra of $\mathcal{M} \rtimes \mathrm{GL}(2, \mathbb{A}_f)^0$ by the projection e_Γ associated to Γ (cf. [8]). With u and v as in Lemma 7 and a and $b \in \mathcal{A}$ one gets

$$RC_n(a\, u,\, b\, v) = \ RC_n(a,\, u\, b\, v) = \ RC_n(a,\, b^u)\, u\ v$$

where $b^u := u\, b\, u^{-1}$, which shows that in the crossed product algebra $\mathcal{A} = \mathcal{M} \ltimes \mathrm{GL}(2, \mathbb{A}_f)^0$ the RC_n are entirely determined by their restriction to $\mathcal{M}$. □

Corollary 9. *When applied to any of the algebras $\mathcal{A}(\Gamma)$ the functor RC_* yields associative deformations.*

Proof. The crossed product of an associative algebra by an automorphism group is associative, as well as its reduced algebras. □

Specifically, the product formula

$$a *_t b := \sum t^n RC_n(a,b)$$

gives an associative deformation. More generally, following [2], for any $\kappa \in \mathbb{C}$ the product rule

$$a *_t^{\kappa} b \;:=\; \sum t^n \, RC_n \left(\mathbf{t}_n^{\kappa}(Y \otimes 1, 1 \otimes Y)(a \otimes b) \right) \,, \tag{2.23}$$

where

$$\mathbf{t}_n^{\kappa}(x,y) := (-\frac{1}{4})^n \sum_j \binom{n}{2j} \frac{\binom{-\frac{1}{2}}{j}\binom{\kappa-\frac{3}{2}}{j}\binom{\frac{1}{2}-\kappa}{j}}{\binom{-x-\frac{1}{2}}{j}\binom{-y-\frac{1}{2}}{j}\binom{n+x+y-\frac{3}{2}}{j}} \tag{2.24}$$

are the twisting coefficients defined in [2], gives an associative deformation.

3 Rankin-Cohen Deformations

We now return to the general case, we let the Hopf algebra $\mathcal{H}_1$ act on an algebra $\mathcal{A}$ and assume that the derivation δ_2' is inner implemented by an element $\Omega \in \mathcal{A}$,

$$\delta_2'(a) = \Omega\, a - a\, \Omega \,, \; \forall a \in \mathcal{A} \tag{3.1}$$

with

$$\delta_k(\Omega) = \; 0 \,, \; \forall k \in \mathbb{N} \tag{3.2}$$

Such an action of $\mathcal{H}_1$ on an algebra $\mathcal{A}$ will be said to define a *projective structure* on $\mathcal{A}$, and the element $\Omega \in \mathcal{A}$ implementing the inner derivation δ_2' will be called its *quadratic differential*.

The main result of this section, extending Corollary 9, can be stated as follows.

Theorem 10. *The functor RC_* applied to any algebra $\mathcal{A}$ endowed with a projective structure yields a family of formal associative deformations of $\mathcal{A}$, whose products are given by formula (2.23).*

In preparation for the proof, we shall extend the scalars in the definition of $\mathcal{H}_1$. Let $\mathcal{P}$ denote the free commutative algebra generated by the indeterminates

$$\{Z_0, Z_1, Z_2, \ldots, Z_n, \ldots\}.$$

We define an action of $\mathcal{H}_1$ on $\mathcal{P}$ by setting on generators

$$Y(Z_j) := (j+2)\, Z_j \,,\; X(Z_j) := Z_{j+1} \,, \quad \forall\, j \geq 0 \,, \tag{3.3}$$

and then extending Y, X as derivations, while

$$\delta_k(P) := 0 \,, \quad \forall\, P \in \mathcal{P} \,. \tag{3.4}$$

Equivalently, the Hopf action of $\mathcal{H}_1$ is lifted from the Lie algebra action defined by (3.3).
We then form the double crossed product algebra

$$\widetilde{\mathcal{H}}_1 = \mathcal{P} \rtimes \mathcal{H}_1 \ltimes \mathcal{P} \,, \tag{3.5}$$

whose underlying vector space is $\mathcal{P} \otimes \mathcal{H}_1 \otimes \mathcal{P}$, with the product defined by the rule

$$P \rtimes h \ltimes Q \,\cdot\, P' \rtimes h' \ltimes Q' \;:=\; \sum_{(h)} P\, h_{(1)}(P') \ltimes h_{(2)} h' \rtimes h_{(3)}(Q')\, Q \,, \tag{3.6}$$

where $P, Q, P', Q' \in \mathcal{P}$ and $h, h' \in \mathcal{H}_1$.
We next proceed to equip $\widetilde{\mathcal{H}}_1$ with the structure of an extended Hopf algebra over $\mathcal{P}$ (comp. [7]). First we turn $\widetilde{\mathcal{H}}_1$ into a (free) $\mathcal{P}$-bimodule, by means of the source and target homorphisms $\alpha \,, \beta : \mathcal{P} \to \widetilde{\mathcal{H}}_1$,

$$\alpha(P) := P \rtimes 1 \ltimes 1 \,, \quad \text{resp.} \quad \beta(Q) := 1 \rtimes 1 \ltimes Q \,, \quad \forall\, P, Q \in \mathcal{P} \,. \tag{3.7}$$

Note that

$$P \rtimes h \ltimes Q \;=\; \alpha(P) \cdot \beta(Q) \cdot h \,, \qquad P, Q \in \mathcal{P}, \quad h \in \mathcal{H}_1 \,. \tag{3.8}$$

Note also that, while the commutation rules of $h \in \mathcal{H}_1$ with $\alpha(P)$ are given by the above action (3.3) of $\mathcal{H}_1$ on $\mathcal{P}$, the commutation rules with $\beta(Q)$ are more subtle ; for example (comp. [7, 1.12]),

$$X\, \beta(Q) \,-\, \beta(Q)\, X \;=\; \beta(X(Q)) \,+\, \beta(Y(Q))\, \delta_1 \,, \quad Q \in \mathcal{P}, \quad h \in \mathcal{H}_1 \,. \tag{3.9}$$

Let $\widetilde{\mathcal{H}}_1 \otimes_{\mathcal{P}} \widetilde{\mathcal{H}}_1$ be the tensor square of $\widetilde{\mathcal{H}}_1$ where we view $\widetilde{\mathcal{H}}_1$ as a bimodule over $\mathcal{P}$, using left multiplication by $\beta(\cdot)$ to define the right module structure and left multiplication by $\alpha(\cdot)$ to define the left module structure.
The coproduct $\Delta : \widetilde{\mathcal{H}}_1 \to \widetilde{\mathcal{H}}_1 \otimes_{\mathcal{P}} \widetilde{\mathcal{H}}_1$ is defined by

$$\Delta(P \rtimes h \ltimes Q) := \sum_{(h)} P \rtimes h_{(1)} \ltimes 1 \otimes 1 \rtimes h_{(2)} \ltimes Q \tag{3.10}$$

and satisfies the properties listed in [7, Prop. 6]. In particular, while the product of two elements in $\widetilde{\mathcal{H}}_1 \otimes_{\mathcal{P}} \widetilde{\mathcal{H}}_1$ is not defined in general, the fact that Δ is multiplicative, i.e. that

$$\Delta(h_1 \cdot h_2) = \Delta(h_1) \cdot \Delta(h_2)\,, \qquad \forall\, h_1, h_2 \in \widetilde{\mathcal{H}}_1\,,$$

makes perfect sense because of the property

$$\Delta\,(h) \cdot (\beta(Q) \otimes 1 \;-\; 1 \otimes \alpha(Q)) = 0\,, \quad \forall\, Q \in \mathcal{P}\,,\, h \in \widetilde{\mathcal{H}}_1\,,$$

which uses only the right action of $\widetilde{\mathcal{H}}_1 \otimes \widetilde{\mathcal{H}}_1$ on $\widetilde{\mathcal{H}}_1 \otimes_{\mathcal{P}} \widetilde{\mathcal{H}}_1$ by right multiplication. In turn, since $\mathcal{H}_1$ is a Hopf algebra, it suffices to check the latter on the algebra generators, i.e. for $h = Y, X$ or δ_1. In that case

$$\begin{aligned} \Delta(h) \cdot (\beta(Q) \otimes 1 \;-\; 1 \otimes \alpha(Q)) \;&=\\ \sum_{(h)} \big(h_{(1)} \cdot \beta(Q) \otimes_{\mathcal{P}} h_{(2)} \;-\; h_{(1)} \cdot \otimes_{\mathcal{P}} h_{(2)} \alpha(Q)\big) \;&=\\ \sum_{(h)} \big([h_{(1)}, \beta(Q)] \otimes_{\mathcal{P}} h_{(2)} \;-\; h_{(1)} \otimes_{\mathcal{P}} [h_{(2)}, \alpha(Q)]\big) \;&=\; 0\,, \end{aligned}$$

where one needs (3.9) to establish the vanishing.
The counit map $\varepsilon : \widetilde{\mathcal{H}}_1 \to \mathcal{P}$ is defined by

$$\varepsilon(P \rtimes h \ltimes Q) :=\; P\,\varepsilon(h)\,Q\,, \quad P, Q \in \mathcal{P}, \quad h \in \mathcal{H}_1$$

and fulfills the conditions listed in [7, Prop. 7].
Finally, the formula for the antipode is

$$S(P \rtimes h \ltimes Q) := S(h)_{(1)}(Q) \rtimes S(h)_{(2)} \ltimes S(h)_{(3)}(P) = S(h) \cdot \alpha(Q) \cdot \beta(P).$$

In the same vein, an algebra $\mathcal{A}$ is a module-algebra over $\widetilde{\mathcal{H}}_1|\mathcal{P}$ if, first of all, $\mathcal{A}$ is gifted with an algebra homomorphism $\rho : \mathcal{P} \to \mathcal{A}$ (playing the role of the unit map over $\mathcal{P}$), which turns $\mathcal{A}$ into a $\mathcal{P}$-bimodule via left and right multiplication by the image of ρ, and secondly $\mathcal{A}$ is endowed with an action $H \otimes a \mapsto H(a)\,, \quad H \in \widetilde{\mathcal{H}}_1\,,\, a \in \mathcal{A}$ satisfying besides the usual action rules

$$\begin{aligned} (H \cdot H')(a) &= H(H'(a))\,, \qquad && H, H' \in \widetilde{\mathcal{H}}_1\,, \\ 1(a) &= a\,, && a \in \mathcal{A}\,, \end{aligned} \tag{3.11}$$

also the compatibility rules

$$\begin{aligned} H(a_1 a_2) &= \sum_{(H)} H_{(1)}(a_1)\, H_{(2)}(a_2)\,, \qquad && a_1, a_2 \in \mathcal{A}\,, \\ H(1) &= \rho(\varepsilon(H))\,, && H \in \widetilde{\mathcal{H}}_1\,. \end{aligned} \tag{3.12}$$

In particular for any $P \in \mathcal{P}$,

$$\alpha(P)(a) = \rho(P)\, a\,, \quad \text{resp.} \quad \beta(P)(a) = a\, \rho(P)\,, \qquad a \in \mathcal{A}\,, \tag{3.13}$$

and therefore more generally for any monomial $H = P \rtimes h \ltimes Q \in \widetilde{\mathcal{H}}_1$ one has

$$P \rtimes h \ltimes Q\,(a) = \rho(P)\, h(a)\, \rho(Q)\,. \tag{3.14}$$

We denote

$$\widetilde{\delta}_2' := \delta_2 - \frac{1}{2}\delta_1^2 - \alpha(Z_0) + \beta(Z_0) \tag{3.15}$$

and remark that it is a primitive element in $\widetilde{\mathcal{H}}_1$:

$$\Delta(\widetilde{\delta}_2') = \widetilde{\delta}_2' \otimes 1 + 1 \otimes \widetilde{\delta}_2'\,. \tag{3.16}$$

We let $\widetilde{\mathcal{H}}_{\rm s}$ denote the quotient of $\widetilde{\mathcal{H}}_1$ by the ideal generated by $\widetilde{\delta}_2'$. In view of (3.16), the latter is also a coideal, and therefore $\widetilde{\mathcal{H}}_{\rm s}$ inherits the structure of an extended Hopf algebra over $\mathcal{P}$. Clearly, the action of $\widetilde{\mathcal{H}}_1|\mathcal{P}$ on an algebra $\mathcal{A}$ endowed with a projective structure descends to an action of $\widetilde{\mathcal{H}}_{\rm s}$ on $\mathcal{A}$.

As already mentioned in Section 3, the prototypical examples of projective structures are furnished by the modular Hecke algebras of [8]. For the purposes of this section it will suffice to consider the 'discrete' modular Hecke algebra, that is the crossed product

$$\mathcal{A}_{G^+(\mathbb{Q})} := \mathcal{M} \rtimes G^+(\mathbb{Q})\,, \qquad G^+(\mathbb{Q}) = \mathrm{GL}^+(2, \mathbb{Q})\,.$$

where $\mathcal{M}$ is the algebra of modular forms of all levels. We recall that $\mathcal{A}_{G^+(\mathbb{Q})}$ consists of finite sums of symbols of the form

$$\sum f\,U_\gamma^*\,, \qquad \text{with} \quad f \in \mathcal{M}\,, \quad \gamma \in G^+(\mathbb{Q})\,,$$

with the product given by the rule

$$f\,U_\alpha^* \cdot g\,U_\beta^* = (f \cdot g|\alpha)\,U_{\beta\alpha}^*\,, \tag{3.17}$$

where the vertical bar denotes the 'slash operation'. Under the customary identification

$$f \in \mathcal{M}_{2k} \longmapsto \widetilde{f} := f\,dz^k$$

of modular forms with higher differentials, the latter is just the pullback

$$f|\gamma \longmapsto \gamma^*(\widetilde{f})\,, \quad \forall\,\gamma \in G^+(\mathbb{Q})\,. \tag{3.18}$$

We further recall that the action of $\mathcal{H}_1$ on $\mathcal{A}_{G^+(\mathbb{Q})}$ is determined by

$$X(f\,U_\gamma^*) = X(f)\,U_\gamma^*\,,\ Y(f\,U_\gamma^*) = Y(f)\,U_\gamma^*\,,\ \delta_1(f\,U_\gamma^*) = \mu_\gamma \cdot f\,U_\gamma^*\,, \tag{3.19}$$

where

$$X(f) = \frac{1}{2\pi i}\left(\frac{df}{dz} - \frac{d}{dz}(\log \eta^4)\cdot Y(f)\right)\,, \tag{3.20}$$

Y stands for the Euler operator,

$$Y(f) = \frac{\ell}{2}\cdot f\,, \tag{3.21}$$

for any f of weight ℓ. Lastly,

$$\mu_\gamma\,(z) = \frac{1}{2\pi i}\,\frac{d}{dz}\log\frac{\eta^4|\gamma}{\eta^4}\,. \tag{3.22}$$

By [8, Prop. 10], the quadratic differential is implemented by the normalized Eisenstein modular form of level 1 and weight 4:

$$\delta_2'(a) = [\,\omega,\,a]\,, \qquad \omega = \frac{E_4}{72}\,. \tag{3.23}$$

More generally, one can conjugate the above action by a $G^+(\mathbb{Q})$-invariant 1-cocycle, as in [8, Prop. 11]. In particular, given any $\sigma \in \mathcal{M}_2$, there exists a unique 1-cocycle $u = u(\sigma) \in \mathcal{L}(\mathcal{H}_1, \mathcal{A}_{G^+(\mathbb{Q})})$ such that

$$u(X) = 0,\ u(Y) = 0,\ u(\delta_1) = \sigma\,. \tag{3.24}$$

The conjugate under $u(\sigma)$ of the above action of $\mathcal{H}_1$ is given on generators as follows:

$$\begin{aligned} Y_\sigma = Y\,, \quad X_\sigma &= X + \sigma\, Y\,, \\ (\delta_1)_\sigma(a) &= \delta_1(a) + [\sigma,\, a]\,, \qquad a \in \mathcal{A}_{G^+(\mathbb{Q})}\,. \end{aligned} \tag{3.25}$$

The conjugate under $u(\sigma)$ of δ_2' is given by the operator

$$(\delta_2')_\sigma(a) = [\,\omega_\sigma,\, a] \quad \text{with} \quad \omega_\sigma = \omega + X(\sigma) + \frac{\sigma^2}{2}\,, \quad a \in \mathcal{A}_{G^+(\mathbb{Q})}. \tag{3.26}$$

Thus, for any 1-cocycle $u = u(\sigma)$ as above, we get an action of $\widetilde{\mathcal{H}}_{\rm s}|\mathcal{P}$ on $\mathcal{A}_{G^+(\mathbb{Q})}$, determined by (3.25) and by the homomorphism $\rho_\sigma : \mathcal{P} \to \mathcal{M}$,

$$\rho_\sigma(Z_k) \;=\; X_\sigma^k(\omega_\sigma)\,, \qquad k = 0, 1, 2, \ldots\,. \tag{3.27}$$

For each n we let $\rho_\sigma^{\otimes n} : \mathcal{P}^{\otimes n} \to \mathcal{M}^{\otimes n}$ be the nth tensor power of ρ_σ. We shall first show that the family ρ_σ is sufficiently large to separate the elements of $\mathcal{P}^{\otimes n}$.

Lemma 11. *For each $n \in \mathbb{N}$, one has* $\displaystyle\bigcap_{\sigma \in \mathcal{M}_2} \operatorname{Ker} \rho_\sigma^{\otimes n} = 0\,.$

Proof. Let us treat the case $n = 1$ first. Let g_2^* be the "quasimodular" ([12]) solution of the equation

$$X(m) + \frac{m^2}{2} + \omega \,=0. \tag{3.28}$$

Note that not only g_2^* fulfills (3.28) but also one has

$$X(f) = \frac{1}{2\pi i}\frac{d}{dz}\, f - g_2^*\, Y\, f\,,$$

for any modular form f. Also

$$\frac{1}{2\pi i}\frac{d}{dz}\,g_2^* - \frac{1}{2}(g_2^*)^2 = -\,\omega$$

Thus, with $\alpha := \sigma - g_2^*$ we get

$$\begin{aligned} X_\sigma &= \frac{1}{2\pi i}\frac{d}{dz} + \alpha\, Y\,, \\ \omega_\sigma &= \frac{1}{2\pi i}\frac{d}{dz}\,\alpha + \frac{\alpha^2}{2} \end{aligned} \tag{3.29}$$

which allows to rewrite (3.27) as

$$\rho_\sigma(Z_k) \;=\; (\frac{1}{2\pi i}\frac{d}{dz} + \alpha\, Y)^k(\frac{1}{2\pi i}\frac{d}{dz}\,\alpha + \frac{\alpha^2}{2})\,, \qquad k = 0,1,2,\ldots. \tag{3.30}$$

Given $0 \neq P \in \mathcal{P}$ the set of α for which $\rho_\sigma(P) = 0$ is seen using (3.30) to be contained in the space of holomorphic solutions of an (autonomous) ODE [1] and these only depend on finitely many parameters. Thus, given $0 \neq P \in \mathcal{P}$, the set of σ for which $\rho_\sigma(P) = 0$ is finite dimensional.

Let us now prove by induction that the same result holds for any n. Given $P \in \mathcal{P}^{\otimes n}$ we write

$$P = \sum P_j \otimes m_j$$

where $P_j \in \mathcal{P}^{\otimes n-1}$ and the m_j belong to the canonical basis of monomials in $\mathcal{P}$. If $P \neq 0$ then $P_j \neq 0$ for some j and by the induction hypothesis the set E_j of σ for which $\rho_\sigma(P_j) = 0$ is finite dimensional. On the complement of E_j any $\alpha := \sigma - g_2^*$ such that $\rho_\sigma(P) = 0$ fulfills a non-trivial (autonomous) ODE of the form

$$\sum \lambda_j \rho_\sigma(m_j) \;= 0 \tag{3.31}$$

where the coefficient $\lambda_j \neq 0$. Since the space of parameters for equations of the form (3.31) is finite dimensional we get the required finite dimensionality. Since the space of modular forms of weight 2 and arbitrary level is infinite dimensional we conclude the proof. □

[1] Any given quasi-modular form α is the solution of a non-trivial autonomous ODE (cf. [12]), but the latter of course depends on α.

We next define a map of bimodules

$$\chi_\sigma^{(n)} : \underbrace{\widetilde{\mathcal{H}}_{\rm s} \otimes_{\mathcal{P}} \ldots \otimes_{\mathcal{P}} \widetilde{\mathcal{H}}_{\rm s}}_{n-\text{times}} \quad \longrightarrow \quad \mathcal{L}(\underbrace{\mathcal{A}_{G^+(\mathbb{Q})} \otimes \ldots \otimes \mathcal{A}_{G^+(\mathbb{Q})}}_{n-\text{times}}, \mathcal{A}_{G^+(\mathbb{Q})})$$

by means of the assignment

$$\chi_\sigma^{(n)}(h^1 \otimes_{\mathcal{P}} \ldots \otimes_{\mathcal{P}} h^n)\,(a_1, \ldots, a_n) \;=\; h_\sigma^1(a_1) \cdots h_\sigma^n(a_n)\,, \tag{3.32}$$

where $a_1, \ldots, a_n \in \mathcal{A}_{G^+(\mathbb{Q})}$ and with $h^1, \ldots, h^n \in \widetilde{\mathcal{H}}_{\rm s}$ acting via ρ_σ.

The following result, which represents a 'modular' analogue of [7, Prop.4], allows to establish the associativity at the Hopf algebraic level.

Proposition 12. *For each $n \in \mathbb{N}$, one has*

$$\bigcap_\sigma \operatorname{Ker} \chi_\sigma^{(n)} = 0\,.$$

Proof. For the sake of clarity, we shall first treat the case $n = 1$. An arbitrary element of $\widetilde{\mathcal{H}}_{\rm s}$ can be represented uniquely as a finite sum of the form

$$H \;=\; \sum_{j,k,m,s \geq 0} \alpha(P_{jkms})\,\beta(Q_{jkms})\,\delta_1^j\,X^k\,Y^m\,,$$

with $P, Q \in \mathcal{P}$. Let us assume $\chi_\sigma^{(1)}(H) = 0$, for any σ.
Evaluating H on a generic monomial in $\mathcal{A}_{G^+(\mathbb{Q})}$ one obtains, for any $f \in \mathcal{M}$ and any $\gamma \in G^+(\mathbb{Q})$,

$$\sum_{j,k,m,s \geq 0} P_{jkms}\,Q_{jkms}|\gamma \cdot \mu_\gamma(\sigma)^j \cdot X^k(Y^m(f)) \;=\; 0\,; \tag{3.33}$$

where,

$$\mu_\gamma(\sigma)\,(z) = \alpha(z) - \alpha|\gamma\,(z) - \frac{1}{\pi i}\;\frac{c}{cz+d} \tag{3.34}$$

with $\alpha := \sigma - g_2^*$ as above. By continuity, the above holds in fact for any $\gamma \in G^+(\mathbb{R})$.
For each fixed l, the differential equation

$$\sum_{j,k,m,s} P_{jkms}Q_{jkms}|\gamma\,\mu_\gamma(\sigma)^j\,l^m\,X^k(f) \;=\; 0$$

is satisfied by all modular forms $f \in \mathcal{M}_{2l}$. In turn, this implies that all its coefficients vanish. Using the freedom in l, it then follows that

$$\sum_{j,s} P_{jkms} Q_{jkms}|\gamma \, \mu_\gamma(\sigma)^j \; = \; 0 \, , \tag{3.35}$$

for each k and m.

Given $z \in H$, the following three functions on $\mathrm{SL}(2,\mathbb{C})$ are defined and independent in a neighborhood of $\mathrm{SL}(2,\mathbb{R})$:

$$\begin{aligned} g_1(a,b,c,d) : &= \frac{az+b}{cz+d}, \quad g_2(a,b,c,d) := \frac{1}{cz+d}, \\ g_3(a,b,c,d) : &= \frac{c}{cz+d}. \end{aligned}$$

Indeed, we recover c from g_2 and g_3 and then d from g_3, then $az+b$ from g_1 and finally a and b from $ad-bc=1$. Now the formula for μ_γ is of the form

$$\mu_\gamma(\sigma) = \alpha(g_1)(g_2)^2 - \alpha(z) - \frac{1}{i\,\pi} g_3 \tag{3.36}$$

and thus involves g_3 nontrivially while the other terms in the formulas do not involve g_3. For fixed z the formula (3.35) remains valid in a neighborhood of $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{SL}(2,\mathbb{C})$. Fixing $z \, , g_1 \, , g_2$ and varying g_3 independently this is enough to show that the coefficient of each power $\mu_\gamma(\sigma)^j$ vanishes identically. Thus,

$$\sum_s P_{jkms} \cdot Q_{jkms}|\gamma \; = \; 0 \, , \qquad \forall \, \gamma \in \mathrm{SL}(2,\mathbb{R}) \, . \tag{3.37}$$

Using the independence of the functions $g_1 \, , g_2$ and the fact that the identity (3.37) holds true for every homogeneous component of Q_{jkms}, in view of the freedom to choose σ, it follows from Lemma 11 that $H = 0$.

The proof of the case $n > 1$ is obtained by combining the above arguments with the proof of the general case in Lemma 11. An arbitrary element of $\widetilde{\mathcal{H}}_{\mathrm{s}}^{\otimes n}$ can be represented uniquely as a finite sum of the form

$$\begin{aligned} H \; = \; \sum_{j,k,m,s} \alpha(P_{1,j_1k_1m_1s}) \, \beta(Q_{1,j_1k_1m_1s}) \, \delta_1^{j_1} \, X^{k_1} \, Y^{m_1} \; \otimes \cdots \\ \cdots \otimes \; \alpha(P_{a,j_ak_am_as}) \, \beta(Q_{a,j_ak_am_as}) \, \delta_1^{j_a} \, X^{k_a} \, Y^{m_a} \; \otimes \cdots \end{aligned}$$

$$\cdots \otimes \, \alpha(P_{n,j_nk_nm_ns})\, \beta(Q_{n,j_nk_nm_ns})\, \delta_1^{j_n}\, X^{k_n}\, Y^{m_n}$$

Evaluating H on a generic monomial in $\mathcal{A}_{G^+(\mathbb{Q})}^{\otimes n}$ one obtains, for any $f_j \in \mathcal{M}$ and any $\gamma_j \in G^+(\mathbb{Q})$,

$$\sum_{j,k,m,s} \prod_a (P_{a,j_ak_am_as} \cdot Q_{a,j_ak_am_as}|\gamma_a \cdot \mu_{\gamma_a}(\sigma)^{j_a} \cdot X_\sigma^{k_a}(Y^{m_a}(f_a)))|\gamma_{a-1}\cdots\gamma_1 \;=\; 0 \tag{3.38}$$

As in the case $n = 1$, the freedom in the choice of the $f_j \in \mathcal{M}$ and $\gamma_j \in G^+(\mathbb{Q})$ shows that for every multiindex $(j_a, k_a, m_a)_{a\in\{1,..,n\}}$, one has

$$\sum_s \prod_a (P_{a,j_ak_am_as} \cdot Q_{a,j_ak_am_as}|\gamma_a)|\gamma_{a-1}\cdots\gamma_1 \;=\; 0\,. \tag{3.39}$$

Using once more the independence of the functions g_1 and g_2 together with the reduction to homogeneous components for $Q_{a,j_ak_am_as}$, and then applying again Lemma 11, one arrives at the conclusion that $H = 0$. ☐

As a direct consequence, one obtains a family of 'universal deformation formulas' (in the sense of [10]) based on $\widetilde{\mathcal{H}}_{\mathrm{s}}$, as follows. From Theorem 8 and Proposition 12, each bilinear operator RC_n uniquely determines an element

$$RC_n^{\mathcal{H}_{\mathrm{s}}} \in \widetilde{\mathcal{H}}_{\mathrm{s}} \otimes_{\mathcal{P}} \widetilde{\mathcal{H}}_{\mathrm{s}}\,. \tag{3.40}$$

We then assemble all of them into the element

$$\mathcal{F} \;:=\; \sum_{n\geq 0} t^n RC_n^{\mathcal{H}_{\mathrm{s}}} \;\in \widetilde{\mathcal{H}}_{\mathrm{s}}[[t]] \otimes_{\mathcal{P}[[t]]} \widetilde{\mathcal{H}}_{\mathrm{s}}[[t]] \tag{3.41}$$

Corollary 13. *The element $\mathcal{F} \in \widetilde{\mathcal{H}}_{\mathrm{s}}[[t]] \otimes_{\mathcal{P}[[t]]} \widetilde{\mathcal{H}}_{\mathrm{s}}[[t]]$ defines a universal deformation formula, i.e. satisfies the identities*

$$(\Delta \otimes \mathrm{Id})(\mathcal{F}) \cdot (\mathcal{F} \otimes 1) \;=\; (\mathrm{Id} \otimes \Delta)(\mathcal{F}) \cdot 1 \otimes \mathcal{F}\,; \tag{3.42}$$

$$(\varepsilon \otimes \mathrm{Id})(\mathcal{F}) \;=\; 1 \otimes 1 \;=\; (\mathrm{Id} \otimes \varepsilon)(\mathcal{F})\,. \tag{3.43}$$

Proof. Indeed, this follows by applying Proposition 12, for $n = 3$, to the associative deformation of the algebra $\mathcal{A}_{G^+(\mathbb{Q})}$ provided by Corollary 9. ☐

This result, in conjunction with the twisting operators $\mathbf{t}_n^\kappa$ of (2.24), immediately imply Theorem 10.

Remark 14. We finally note that by evaluating $\mathcal{F}$ at $Z_k = 0\,, k \geq 0\,$, one obtains a completely explicit twisting element $F \in \mathcal{H}_{\rm s}[[t]] \otimes_{\mathbb{C}[[t]]} \mathcal{H}_{\rm s}[[t]]$,

$$F \;=\; \sum_{n\geq 0} t^n \sum_{k=0}^{n} \frac{S(X)^k}{k!}\,(2Y+k)_{n-k} \otimes \frac{X^{n-k}}{(n-k)!}\,(2Y+n-k)_k\,, \tag{3.44}$$

which defines a deformation of the Hopf algebra $\mathcal{H}_{\rm s}$ in the direction of the Hochschild 2-cocycle $-2T = -X \otimes 2Y + 2Y \otimes X + \delta_1 \cdot Y \otimes 2Y$. Although the expressions of its components were found early on, cf. (1.5), the proof of their universal associativity property passes through the treatment of the more general case of an arbitrary quadratic differential.

4 Appendix: Explicit Formulas

We display below the formulas of RC_n for the first three values of n, illustrating the rapid increase of their complexity. They are in reduced form, with the terms in $\alpha[\cdot]$ followed by those in $\beta[\cdot]$ appearing in the role of coefficients, and then followed by terms in δ_1, in X and finally in Y; the last three types form the analogue of a Poincaré-Birkhoff-Witt basis of $\widetilde{\mathcal{H}}_{\rm s}$, viewed as a module over $\mathcal{P} \otimes \mathcal{P}$. Moreover, taking advantage of the tensoring over $\mathcal{P}$, no term in $\beta[\cdot]$ appears in the first argument of the tensor square.

In the formulas that follow we shall lighten the notation, using the symbol RC_* instead of $RC_*^{\mathcal{H}_{\rm s}}$, and $\otimes$ instead of $\otimes_{\mathcal{P}}$; we shall also write Ω instead of Z_0.

$$\frac{1}{2} RC_1 \;=\; -X \otimes Y + Y \otimes X + \delta_1 \cdot Y \otimes Y\;. \tag{4.1}$$

Quite remarkably, the formula for A_n appears to contain no term in $\beta[.]$, unlike the formula for $S(X)^n$. The latter, once put in reduced form, is much more complicated than the expression of A_n.

The associativity for RC_1 follows directly from its Hochschild property. It is already harder to check it directly for RC_2 which is given by the following expression:

$$
\begin{aligned}
RC_2 \;=\; & -X \otimes X - 2\,X \otimes X \cdot Y + X^2 \otimes Y + 2\,X^2 \otimes Y^2 + Y \otimes X^2 \\
& - Y \otimes \alpha[\Omega] \cdot Y + 2\,Y^2 \otimes X^2 - 2\,Y^2 \otimes \alpha[\Omega] \cdot Y - 2\,X \cdot Y \otimes X \\
& -4\,X \cdot Y \otimes X \cdot Y - \delta_1 \cdot X \otimes Y - 2\,\delta_1 \cdot X \otimes Y^2 + \delta_1 \cdot Y \otimes X \\
& + 2\,\delta_1 \cdot Y \otimes X \cdot Y + 2\,\delta_1 \cdot Y^2 \otimes X + 4\,\delta_1 \cdot Y^2 \otimes X \cdot Y + \frac{1}{2}\,\delta_1^2 \cdot Y \otimes Y \\
& + \delta_1^2 \cdot Y \otimes Y^2 + \delta_1^2 \cdot Y^2 \otimes Y + 2\,\delta_1^2 \cdot Y^2 \otimes Y^2 - \alpha[\Omega] \cdot Y \otimes Y \\
& - 2\,\alpha[\Omega] \cdot Y \otimes Y^2 - 2\,\delta_1 \cdot X \cdot Y \otimes Y - 4\,\delta_1 \cdot X \cdot Y \otimes Y^2\,.
\end{aligned}
$$

We did check directly Corollary 13 up to order 4 included, (i.e. the associativity for RC_3 and RC_4) with the help of a computer. This is beyond the reach of any 'bare hands' computation, as witnessed by the complexity of the following formula for RC_3. (The expression of RC_4 is much longer, it would occupy several pages.)

$$
\begin{aligned}
RC_3 \;=\; & -2\,X \otimes X^2 - 2\,X \otimes X^2.Y + 2\,X \otimes \alpha[\Omega].Y + 2\,X \otimes \alpha[\Omega].Y^2 + 2\,X^2 \otimes X + 6\,X^2 \otimes X.Y + \\
& + 4\,X^2 \otimes X.Y^2 - \frac{2\,X^3 \otimes Y}{3} - 2\,X^3 \otimes Y^2 - \frac{4\,X^3 \otimes Y^3}{3} + \frac{2\,Y \otimes X^3}{3} - \frac{2}{3}\,Y \otimes \alpha[\Omega].X \\
& -\frac{2}{3}\,Y \otimes \alpha[X[\Omega]].Y - 2\,Y \otimes \alpha[\Omega].X.Y + 2\,Y^2 \otimes X^3 - 2\,Y^2 \otimes \alpha[\Omega].X - 2\,Y^2 \otimes \alpha[X[\Omega]].Y \\
& -6\,Y^2 \otimes \alpha[\Omega].X.Y + \frac{4\,Y^3 \otimes X^3}{3} - \frac{4}{3}\,Y^3 \otimes \alpha[\Omega].X - \frac{4}{3}\,Y^3 \otimes \alpha[X[\Omega]].Y - 4\,Y^3 \otimes \alpha[\Omega].X.Y \\
& -6\,X.Y \otimes X^2 - 6\,X.Y \otimes X^2.Y + 6\,X.Y \otimes \alpha[\Omega].Y + 6\,X.Y \otimes \alpha[\Omega].Y^2 - 4\,X.Y^2 \otimes X^2 \\
& -4\,X.Y^2 \otimes X^2.Y + 4\,X.Y^2 \otimes \alpha[\Omega].Y + 4\,X.Y^2 \otimes \alpha[\Omega].Y^2 + 2\,X^2.Y \otimes X + 6\,X^2.Y \otimes X.Y \\
& +4\,X^2.Y \otimes X.Y^2 - 2\,\delta_1.X \otimes X - 6\,\delta_1.X \otimes X.Y - 4\,\delta_1.X \otimes X.Y^2 + 2\,\delta_1.X^2 \otimes Y + 6\,\delta_1.X^2 \otimes Y^2 \\
& +4\,\delta_1.X^2 \otimes Y^3 + 2\,\delta_1.Y \otimes X^2 + 2\,\delta_1.Y \otimes X^2.Y - 2\,\delta_1.Y \otimes \alpha[\Omega].Y - 2\,\delta_1.Y \otimes \alpha[\Omega].Y^2 \\
& +6\,\delta_1.Y^2 \otimes X^2 + 6\,\delta_1.Y^2 \otimes X^2.Y - 6\,\delta_1.Y^2 \otimes \alpha[\Omega].Y - 6\,\delta_1.Y^2 \otimes \alpha[\Omega].Y^2 + 4\,\delta_1.Y^3 \otimes X^2 \\
& +4\,\delta_1.Y^3 \otimes X^2.Y - 4\,\delta_1.Y^3 \otimes \alpha[\Omega].Y - 4\,\delta_1.Y^3 \otimes \alpha[\Omega].Y^2 - \delta_1^2.X \otimes Y - 3\,\delta_1^2.X \otimes Y^2 \\
& -2\,\delta_1^2.X \otimes Y^3 + \delta_1^2.Y \otimes X + 3\,\delta_1^2.Y \otimes X.Y + 2\,\delta_1^2.Y \otimes X.Y^2 + 3\,\delta_1^2.Y^2 \otimes X + 9\,\delta_1^2.Y^2 \otimes X.Y \\
& +6\,\delta_1^2.Y^2 \otimes X.Y^2 + 2\,\delta_1^2.Y^3 \otimes X + 6\,\delta_1^2.Y^3 \otimes X.Y + 4\,\delta_1^2.Y^3 \otimes X.Y^2 + \frac{1}{3}\,\delta_1^3.Y \otimes Y + \delta_1^3.Y \otimes Y^2
\end{aligned}
$$

$$+\frac{2}{3}\,\delta_1^3.Y\otimes Y^3+\delta_1^3.Y^2\otimes Y+3\,\delta_1^3.Y^2\otimes Y^2+2\,\delta_1^3.Y^2\otimes Y^3+\frac{2}{3}\,\delta_1^3.Y^3\otimes Y+2\,\delta_1^3.Y^3\otimes Y^2$$
$$+\frac{4}{3}\,\delta_1^3.Y^3\otimes Y^3+\ \frac{2}{3}\,\alpha[\Omega].X\otimes Y+2\,\alpha[\Omega].X\otimes Y^2+\frac{4}{3}\,\alpha[\Omega].X\otimes Y^3-2\,\alpha[\Omega].Y\otimes X$$
$$-6\,\alpha[\Omega].Y\otimes X.Y-4\,\alpha[\Omega].Y\otimes X.Y^2-2\,\alpha[\Omega].Y^2\otimes X-6\,\alpha[\Omega].Y^2\otimes X.Y-4\,\alpha[\Omega].Y^2\otimes X.Y^2$$
$$+\frac{2}{3}\,\alpha[X[\Omega]].Y\otimes Y+2\,\alpha[X[\Omega]].Y\otimes Y^2+\frac{4}{3}\,\alpha[X[\Omega]].Y\otimes Y^3-6\,\delta_1.X.Y\otimes X-18\,\delta_1.X.Y\otimes X.Y$$
$$-12\,\delta_1.X.Y\otimes X.Y^2-4\,\delta_1.X.Y^2\otimes X-12\,\delta_1.X.Y^2\otimes X.Y-8\,\delta_1.X.Y^2\otimes X.Y^2+2\,\delta_1.X^2.Y\otimes Y$$
$$+\,6\,\delta_1.X^2.Y\otimes Y^2+4\,\delta_1.X^2.Y\otimes Y^3-3\,\delta_1^2.X.Y\otimes Y-9\,\delta_1^2.X.Y\otimes Y^2-6\,\delta_1^2.X.Y\otimes Y^3$$
$$-2\,\delta_1^2.X.Y^2\otimes Y-6\,\delta_1^2.X.Y^2\otimes Y^2-4\,\delta_1^2.X.Y^2\otimes Y^3+2\,\alpha[\Omega].X.Y\otimes Y+6\,\alpha[\Omega].X.Y\otimes Y^2$$
$$+4\,\alpha[\Omega].X.Y\otimes Y^3-2\,\alpha[\Omega].\delta_1.Y\otimes Y-6\,\alpha[\Omega].\delta_1.Y\otimes Y^2-4\,\alpha[\Omega].\delta_1.Y\otimes Y^3-2\,\alpha[\Omega].\delta_1.Y^2\otimes Y$$
$$-6\,\alpha[\Omega].\delta_1.Y^2\otimes Y^2-4\,\alpha[\Omega].\delta_1.Y^2\otimes Y^3\ .$$

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