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Riemann-Roch for $\overline{\operatorname{Spec}\mathbb{Z}}$

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We prove a Riemann-Roch theorem of an entirely novel nature for divisors on the Arakelov compactification of the algebraic spectrum of the integers. This result relies on the introduction of three key concepts: the cohomologies (attached to a divisor), their *integer* dimension, and Serre duality. These notions directly extend their classical counterparts for function fields. The Riemann-Roch formula equates the (integer valued) Euler characteristic of a divisor with a slight modification of the traditional expression in terms of the sum of the degree of the divisor and the logarithm of 2. Both the definitions of the cohomologies and of their dimensions rely on a universal arithmetic theory over the sphere spectrum that we had previously introduced using Segal's Gamma rings. By adopting this new perspective we can parallel Weil's adelic proof of the Riemann-Roch formula for function fields including the use of Pontryagin duality.

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1. Introduction

The development of the theory of curves from complex Riemann surfaces to algebraic curves over arbitrary fields, led F. K. Schmidt, in the 1930s, to the first proof of the Riemann-Roch theorem for function fields over finite fields. In nowadays terminology, that result involves (in complete analogy to the classical formula holding for compact, complex Riemann surfaces of genus g) the integer dimensions of the two cohomologies $H^0(X, \mathcal{O}_X(D))$ and $H^1(X, \mathcal{O}_X(D))$ of a divisor D on the algebraic curve X associated to the function field, as vector spaces over the (finite) field of constants.

Theorem 1.1 *(Riemann-Roch). One has the following equality*

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D)) = 1 - g + \deg D$$

The invertible sheaf ω_X of (regular) differentials on X corresponds to the so called canonical divisor K on the curve, while the invertible sheaf $\omega_X \otimes \mathcal{O}_X(D)^*$ is isomorphic to $\mathcal{O}_X(K-D)$. Furthermore, by Serre's duality theorem $H^0(X, \mathcal{O}_X(D))$ is isomorphic to the dual of $H^1(X, \mathcal{O}_X(D))$, so that $\dim H^1(X, \mathcal{O}_X(D)) = \dim H^0(X, \mathcal{O}_X(K-D))$. The analogy between function fields in one variable over fields of positive characteristic and number fields suggests, as pointed out by A. Weil [12], to investigate the existence of a similar Riemann-Roch formula for number fields. Both the notions of a divisor D with its degree, and the finite set underlying $H^0(D)$ are easy to define. These ideas led S. Lang [9] to an asymptotic formula that relates for $\mathbb{Q}$, $\log \# H^0(D)$ and $\deg D + \log 2$, when $\deg D \to \infty$. Another attempt to a Riemann-Roch formula for a number field was promoted two decades ago by G. van der Geer and R. Shoof [10] and more recently, in the work of J.B. Bost [1]. In this approach one starts from the functional equation in terms of the theta function and rewrites that equation as a Riemann-Roch formula for the log-theta number, which is thus promoted to the status of a dimension. This view of the log-theta number as a dimension remains virtual for the obvious reason that it outputs real numbers rather than integers. An interpretation of these numbers as Murray-von Neumann dimensions could greatly improve their status as dimensions, but this step has not been achieved so far.

The most fundamental number field is the field $\mathbb{Q}$ of rational numbers that governs elementary arithmetics through its ring $\mathbb{Z}$ of integers. In this paper we prove a Riemann-Roch theorem of an entirely novel nature for this basic arithmetic structure. This result relies on the introduction of three key concepts: the two cohomologies $\left(H^0(D), H^1(D)\right)$ being suitable pointed covariant functors, their (integer) dimension, and Serre duality. These three notions directly extend their classical analogues for function fields. We recall that the arithmetic degree of a divisor $D = \sum_j a_j\{p_j\} + a\{\infty\}$ is defined as $\deg D :=$

$\sum_j a_j \log(p_j) + a$. Let denote by $\lceil x \rceil'$ the odd function on $\mathbb{R}$ that agrees with the ceiling function[2] $\lceil x \rceil$ on positive reals, then our main result is the following theorem.

Theorem 1.2. *Let D be an Arakelov divisor on $\overline{\operatorname{Spec} \mathbb{Z}}$. Then*

$$\dim_{\mathbb{S}[\pm 1]} H^0(D) - \dim_{\mathbb{S}[\pm 1]} H^1(D) = \left\lceil \frac{\deg D + \log 2}{\log 3} \right\rceil' - \mathbf{1}_L(\deg D). \tag{1.1}$$

Here, $\mathbf{1}_L$ is the characteristic function of the exceptional set L of finite Lebesgue measure which is the union[3] *of the open intervals $\left(\log \frac{3^k}{2}, \log \frac{3^k+1}{2}\right)$.*

Let us now comment on the key role played by the number 3. In the proof of Theorem 1.2, the ring $\mathbb{Z}$ appears in a new light as a ring of *polynomials* $\sum a_j 3^j$ with coefficients in the absolute base $\mathbb{S}[\pm 1] = \mathbb{S}[\{0, \pm 1\}]$. At an elementary level, it is not difficult to check that any integer $n \in \mathbb{Z}$ can be written uniquely as a finite sum of powers of 3 with coefficients in $\{0, \pm 1\}$. The deep reason behind this fact is that, for $p = 3$ (and only for this rational prime) the Witt vectors with only finitely many non-zero components form a subring inside the ring of p-adic integers, which turns out to be isomorphic to $\mathbb{Z}$. In particular, the formula for the addition of integers written as polynomials $\sum a_j 3^j$ coincides with the formula for the addition of Witt vectors over $\mathbb{F}_3$.

Next, we describe the general formalism that allows us to give a precise mathematical meaning both to the base $\mathbb{S}[\pm 1] = \{0, \pm 1\}$ and (to) the two cohomologies $H^\bullet(D)$ with their integer-valued dimension. The basic step underlying this formalism is taken by going beyond the traditional use of abelian groups and rings in algebra using Segal's Γ-rings. In [2,5] we developed the fundamentals of a geometric theory where the initial ring $\mathbb{Z}$ of the category of rings is replaced by the sphere spectrum $\mathbb{S}$. A similar change of structures is familiar in homotopy theory, but in our work we insist on staying at a concrete computational level in the definition of the basic objects, while the link with the Γ-spaces of homotopy theory [6] enters only when doing homological algebra. The extension of the category of abelian groups is obtained by embedding faithfully and fully this category in the category $\mathbb{S} - \mathrm{Mod}$ of pointed covariant functors from finite pointed sets to pointed sets. Given an abelian group H one assigns to a finite pointed set X the pointed set of H-valued divisors on X. These divisors push forward by summing over the pre-image of a point. In $\mathbb{S} - \mathrm{Mod}$ the traditional notion of ring becomes that of Segal's Γ-ring, and the (classical) base $\mathbb{Z}$ is replaced by the Γ-ring $\mathbb{S}$ here understood in its most elementary form of identity functor (from finite pointed sets to pointed sets). The base $\mathbb{S}$ is in fact the most elementary categorical form of the sphere spectrum in homotopy theory [6]. In analogy with the theory of algebraic curves over finite fields, we have (previously) defined in [2] the structure sheaf of the one-point compactification $\overline{\operatorname{Spec} \mathbb{Z}}$ of

[2] defined as the smallest integer $\geq x$.

[3] For $\deg D$ in the interval $\left(\log \frac{3^k}{2}, \log \frac{3^k+1}{2}\right)$ for $k = 0$, the right hand side is 0.

the algebraic spectrum of the integers by suitably extending, at the archimedean place, the structure sheaf of $\operatorname{Spec}\mathbb{Z}$ as a subsheaf of the constant sheaf $\mathbb{Q}$. The global sections of this sheaf determine the $\mathbb{S}$-algebra $\mathbb{S}[\pm 1]$. In [3] we have generalized the notion of homology for simplicial complexes, when group coefficients are replaced by $\mathbb{S}$-modules. In [4] we implemented homological algebra in this context by showing an extension of the Dold-Kan correspondence which associates a Γ-space to a short complex of $\mathbb{S}$-modules.

Let us now introduce the cohomologies and their dimensions. An Arakelov divisor $D = \sum_j a_j\{p_j\} + a\{\infty\}$ on $\overline{\operatorname{Spec}\mathbb{Z}}$ determines a compact *subset* $\mathcal{O}(D) \subset \mathbb{A}_{\mathbb{Q}}$ of the adeles of $\mathbb{Q}$. This is the product, indexed by the places of $\mathbb{Q}$, of the abelian group $\mathbb{Z}_p \subset \mathbb{Q}_p$ for each finite prime $p \notin \{p_j\}_j$, the abelian groups $p_j^{-a_j}\mathbb{Z}_{p_j}$, and the interval $[-e^a, e^a] \subset \mathbb{R}$. This last component is implemented by a sub-module of the Eilenberg-MacLane $\mathbb{S}[\pm 1]$-module $H\mathbb{R}$ which functorially encodes the additive structure of the group $\mathbb{R}$. The morphisms of abelian groups used by Weil in the development of the adelic geometry of function fields still retain a meaning in this absolute set-up and they are viewed as morphisms of $\mathbb{S}[\pm 1]$-modules. We focus, in particular, on the morphism

$$\psi : \mathbb{Q} \times \mathcal{O}(D) \to \mathbb{A}_{\mathbb{Q}} \quad \psi(q, a) = q + a \quad \forall q \in \mathbb{Q},\ a \in \mathcal{O}(D), \tag{1.2}$$

where $\mathbb{Q}$ is embedded diagonally in the adeles. The Γ-space $\mathbf{H}(D)$ associated by the Dold-Kan correspondence to (1.2) (ψ is viewed here as a short complex of $\mathbb{S}[\pm 1]$-modules: see appendix C provides the absolute incarnation of the Riemann-Roch problem for the divisor D. By implementing linear equivalence i.e. the multiplicative action of $\mathbb{Q}^\times$ on $\mathbb{A}_{\mathbb{Q}}$, one is reduced to consider only the case $D = a\{\infty\}$, for which $\ker(\psi)$ is the $\mathbb{S}[\pm 1]$-module $H^0(D) = \|H\mathbb{Z}\|_{e^a}$ that associates to a finite pointed set X the $\mathbb{Z}$-valued divisors $\sum_i n_i x_i$, $x_i \in X$, fulfilling the condition $\sum_i |n_i| \le e^a$ ($|\cdot|$ = euclidean absolute value). By definition, $H^0(D)$ is a covariant functor $\Gamma^o \longrightarrow \mathfrak{Sets}_*$ from the small category Γ^o (a skeleton of the category of finite pointed sets) to pointed sets. It keeps track of the partially defined addition in $I = \|H\mathbb{Z}\|_{e^a}(1_+) = [-e^a, e^a] \cap \mathbb{Z}$, so that for $n_i \in I$, the sum $\sum n_i$ is meaningful provided $\sum_i |n_i| \le e^a$. The dimension $\dim_{\mathbb{S}[\pm 1]} H^0(D)$ is defined to be the smallest number of *linear* generators of I. More precisely, a subset $F \subset I$ linearly generates if and only if for every $m \in I$ there exist coefficients $\alpha(f) \in \{-1, 0, 1\} = \mathbb{S}[\pm 1](1_+)$ such that $m = \sum \alpha(f) f$, $f \in F$, and $\sum |\alpha(f) f| \le e^a$ (see section 3). The $\mathbb{S}[\pm 1]$-module $H^0(D)$ only depends upon the integer part $n = \lfloor e^a \rfloor$ of e^a and Proposition 3.3 determines its dimension to be[4]

$$\dim_{\mathbb{S}[\pm 1]}(\|H\mathbb{Z}\|_n) = \left\lceil \frac{\log(2n+1)}{\log 3} \right\rceil. \tag{1.3}$$

The cokernel of ψ in (1.2), that is $H^1(D)$, is defined in terms of the Γ-space $\mathbf{H}(D)$. The lack of transitivity of the homotopy relation in non-Kan complexes is dealt with

[4] For the values $n = 4, 13, 40, \ldots, (3^k - 1)/2$, every element of I is uniquely written in terms of the generating set $F = \{3^i \mid 0 \le i < k\}$ (Proposition 3.3 (*iii*)).

using the classical notion of tolerance relation [11]. Thus $H^1(D)$ is the couple of the $\mathbb{S}[\pm 1]$-module $H\mathbb{A}_\mathbb{Q}$ and a suitable tolerance relation $\mathcal{R}$ on it (see Proposition 2.2 and appendix B). This relation is defined by pairs whose difference belongs to the image of ψ. More specifically, $H^1(D)$ defines an object of the category $\Gamma\mathcal{T}_*$ of tolerance $\mathbb{S}$-modules (see appendix B). Its dimension $\dim_{\mathbb{S}[\pm 1]} H^1(D)$ is defined to be the smallest number of linear generators where a subset $F \subset H\mathbb{A}_\mathbb{Q}(1_+) = \mathbb{A}_\mathbb{Q}$ linearly generates if and only if for every $x \in H\mathbb{A}_\mathbb{Q}(1_+)$ there exist coefficients $\alpha(f) \in \{-1, 0, 1\}$ such that the pair $(x, \sum \alpha(f) f)$ belongs to the relation $\mathcal{R}$ (see section 4).
An attentive reader should readily recognize that our construction strictly parallels Weil's adelic proof of the Riemann-Roch formula for function fields and, in particular, that our notion of dimension is a straightforward generalization of the classical dimension for vector spaces. For an archimedean divisor $D = a\{\infty\}$, the dimension of $H^1(D)$ is the same as the dimension of the pair $(H(\mathbb{R}/\mathbb{Z}), \mathcal{R})$, where the relation $\mathcal{R}$ on $H(\mathbb{R}/\mathbb{Z})(1_+) = \mathbb{R}/\mathbb{Z}$ is given by

$$(x, y) \in \mathcal{R} \iff d(x, y) \leq e^a, \tag{1.4}$$

for d the translation invariant Riemannian metric of length 1 on $\mathbb{R}/\mathbb{Z}$. In Proposition 4.1 we show that this dimension is the integer

$$\dim_{\mathbb{S}[\pm 1]}(H(\mathbb{R}/\mathbb{Z}), \mathcal{R}) = \left\lceil \frac{-a - \log 2}{\log 3} \right\rceil. \tag{1.5}$$

Theorem 1.2 follows from (1.3) when $\deg D \geq -\log 2$, since in this case $\dim_{\mathbb{S}[\pm 1]} H^1(D) = 0$. The case $\deg D < -\log 2$ is proved using (1.5). The antisymmetry of the term $\lceil \frac{\deg D + \log 2}{\log 3} \rceil'$ under the replacement of D by the divisor $K - D$, with $K := -2\{2\}$, is the numerical evidence of a geometric duality holding on the curve $\overline{\operatorname{Spec}\mathbb{Z}}$ over $\mathbb{S}[\pm 1]$ (see section 5). This is precisely stated by the following (see Theorem 5.3)

Theorem 1.3. *Let D be an Arakelov divisor on $\overline{\operatorname{Spec}\mathbb{Z}}$ and $K = -2\{2\}$, with $\deg K = -2\log 2$. Then there is an isomorphism of $\mathbb{S}[\pm 1]$-modules.*

$$H^0(K - D) \simeq \underline{\operatorname{Hom}}_{\Gamma\mathcal{T}_*}(H^1(D), U(1)_{\frac{1}{4}}), \tag{1.6}$$

where $U(1)_{\frac{1}{4}}$ is defined in Proposition 4.1 (see also Proposition B.3) and coincides with $H^1(K)$, playing the role of the dualizing module in Pontryagin duality.

The isomorphism $H^0(K - D) \simeq \underline{\operatorname{Hom}}_{\Gamma\mathcal{T}_*}(H^1(D), H^1(K))$ is the analogue of Grothendieck's version of Serre's duality [7] (12.15.1955). The duality (1.6) derives from Pontryagin's duality holding for tolerant $\mathbb{S}[\pm 1]$-modules (Proposition 5.2), once again in analogy with the corresponding result proven in [12] for curves over fields of positive characteristic. The divisor K plays the role of the canonical divisor.

The paper is organized as follows, in section 2 we adapt to the adelic framework the construction of the Γ-space $\mathbf{H}(D)$ naturally associated to an Arakelov divisor D. This gives, in Proposition 2.2, the cohomologies $H^0(D)$ and $H^1(D)$. We compute the dimension of $H^0(D)$ in section 3 (Theorem 3.4), and of $H^1(D)$ in section 4 which concludes with the proof of the main Theorem 4.3. Section 5 is devoted to Serre and Pontryagin dualities. In appendices A, B we recall some basic definitions on $\mathbb{S}$-modules and develop the needed generalities on tolerance $\mathbb{S}$-modules, while in appendix C we report a few technical details of the construction, through the Dold-Kan correspondence, of the Γ-space $\mathbf{H}(D)$.

2. The cohomology $H^\bullet(D)$

We recall, from [4], the construction of the Γ-space $\mathbf{H}(D)$ naturally associated to an Arakelov divisor D on $\overline{\operatorname{Spec}\mathbb{Z}}$. This space is the absolute homological incarnation of the Riemann-Roch problem for the divisor D. We formulate this construction in terms of the adeles of $\mathbb{Q}$.

Let $D = \sum_j a_j\{p_j\} + a\{\infty\}$ be an Arakelov divisor on $\overline{\operatorname{Spec}\mathbb{Z}}$. This is a formal finite sum with $a_j \in \mathbb{Z}$ and $a \in \mathbb{R}$, where p_j are rational primes in $\mathbb{Z}$ and the symbol ∞ stands for the restriction $|\cdot| : \mathbb{Q} \to \mathbb{R}$ of the euclidean absolute value $|\cdot|_\infty$. Let $\Sigma_\mathbb{Q}$ denote the full set of places ν of $\mathbb{Q}$. To D is naturally associated the following idele $\exp(D)$ of $\mathbb{Q}$ (v_j = the usual p_j-adic valuation on $\mathbb{Q}$)

$$\exp(D)_\nu := \begin{cases} p_j^{-a_j} & \text{if } \nu = \nu_j \\ 1 & \forall \nu \neq \nu_j,\ \forall j \\ e^a & \text{if } \nu = \infty. \end{cases}$$

The equality $\exp(D)(\nu) = |\exp(D)_\nu|_\nu$ defines a map $\exp(D) : \Sigma_\mathbb{Q} \to \mathbb{R}_+^*$ such that $\exp(D)(\nu) \in \operatorname{mod}(\mathbb{Q}_\nu)$, $\forall \nu \in \Sigma_\mathbb{Q}$, and $\exp(D)(\nu) = 1$, for $\nu \neq \infty, \nu_j, \forall j$. To the divisor D corresponds the compact *subset* of the adeles of $\mathbb{Q}$

$$\mathcal{O}(D) := \left\{(a_\nu) \in \mathbb{A}_\mathbb{Q} \mid |a_\nu|_\nu \le \exp(D)(\nu), \forall \nu \in \Sigma_\mathbb{Q}\right\} \subset \mathbb{A}_\mathbb{Q}.$$

By definition $\mathcal{O}(D) = \mathcal{O}(D)_f \times \mathcal{O}(D)_\infty = \prod_\nu \mathcal{O}(D)_\nu$ with

$$\mathcal{O}(D)_\nu = \begin{cases} p_j^{-a_j}\mathbb{Z}_{p_j} & \text{if } \nu = \nu_j \\ \mathbb{Z}_\nu & \forall \nu \neq \nu_j,\ \forall j,\ \nu < \infty \\ [-e^a, e^a] & \text{if } \nu = \infty. \end{cases}$$

In particular, the product $\mathcal{O}(D)_f = \prod_{\nu\neq\infty} \mathcal{O}(D)_\nu$ is a compact abelian group. The archimedean component, on the other hand, *i.e.* the real interval $[-e^a, e^a]$, is not, and for this reason we implement the functorial viewpoint to suitably understand $\mathcal{O}(D)$. Indeed, to $\mathcal{O}(D)_\infty$ we associate the covariant functor

$$\|H\mathbb{R}\|_{e^a}:\Gamma^o\longrightarrow\mathfrak{Sets}_*\qquad \|H\mathbb{R}\|_{e^a}(F):=\left\{\phi\in H\mathbb{R}(F)\mid\sum_{F\setminus\{*\}}|\phi(x)|\leq e^a\right\}$$

from the opposite of the Segal category (see *e.g.* [6] Chpt. 2 and [2]) to pointed sets, which maps a finite pointed set F to the pointed set of $\mathbb{R}$-valued divisors on F vanishing on the base point and whose total mass $\sum_{F\setminus\{*\}}|\phi(x)|$ is bounded by e^a. Covariant functors $\Gamma^o\longrightarrow\mathfrak{Sets}_*$ and their natural transformations determine the category $\Gamma\mathfrak{Sets}_*$ of Γ-sets (aka $\mathbb{S}$-modules: see appendix A). In particular, the Eilenberg-MacLane functor H encodes an abelian group A as the covariant functor $HA:\Gamma^o\longrightarrow\mathfrak{Sets}_*$ that associates to F the pointed set of A-valued divisors on F vanishing on the base point. Functoriality holds by taking sums on the inverse images of a point. The functor HA encodes the addition on $A=HA(1_+)$. In general, let $\sigma\in\mathrm{Hom}_{\Gamma^o}(k_+,1_+)$ with $\sigma(\ell)=1\ \forall\ell\neq *$ and

$$\delta(j,k)\in\mathrm{Hom}_{\Gamma^o}(k_+,1_+),\quad \delta(j,k)(\ell):=\begin{cases}1 & \text{if } \ell=j\\ * & \text{if } \ell\neq j.\end{cases}\tag{2.1}$$

Given an $\mathbb{S}$-module $\mathcal{F}$ and elements $x,x_j\in\mathcal{F}(1_+)$, $j=1,\ldots,k$, one writes

$$x=\sum_j x_j\iff \exists z\in\mathcal{F}(k_+)\text{ s.t. }\mathcal{F}(\sigma)(z)=x,\ \mathcal{F}(\delta(j,k))(z)=x_j,\ \forall j.\tag{2.2}$$

For $A=\mathbb{R}$ the key point is that this general construction respects the bound on the total mass while encoding the addition. One easily sees that $\|H\mathbb{R}\|_{e^a}$ is an $\mathbb{S}[\pm1]$-module (equivalently an "$(\mathbb{S}[\pm1]\wedge -)$-algebra" as in [6] 2.1.5), where $\mathbb{S}[\pm1]:\Gamma^o\longrightarrow\mathfrak{Sets}_*$, $\mathbb{S}[\pm1](F):=\{-1,0,1\}\wedge F$ is the spherical monoid algebra of the multiplicative monoid $\{\pm1\}$. In particular, the inclusion functor $\|H\mathbb{R}\|_{e^a}\longrightarrow H\mathbb{R}$ determines $\|H\mathbb{R}\|_{e^a}$ as a sub $\mathbb{S}[\pm1]$-module of $H\mathbb{R}$, where $\mathbb{S}$ is the identity functor $\mathbb{S}:\Gamma^o\longrightarrow\mathfrak{Sets}_*$.

Given two $\mathbb{S}$-modules $\mathcal{F}_j$, $j=1,2$, one defines their product as the functor

$$\mathcal{F}_1\times\mathcal{F}_2:\Gamma^o\longrightarrow\mathfrak{Sets}_*,\qquad (\mathcal{F}_1\times\mathcal{F}_2)(F)=\mathcal{F}_1(F)\times\mathcal{F}_2(F)$$

with the base point of $\mathcal{F}_1(F)\times\mathcal{F}_2(F)$ taken to be $(*,*)$. For abelian groups A,B one has a canonical isomorphism $H(A\times B)\simeq HA\times HB$. In particular, the morphism of addition in adeles $\alpha:\mathbb{Q}\times\mathbb{A}_\mathbb{Q}\to\mathbb{A}_\mathbb{Q}$, $\alpha(q,a)=q+a$ (using the diagonal embedding of $\mathbb{Q}$ in $\mathbb{A}_\mathbb{Q}$) determines a morphism of $\mathbb{S}[\pm1]$-modules $H\alpha:H\mathbb{Q}\times H\mathbb{A}_\mathbb{Q}\longrightarrow H\mathbb{A}_\mathbb{Q}$. Next proposition shows how this functorial construction is well combined with the adelic formalism.

Proposition 2.1.

(i) *The* $\mathbb{S}[\pm1]$*-module*

$$H\mathcal{O}(D):=H(\mathcal{O}(D)_f)\times\|H\mathbb{R}\|_{e^a}$$

determines, canonically, a sub $\mathbb{S}[\pm1]$*-module* $\iota:H\mathcal{O}(D)\longrightarrow H\mathbb{A}_\mathbb{Q}$.

(ii) *The restriction of $H\alpha$ to $H\mathbb{Q} \times H\mathcal{O}(D)$ defines a morphism of $\mathbb{S}[\pm 1]$-modules*

$$\psi : H\mathbb{Q} \times H\mathcal{O}(D) \longrightarrow H\mathbb{A}_{\mathbb{Q}}. \tag{2.3}$$

Proof. (i) With $\mathbb{A}_{\mathbb{Q}} = \mathbb{A}_f \times \mathbb{R}$, let $\iota_f : \mathcal{O}(D)_f \to \mathbb{A}_f$ be the inclusion of abelian groups. Then $\iota : H\mathcal{O}(D) \to H\mathbb{A}_{\mathbb{Q}}$ is the product of $H\iota_f$ and the inclusion $\|H\mathbb{R}\|_{e^a} \to H\mathbb{R}$.
(ii) By composition one obtains $\psi = H\alpha \circ (H\mathrm{id}_{\mathbb{Q}} \times \iota)$. $\square$

The morphism ψ in (2.3) is obtained by restricting $H\alpha$ to the (adelic) divisor D, where α is a morphism of abelian groups. The theory developed in [4] associates to ψ a Γ-space $\mathbf{H}(D)$, by implementing the Dold-Kan correspondence in the special case of a short complex of abelian groups and its restriction to a sub-$\mathbb{S}$-module. The Γ-space $\mathbf{H}(D)$ supplies the absolute cohomology of the divisor D. We recall this construction in appendix C. Moreover, we review how the generalized homotopy $\pi_*^{\mathcal{T}}(\mathbf{H}(D))$, where $\mathcal{T}$ is the category of tolerance relations defined in B.1, is meaningful in this context supplying, with $\pi_1^{\mathcal{T}}(\mathbf{H}(D))$ and $\pi_0^{\mathcal{T}}(\mathbf{H}(D))$, the (tolerant) $\mathbb{S}$-modules descriptions of *resp.* the kernel and the cokernel of ψ. In other words, one obtains the following result

Proposition 2.2.

(i) *The kernel of ψ is the $\mathbb{S}[\pm 1]$-module:*

$$H^0(D) = (H\mathrm{id}_{\mathbb{Q}} \times \iota)^{-1}(H\ker(\alpha)) \simeq \iota^{-1}H\mathbb{Q} = \|H^0(\operatorname{Spec}\mathbb{Z}, \mathcal{O}(D)_f)\|_{e^a}. \tag{2.4}$$

(ii) *The cokernel of ψ is the $\mathbb{S}[\pm 1]$-module $H^1(D) = (H\mathbb{A}_{\mathbb{Q}}, \mathcal{R})$ endowed with the relations*

$$\mathcal{R}_k := \{(x, y) \in H\mathbb{A}_{\mathbb{Q}}(k_+) \times H\mathbb{A}_{\mathbb{Q}}(k_+) \mid x - y \in \operatorname{Image}\psi(k_+)\}. \tag{2.5}$$

(iii) *Both $H^0(D)$ and $H^1(D)$ only depend, up to isomorphism, on the linear equivalence class of the divisor D.*

Proof. We refer to appendix C.1. $\square$

3. The dimension of $H^0(D)$

In this section we consider, for each integer $n > 0$, the $\mathbb{S}[\pm 1]$-module $\|H\mathbb{Z}\|_n$

$$\|H\mathbb{Z}\|_n : \Gamma^o \longrightarrow \mathfrak{Sets}_* \qquad \|H\mathbb{Z}\|_n(F) := \{\phi \in H\mathbb{Z}(F) \mid \sum_{F\setminus\{*\}} |\phi(x)| \leq n\}$$

and evaluate its dimension. It follows from the $\mathbb{S}[\pm 1]$-module structure, that (2.2) holds for a sum of the form

$$\sum_F \alpha_j j \in [-n,n]\cap\mathbb{Z} = \|H\mathbb{Z}\|_n(1_+) \quad \forall \alpha_j \in \{-1,0,1\} \text{ s.t. } \sum_F |\alpha_j j| \le n.$$

By applying Definition B.2, a subset $F \subset [-n,n]\cap\mathbb{Z} = \|H\mathbb{Z}\|_n(1_+)$ generates[5] $\|H\mathbb{Z}\|_n(1_+)$ if and only if for every element $x \in \|H\mathbb{Z}\|_n(1_+)$ there exists coefficients $\alpha_j \in \{-1,0,1\}$, $j \in F$, such that $x = \sum_F \alpha_j j$ and $\sum_F |\alpha_j j| \le n$. The *dimension* $\dim_{\mathbb{S}[\pm 1]} \|H\mathbb{Z}\|_n$ is the minimal cardinality of a generating set.

The explicit computation of $\dim_{\mathbb{S}[\pm 1]} \|H\mathbb{Z}\|_n$ follows from the next two statements.

Lemma 3.1. *For $k \in \mathbb{N}$, let $n = (3^k - 1)/2$. Consider the subset $F := \{3^i, 0 \le i \le k-1\} \subset [-n,n]$. Then F is a generating set and the map*

$$\theta : \{-1,0,1\}^{\times k} \to [-n,n]\cap\mathbb{Z}, \qquad \theta((\alpha_i)) := \sum_{i=0}^{k-1} \alpha_i 3^i \tag{3.1}$$

is bijective. Moreover $\sum_{i=0}^{k-1} |\alpha_i 3^i| \le n$, for all $\alpha_i \in \{-1,0,1\}^{\times k}$.

Proof. First of all, note that for all $\alpha \in \{-1,0,1\}^{\times k}$ one has

$$\sum_{i=0}^{k-1} |\alpha_i 3^i| \le \sum_{i=0}^{k-1} 3^i = n.$$

This shows that for any $\alpha \in \{-1,0,1\}^{\times k}$ one has $z = (\alpha_i 3^i) \in \|H\mathbb{Z}\|_n(k_+)$, and, as in (2.2), $\|H\mathbb{Z}\|_n(\sigma)(z) = \theta((\alpha_i))$. We prove the bijectivity of θ by induction on k. It is clear for $k = 1$. For $k = 2$, $n = 4$ and one writes the set $[-4,4]\cap\mathbb{Z}$ as the disjoint union of three subsets as follows

$$\begin{aligned}[-4,4]\cap\mathbb{Z} &= \{-4,-3,-2\} \cup \{-1,0,1\} \cup \{2,3,4\} = \\ &= (-3 + \{-1,0,1\}) \cup \{-1,0,1\} \cup (3 + \{-1,0,1\}).\end{aligned}$$

This shows that θ is bijective. Next, by assuming to have shown the bijectivity up to $k-1$, we deduce it for k. Indeed, with $n = (3^k - 1)/2$ let $m = (3^{k-1} - 1)/2$. The induction hypothesis ensures that every element of the set $[-m,m]\cap\mathbb{Z}$ can be uniquely written as a sum $\sum_{i=0}^{k-2} \alpha_i 3^i$. We divide the set $[-n,n]\cap\mathbb{Z}$ into three disjoint parts of the form

$$J_{-1} := [-n, -n+2m]\cap\mathbb{Z}, \quad J_0 := [-m,m]\cap\mathbb{Z}, \quad J_1 := [n-2m, n]\cap\mathbb{Z}.$$

One has $n - 2m = m + 1$ and thus the union of these three disjoint sets is $[-n,n]\cap\mathbb{Z}$. Moreover one has $J_1 = (n-m) + J_0$ with $n - m = 3^{k-1}$, and similarly $J_{-1} = J_0 - 3^{k-1}$. Thus, given $x \in [-n,n]\cap\mathbb{Z}$ one determines uniquely the coefficients $\alpha_i \in \{-1,0,1\}$ such

[5] For simplicity we use from now on the term “generates” for “linearly generates”.

that $\theta((\alpha_i)) = x$. Indeed, one uses the above partition in three intervals to determine the coefficient α_{k-1} and then one applies the induction hypothesis to determine the others. □

Remark 3.2.

1. The conceptual explanation of Lemma 3.1 derives from the following peculiar property of the Teichmüller lift $\tau : \mathbb{F}_3 \to \mathbb{Z}_3$. One has $\tau(\mathbb{F}_3) = \{-1, 0, 1\} \subset \mathbb{Z} \subset \mathbb{Z}_3$ and τ extends to a canonical map $\tilde{\tau} : W(\mathbb{F}_3) \to \mathbb{Z}$ from (isotypical) Witt vectors to $\mathbb{Z}$

$$\xi = (\xi_j) \in W(\mathbb{F}_3) \mapsto \tilde{\tau}(\xi) := \sum_0^{k-1} \tau(\xi_j) 3^j \in \mathbb{Z}.$$

   This map coincides with θ in (3.1) and one can compute the partially defined sums (2.2) in $\|H\mathbb{Z}\|_n$ using the addition in $W_k(\mathbb{F}_3)$. The prime $p = 3$ is the only prime such that the subring $\mathbb{Z} \subset \mathbb{Z}_p = W(\mathbb{F}_p)$ coincides with the ring of Witt vectors with finitely many non-zero components. When $p > 3$ one has $\tau(\mathbb{F}_p) \not\subset \mathbb{Z}$, while for $p = 2$ the integer $-1 \in \mathbb{Z} \subset \mathbb{Z}_2$ is the Witt vector whose components are all equal to 1. In general the Witt vectors with finitely many non-zero components do not even form a subgroup of the additive group of Witt vectors.
2. The Teichmüller lift τ, as a morphism of multiplicative pointed monoids, induces a morphism $\mathbb{S}[\pm 1] \to H\mathbb{Z}$ of $\mathbb{S}$-algebras.
3. The ordering of the natural numbers encoded by θ coincides with the lexicographic ordering of the coefficients $(\alpha_i) \in \{-1, 0, 1\}^{\times k}$

$$\theta((\alpha_i)) < \theta((\beta_i)) \iff \exists j \text{ s.t. } \alpha_j < \beta_j \text{ and } \alpha_\ell = \beta_\ell \quad \forall \ell > j.$$

We recall that the ceiling function $\lceil x \rceil$ associates to a positive real number x the smallest integer $n \geq x$.

Proposition 3.3.

(i) *For any integer $n \geq 0$ one has*

$$\dim_{\mathbb{S}[\pm 1]} \|H\mathbb{Z}\|_n = \left\lceil \frac{\log(2n+1)}{\log 3} \right\rceil. \tag{3.2}$$

(ii) *For $n \notin \{2, 5\}$, there exists a generating subset $F \subset \{1, \ldots, n\}$ of cardinality $\#F = \lceil \frac{\log(2n+1)}{\log 3} \rceil$, with $\sum_{j \in F} j = n$, such that the following map surjects onto $[-n, n] \cap \mathbb{Z}$*

$$\theta : \{-1, 0, 1\}^{\times(\#F)} \to [-n, n] \cap \mathbb{Z}, \qquad \theta((\alpha_i)) := \sum_{j \in F} \alpha_j j.$$

Proof. (i) Set $\kappa(n) := \dim_{\mathbb{S}[\pm 1]} \|H\mathbb{Z}\|_n$. Since the cardinality of $[-n,n] \cap \mathbb{Z}$ is $2n+1$ and the cardinality of $\{-1,0,1\}^{\times \# F}$ is $3^{\# F}$, it follows that if F is a generating set one must have $3^{\# F} \geq 2n+1$. Thus one has $\# F \geq \frac{\log(2n+1)}{\log 3}$, and since $\# F$ is an integer one gets $\# F \geq \lceil \frac{\log(2n+1)}{\log 3} \rceil$. This shows that $\kappa(n) \geq \lceil \frac{\log(2n+1)}{\log 3} \rceil$. It remains to prove that for each $n \geq 0$ one can find a generating set F with $\# F = \lceil \frac{\log(2n+1)}{\log 3} \rceil$. One uses Lemma 3.1, and distinguishes several cases starting with the easiest one. Set $F(k) := \{3^i, 0 \leq i \leq k-1\}$ and let $E \subset \mathbb{N}$ be the subset

$$E = \left\{3^\ell + \frac{1}{2}\left(3^m - 1\right) \mid m \in \mathbb{N}, \ell < m\right\}. \tag{3.3}$$

We list the first elements of E as follows

$$2, 5, 7, 14, 16, 22, 41, 43, 49, 67, 122, 124, 130, 148, 202, 365, 367, 373, 391, 445, 607 \ldots$$

First, assume that $n \notin E$. Then let m be the largest integer such that $3^m \leq 2n+1$. If $3^m = 2n+1$, then Lemma 3.1 provides one with a generating set of cardinality $m = \frac{\log(2n+1)}{\log 3}$. Assume now that $3^m < 2n+1$. Then, the map θ for $F(m)$ covers the set $[-q,q] \cap \mathbb{Z}$, where $q = \frac{1}{2}\left(3^m - 1\right)$. Thus by adjoining to $F(m)$ the element $n - q$ one covers $[-n,n] \cap \mathbb{Z}$. More precisely, let $F = F(m) \cup \{n - \frac{1}{2}\left(3^m - 1\right)\}$, then we show that F fulfills the conditions in (ii). One has $\# F = \# F(m) + 1$ since the additional element $n - \frac{1}{2}\left(3^m - 1\right)$ does not belong to $F(m)$ as, by hypothesis, $n \notin E$. Since $\# F(m) = m$, $\# F = m + 1 = \lceil \frac{\log(2n+1)}{\log 3} \rceil$. The sum of elements in $F(m)$ is $\frac{1}{2}\left(3^m - 1\right)$, so the sum of elements in F is equal to n. Next, we show that θ is surjective. Its image contains the three sets $\theta(F(m))$, $\theta(F(m)) + n - \frac{1}{2}\left(3^m - 1\right)$, and $\theta(F(m)) - (n - \frac{1}{2}\left(3^m - 1\right))$. Using Lemma 3.1 one obtains, with $q = \frac{1}{2}\left(3^m - 1\right)$

$$\begin{aligned}
&\theta F(m) = [-q,q] \cap \mathbb{Z}, \quad \theta F(m) + n - q = [n - 2q, n] \cap \mathbb{Z}, \\
&\theta F(m) - (n - q) = [-n, -n + 2q] \cap \mathbb{Z}.
\end{aligned}$$

One has $3(2q+1) = 3^{m+1} > 2n+1$. This inequality prevents the three subsets from being disjoint, thus the upper limit q of $\theta F(m)$ is greater or equal to the lower limit $n - 2q$ of the interval $\theta F(m) + n - q$. Thus θ is surjective.

The next case is for $n \in E$ of the form $n = 1 + \frac{1}{2}\left(3^m - 1\right)$, for some m ($\ell = 0$ in (3.3)). We assume that $m > 2$ (this excludes the cases $n = 2$ for $m = 1$, and $n = 5$ for $m = 2$). Let

$$F := \{3^k \mid 0 \leq k \leq m - 2\} \cup \{2, 3^{m-1} - 1\}.$$

(This choice avoids the repetition of the element $1 = n - q$ while keeping the sum of elements of F equal to n.) We show that F fulfills the conditions in (ii). We have $\# F = m - 1 + 2 = m + 1$ since $3^{m-1} - 1 \notin \{1, 2\}$. Moreover $2n + 1 = 3^m + 2$, thus $\#F = m+1 = \lceil \frac{\log(2n+1)}{\log 3} \rceil$. The sum of terms in F is equal to one plus the sum of the powers of 3 up to 3^{m-1} and thus is equal to n. Let us now show that the map θ is surjective. By construction $\theta(F)$ is the union of nine intervals obtained, with $q' = \frac{1}{2}(3^{m-1}-1)$, by adding to the set $[-q', q'] \cap \mathbb{Z}$ the terms in $\{2, 3^{m-1}-1\}$ with coefficients in $\{-1, 0, 1\}$. With the term $3^{m-1}-1$ one covers the set $[-q+1, q-1] \cap \mathbb{Z}$, where $q = \frac{1}{2}(3^m - 1)$ as the union of the three sets

$$[-q', q'] \cap \mathbb{Z}, \quad ([-q', q'] + 3^{m-1} - 1) \cap \mathbb{Z}, \quad ([-q', q'] - (3^{m-1} - 1)) \cap \mathbb{Z}.$$

One has $q' + 3^{m-1} - 1 = n - 2$, and thus it remains to cover the two elements $n-1, n$ and their opposite. This is done using the set $([-q', q'] + 3^{m-1} - 1 + 2) \cap \mathbb{Z}$ whose upper limit is n, and appealing to the fact that since $q' > 1$ this interval contains $n-1$.
Finally, we consider the case $n \in E$ of the form $n = 3^\ell + \frac{1}{2}(3^m - 1)$, with $m \in \mathbb{N}$, $0 < \ell < m$. Here, the additional element $n - \frac{1}{2}(3^m - 1)$ does belong to $F(m)$ so the proof of the first case considered above does not apply since 3^ℓ appears twice in $F(m) \cup \{n - \frac{1}{2}(3^m - 1)\}$. We replace this double occurrence of 3^ℓ by the two distinct elements $3^\ell \pm 1$. This gives

$$F = (F(m) \setminus \{3^\ell\}) \cup \{3^\ell - 1, 3^\ell + 1\}.$$

By construction $\#F(m) + 1 = m + 1$. Furthermore one has: $n = 3^\ell + \frac{1}{2}(3^m - 1)$, $2n+1 = 3^m + 2 \times 3^\ell$ and since $0 < \ell < m$ one gets $\#F = m + 1 = \lceil \frac{\log(2n+1)}{\log 3} \rceil$. The sum of elements of F is the same as the sum of elements of $F(m)$ plus 3^ℓ which is equal to n. It remains to show that, for such F, θ is surjective. We first deal with the set $G(\ell) := F(\ell) \cup \{3^\ell - 1, 3^\ell + 1\}$ and show that the map $\theta_{G(\ell)}$, for this set, surjects onto the interval $[-t(\ell), t(\ell)] \cap \mathbb{Z}$, where $t(\ell) = \sum_{G(\ell)} j = 3^\ell + \frac{1}{2}(3^{\ell+1} - 1)$. Lemma 3.1 shows that the image of $\theta_{F(\ell)}$ is the set $J \cap \mathbb{Z}$, with $J = [-\frac{1}{2}(3^\ell - 1), \frac{1}{2}(3^\ell - 1)]$. To show the surjectivity of $\theta_{G(\ell)}$ it is enough to show that the following intervals cover $[0, t(\ell)]$:

$$J, \quad J + 3^\ell - 1, \quad J + 3^\ell + 1, \quad J + 2 \times 3^\ell.$$

The upper limit of $J + 2 \times 3^\ell$ is $\frac{1}{2}(3^\ell - 1) + 2 \times 3^\ell = t(\ell)$, its lower limit is equal to the upper limit of $J + 3^\ell + 1$. The lower limit of $J + 3^\ell - 1$ is $3^\ell - 1 - \frac{1}{2}(3^\ell - 1) = \frac{1}{2}(3^\ell - 1)$ which is the upper limit of J. Moreover, the length of J is $3^\ell - 1 \geq 2$ so that two translates of J of the form $J + a$, $J + a + 2$ necessarily overlap. This shows that the map $\theta_{G(\ell)}$ surjects onto the interval $[-t(\ell), t(\ell)] \cap \mathbb{Z}$. Thus $\theta_{G(\ell)}$ is the symmetric enlargement of the interval $\theta(F(\ell + 1))$ obtained by adding 3^ℓ to the upper limit. Then, as in the proof of Lemma 3.1, one obtains by induction that when one adjoins the higher powers $3^{\ell'}$, where $\ell < \ell' < m$, one achieves the enlargement of the interval $\theta(F(m))$ obtained by adding 3^ℓ to the upper limit. This shows that the map θ for F is surjective. In fact the above arguments prove (ii) and (i), except that one has to take care of the special cases $n = 2, 5$. For $n = 2$, $\{1, 2\}$ is a generating set but the sum of its terms is > 2. For $n = 5$, $\{1, 2, 3\}$ is a generating set but the sum of its terms is > 5. $\square$

We are now ready to determine the dimension of the $\mathbb{S}[\pm 1]$-module $H^0(D)$, for any Arakelov divisor D. It is a non-decreasing function of $\deg D$ but the formula for this dimension drops by 1 on an exceptional set L. It is the open set defined as the union of open intervals as follows

$$L + \log 2 = \bigcup_{k\in\mathbb{N}} (k\log 3, k\log 3 + \epsilon_k), \quad \epsilon_k := \log(1 + 3^{-k}). \tag{3.4}$$

It has finite Lebesgue measure

$$|L| = \sum_{k\geq 0} \epsilon_k = \log\left(-1; \frac{1}{3}\right)_\infty = 1.14099\ldots$$

where $\left(-1; \frac{1}{3}\right)_\infty = \prod_{n=0}^{\infty}(1 + 3^{-n})$ is the -1 Pochhammer symbol at $\frac{1}{3}$.

Theorem 3.4. *Let D be an Arakelov divisor on $\overline{\operatorname{Spec}\mathbb{Z}}$. If $\deg D \geq -\log 2$, then*

$$\dim_{\mathbb{S}[\pm 1]} H^0(D) = \left\lceil \frac{\deg D + \log 2}{\log 3} \right\rceil - \mathbf{1}_L \tag{3.5}$$

where $\mathbf{1}_L$ is the characteristic function of the open set L.

Proof. Let $a = \deg D$ and n be the integer-part $n = \lfloor \exp(\deg(D)\rfloor$, then $H^0(D) = \|H\mathbb{Z}\|_n$. By Proposition 3.3 one gets

$$\dim_{\mathbb{S}[\pm 1]} H^0(D) = \left\lceil \frac{\log(2n+1)}{\log 3} \right\rceil, \qquad n := \lfloor \exp(a) \rfloor.$$

Let us compare this formula with (3.5). Assume first that $a \in L$, thus there exists $k \in \mathbb{N}$ such that $a + \log 2 \in (k\log 3, k\log 3 + \epsilon_k)$. Then, $2\exp a \in (3^k, 3^k + 1)$ and with $n = \lfloor \exp a \rfloor$, one has $n = \frac{1}{2}(3^k - 1)$. Thus $2n + 1 = 3^k$ and $\lceil \frac{\log(2n+1)}{\log 3} \rceil = k$. Moreover, $\frac{a+\log 2}{\log 3} \in (k, k + \frac{\epsilon_k}{\log 3})$, where $0 < \frac{\epsilon_k}{\log 3} < 1$, hence one has $\lceil \frac{a+\log 2}{\log 3} \rceil = k+1$, so that (3.5) holds since $\mathbf{1}_L(a) = 1$. Now, assume that $a \notin L$, equivalently that $\frac{a+\log 2}{\log 3} \in [k + \frac{\epsilon_k}{\log 3}, k+1]$ for some integer $k \geq 0$ (since $a \geq -\log 2$ by hypothesis). Then $\lceil \frac{a+\log 2}{\log 3} \rceil = k + 1$. One has

$$2\exp a \in [\exp(k\log 3)(1 + 3^{-k}), \exp((k+1)\log 3)] = [3^k + 1, 3^{k+1}].$$

In this case $n = \lfloor \exp a \rfloor \in [\frac{1}{2}(3^k + 1), \frac{1}{2}(3^{k+1} - 1)]$ and thus $2n + 1 \in [3^k + 2, 3^{k+1}]$. This implies that $\frac{\log(2n+1)}{\log 3} \in (k, k + 1]$ and hence that $\lceil \frac{\log(2n+1)}{\log 3} \rceil = k + 1$. In this case (3.5) holds since $\mathbf{1}_L(a) = 0$. $\square$

4. The dimension of $H^1(D)$

Let (A, d) be an abelian group endowed with a translation invariant metric d. For $\lambda \in \mathbb{R}_{>0}$, we shall refer to $(A, d)_\lambda$ as the associated object of the category $\Gamma\mathcal{T}_*$ as in Proposition B.3. The $\mathbb{S}$-module structure determines the addition in $A = (A, d)_\lambda(1_+)$, as well as the action of $\mathbb{S}[\pm 1]$ on it, where for $x \in A = (A, d)_\lambda(1_+)$, $-x$ is the additive inverse of x. The metric d determines the tolerance relation $\mathcal{R}$ on $A = (A, d)_\lambda(1_+)$. Next result computes the dimension (Definition B.2) of the tolerant $\mathbb{S}[\pm 1]$-module $(\mathbb{R}/\mathbb{Z}, d)_\lambda$.

Proposition 4.1. *Let $\lambda \in \mathbb{R}_{>0}$, and $U(1)_\lambda = (U(1), d)_\lambda$ where $U(1)$ is the abelian group $\mathbb{R}/\mathbb{Z}$ endowed with the canonical metric d of length 1. Then*

$$\dim_{\mathbb{S}[\pm 1]} U(1)_\lambda = \begin{cases} \left\lceil \frac{-\log\lambda - \log 2}{\log 3} \right\rceil & \textit{if } \lambda < \frac{1}{2} \\ 0 & \textit{if } \lambda \geq \frac{1}{2}. \end{cases} \tag{4.1}$$

Proof. For $\lambda \geq \frac{1}{2}$, any element of $U(1)_\lambda = (\mathbb{R}/\mathbb{Z}, d)_\lambda$ is at distance $\leq \lambda$ from 0, thus one can take $F = \emptyset$ as generating set since, by convention, $\sum_\emptyset = 0$. Thus $\dim_{\mathbb{S}[\pm 1]} U(1)_\lambda = 0$. Next, we assume $\lambda < \frac{1}{2}$. Let $F \subset U(1)$ be a generating set and let $k = \#F$. One easily sees that there are at most 3^k elements of the form $\sum_F \alpha_j j$. The subsets $\{x \in U(1) \mid d(x, \alpha_j j) \leq \lambda\}$ cover $U(1)$, and since each of them has measure 2λ one gets the inequality $2\lambda \cdot 3^k \geq 1$. Thus $k \geq \frac{-\log\lambda - \log 2}{\log 3}$. For $\lambda = \frac{1}{6}$ one has $k \geq 1$ and the subset $F = \{\frac{1}{3}\}$ generates, thus $\dim_{\mathbb{S}[\pm 1]} U(1)_\lambda = 1$. When $\frac{-\log\lambda - \log 2}{\log 3} = m$ is an integer, one has $\lambda = \frac{1}{2}3^{-m}$. Let $F(m) = \{\frac{1}{3}, \ldots, \frac{1}{3^m}\}$. The minimal distance between two elements of $F(m)$ is the distance between 3^{-m} and $3^{-m+1} = 3 \cdot 3^{-m}$ which is $2 \cdot 3^{-m} = 4\lambda$. Let us show that $F(m)$ is a generating set. By Lemma 3.1 any integer q in the interval $[-N, N]$, for $N = \frac{1}{2}(3^m - 1)$ can be written as $q = \sum_{i=0}^{m-1} \alpha_i 3^i$, with $\alpha_i \in \{-1, 0, 1\}$. One then gets

$$q \cdot 3^{-m} = \sum_{i=0}^{m-1} \alpha_i 3^{i-m} = \sum_{j=1}^{m} \alpha_{m-j} 3^{-j}.$$

Let $x \in [-\frac{1}{2}, \frac{1}{2}]$, then $3^m x \in [-\frac{3^m}{2}, \frac{3^m}{2}]$ and there exists an integer $q \in [-N, N]$ such that $|3^m x - q| \leq \frac{1}{2}$. Hence $d(x, q \cdot 3^{-m}) \leq \lambda$. This proves that $F(m)$ is a generating set (see Definition B.2) and one derives $\dim_{\mathbb{S}[\pm 1]} U(1)_\lambda = m$. Assume now that $\frac{-\log\lambda - \log 2}{\log 3} \in (m, m+1)$, where m is an integer. For any generating set F of cardinality k one has $k \geq \frac{-\log\lambda - \log 2}{\log 3} > m$ so that $k \geq m + 1$. The subset $F(m+1) = \{\frac{1}{3}, \ldots, \frac{1}{3^{m+1}}\}$ fulfills the first condition of Definition B.2 since the minimal distance between two elements of $F(m+1)$ is $2 \cdot 3^{-m-1}$ which is larger than $\lambda < \frac{1}{2}3^{-m}$. As shown above, the subset $F(m+1)$ is generating for $\lambda = \frac{1}{2}3^{-m-1}$ and a fortiori for $\lambda > \frac{1}{2}3^{-m-1}$ (as by assumption $\frac{-\log\lambda - \log 2}{\log 3} < m + 1$). Thus one obtains $\dim_{\mathbb{S}[\pm 1]} U(1)_\lambda = m + 1$ and (4.1) is proven. $\square$

Remark 4.2. The proof of Proposition 4.1 relies on a definite advantage of the triadic expansion of numbers [14]: truncating a number is identical to rounding it. This property does not hold in the decimal system where rounding a number requires the knowledge of the next digit.

We extend the ceiling function to negative values of the variable as follows

$$\lceil x\rceil' = \begin{cases} \lceil x\rceil \text{ if } x \geq 0 \\ -\lceil -x\rceil \text{ if } x \leq 0. \end{cases} \tag{4.2}$$

We can now state and prove the absolute Riemann-Roch theorem for $\overline{\operatorname{Spec}\mathbb{Z}}$ over $\mathbb{S}[\pm 1]$. With L the exceptional set defined in (3.4) one has

Theorem 4.3. *Let D be an Arakelov divisor on $\overline{\operatorname{Spec}\mathbb{Z}}$. Then*

$$\dim_{\mathbb{S}[\pm 1]} H^0(D) - \dim_{\mathbb{S}[\pm 1]} H^1(D) = \left\lceil \frac{\deg D + \log 2}{\log 3} \right\rceil' - \mathbf{1}_L(\deg D). \tag{4.3}$$

Proof. By appealing to the invariance under linear equivalence (Proposition 2.2 (iii)), one may assume that $D = a\{\infty\}$. Then it follows from Proposition B.5 (ii) that $\dim_{\mathbb{S}[\pm 1]} H^1(D) = \dim_{\mathbb{S}[\pm 1]} U(1)_\lambda$ for $\lambda = \exp(\deg(D))$. Assume first that $\exp(\deg(D)) \geq \frac{1}{2}$, then by Proposition 4.1: $\dim_{\mathbb{S}[\pm 1]} H^1(D) = 0$, thus for $\deg(D) \geq -\log 2$ (4.3) follows from (3.5). Let us now assume that $\deg(D) < -\log 2$. Then $\dim_{\mathbb{S}[\pm 1]} H^0(D) = 0$ since $H^0(D) = \{*\}$ when $\deg D < 0$. Moreover $\deg D \notin L$ since L is lower bounded by $-\log 2$. Thus (4.3) follows from (4.1) which gives, using (4.2)

$$-\dim_{\mathbb{S}[\pm 1]} H^1(D) = -\dim_{\mathbb{S}[\pm 1]} U(1)_\lambda = -\left\lceil \frac{-\log \lambda - \log 2}{\log 3} \right\rceil = \left\lceil \frac{\deg D + \log 2}{\log 3} \right\rceil'.$$

This ends the proof of (4.3). $\square$

5. Duality

In this section we prove an absolute analogue of Serre's duality, namely the following isomorphism of $\mathbb{S}[\pm 1]$-modules, for any divisor D on $\overline{\operatorname{Spec}\mathbb{Z}}$:

$$H^0(D) \simeq \underline{\operatorname{Hom}}_{\Gamma\mathcal{T}_*}(H^1(K - D), U(1)_{\frac{1}{4}}).$$

Here, the divisor $K = -2\{2\}$ plays the role of the canonical divisor. The choice of the tolerant $\mathbb{S}[\pm 1]$-module $(U(1), d)_{\frac{1}{4}}$ as dualizing module is motivated by Pontryagin duality (see 5.2). One has $\dim_{\mathbb{S}[\pm 1]}(U(1)_{\frac{1}{4}}) = 1$. This equality, in fact, also holds for the tolerant $\mathbb{S}[\pm 1]$-module $U(1)_\lambda$ for $\frac{1}{6} \leq \lambda < \frac{1}{2}$: the specific choice $\lambda = \frac{1}{4}$ is dictated by the invariance of the Riemann-Roch formula (4.3) when one switches H^0 and H^1 and replaces D by $K - D$ (ignoring the exceptional set L).

5.1. $\underline{\mathrm{Hom}}_{\mathbb{S}}(\|H\mathbb{R}\|_\lambda, \|H\mathbb{R}\|_\mu)$

We start with the following general statement.

Lemma 5.1. *Let $\lambda, \mu \in \mathbb{R}_{>0}$. The $\mathbb{S}$-algebra structure of $H\mathbb{R}$ induces an isomorphism of $\mathbb{S}$-modules*

$$\|H\mathbb{R}\|_{\mu/\lambda} \simeq \underline{\mathrm{Hom}}_{\mathbb{S}}(\|H\mathbb{R}\|_\lambda, \|H\mathbb{R}\|_\mu). \tag{5.1}$$

Proof. One starts by defining the morphism of $\mathbb{S}$-modules

$$\eta : \|H\mathbb{R}\|_{\mu/\lambda} \to \underline{\mathrm{Hom}}_{\mathbb{S}}(\|H\mathbb{R}\|_\lambda, \|H\mathbb{R}\|_\mu). \tag{5.2}$$

Precisely, the multiplication in the $\mathbb{S}$-algebra $H\mathbb{R}$ determines natural maps

$$m : H\mathbb{R}(X) \wedge H\mathbb{R}(Y) \to H\mathbb{R}(X \wedge Y)$$

inducing, for a fixed pointed set X, the map $m_X : H\mathbb{R}(X) \to \mathrm{Hom}_{\mathbb{S}}(H\mathbb{R}, H\mathbb{R}(X \wedge -))$. The morphism η is defined as the natural transformation of functors which associates to $X = k_+$, the map $\eta_X : \|H\mathbb{R}\|_{\mu/\lambda}(X) \to \mathrm{Hom}_{\mathbb{S}}(\|H\mathbb{R}\|_\lambda, \|H\mathbb{R}\|_\mu(X \wedge -))$ defined as the restriction of m_X. This restriction is meaningful in view of [2] (Proposition 6.1).
Next, we show that η is an isomorphism. First, we determine the $\mathbb{S}$-module $\mathrm{Hom}_{\mathbb{S}}(\|H\mathbb{R}\|_\lambda, \|H\mathbb{R}\|_\mu)$. For integers $1 \le j \le k$ we let as in (2.1)

$$\delta(j,k) \in \mathrm{Hom}_{\Gamma^o}(k_+, 1_+), \quad \delta(j,k)(\ell) := \begin{cases} 1 & \text{if } \ell = j \\ * & \text{if } \ell \neq j. \end{cases}$$

Let $\phi \in \mathrm{Hom}_{\mathbb{S}}(\|H\mathbb{R}\|_\lambda, \|H\mathbb{R}\|_\mu)$. By construction, the natural transformation ϕ reads, for each k, as a map $\phi(k_+) : \|H\mathbb{R}\|_\lambda(k_+) \to \|H\mathbb{R}\|_\mu(k_+)$ and by naturality we have

$$\|H\mathbb{R}\|_\mu(\delta(j,k)) \circ \phi(k_+) = \phi(1_+) \circ \|H\mathbb{R}\|_\lambda(\delta(j,k)). \tag{5.3}$$

Since an element $y \in \|H\mathbb{R}\|_\mu(k_+)$ is determined by its components $y_j \in \mathbb{R}$, $1 \le j \le k$, with $y_j = \|H\mathbb{R}\|_\mu(\delta(j,k))(y)$, (5.3) shows that $\phi(k_+)$ is uniquely determined by the map $\phi(1_+)$ acting componentwise, *i.e.*

$$\phi(k_+)((x_j)) = (\phi(1_+)(x_j)). \tag{5.4}$$

Moreover, the map $f = \phi(1_+) : [-\lambda, \lambda] \to [-\mu, \mu]$ fulfills

$$f(x+y) = f(x) + f(y) \quad \forall x, y \ s.t. \ |x| + |y| \le \lambda, \tag{5.5}$$

as one sees using the naturality of ϕ for the map $\sigma \in \mathrm{Hom}_{\Gamma^o}(2_+, 1_+)$, *i.e.* using

$$(\|H\mathbb{R}\|_\mu(\sigma)\circ\phi(2_+))(x,y)=(\phi(1_+)\circ\|H\mathbb{R}\|_\lambda(\sigma))(x,y)\Longrightarrow f(x)+f(y)=f(x+y).$$

By (5.5) one has $f(x2^{-n})=2^{-n}f(x)$, for any $x\in[-\lambda,\lambda]$ and $n\geq 0$. Thus the 'germ of map' f uniquely extends to a map $\tilde f:\mathbb{R}\to\mathbb{R}$ defined by $\tilde f(x):=2^n f(x2^{-n})$ for any n such that $|x2^{-n}|\leq\lambda$. Moreover, again by (5.5), the map $\tilde f$ is additive and since $\tilde f([-2^{-n}\lambda,2^{-n}\lambda])\subset[-2^{-n}\mu,2^{-n}\mu]$, it is also continuous and hence determined by the multiplication by a real number r. One has $|r\lambda|\leq\mu$ and hence $r\in\|H\mathbb{R}\|_{\mu/\lambda}(1_+)$. Thus one gets $\phi(1_+)(x)=rx$, $\forall x\in[-\lambda,\lambda]$. By (5.4) one obtains $\phi(k_+)((x_j))=(rx_j)$, $\forall(x_j)\in\|H\mathbb{R}\|_\lambda(k_+)$. This shows that for $X=1_+$, $\eta_X:\|H\mathbb{R}\|_{\mu/\lambda}(X)\to\mathrm{Hom}_{\mathbb{S}}(\|H\mathbb{R}\|_\lambda,\|H\mathbb{R}\|_\mu(X\wedge -))$ is bijective. The next step is to determine $\mathrm{Hom}_{\mathbb{S}}(\|H\mathbb{R}\|_\lambda,\|H\mathbb{R}\|_\mu(X\wedge -))$ for $X=\ell_+$. Let $\phi\in\mathrm{Hom}_{\mathbb{S}}(\|H\mathbb{R}\|_\lambda,\|H\mathbb{R}\|_\mu(\ell_+\wedge -))$. One has $\ell_+\wedge k_+=(\ell k)_+$ and an element $z\in\|H\mathbb{R}\|_\mu(\ell_+\wedge k_+)$ is determined by its components $z(i,j)\in\mathbb{R}$ for $1\leq i\leq\ell$, $1\leq j\leq k$ such that $\sum|z(i,j)|\leq\mu$. In particular, for each j, the $z(i,j)$, $1\leq i\leq\ell$, are the components of $z_j=\|H\mathbb{R}\|_\mu(\mathrm{id}\wedge\delta(j,k)))(z)\in\|H\mathbb{R}\|_\mu(\ell_+)$. This implies by applying the naturality of ϕ, that

$$\phi(k_+)(z)=(\phi(1_+)(z_j))\,. \tag{5.6}$$

As shown above the map $f=\phi(1_+):[-\lambda,\lambda]\to\|H\mathbb{R}\|_\mu(\ell_+)$ fulfills

$$f(x+y)=f(x)+f(y)\quad\forall x,y\ s.t.\ |x|+|y|\leq\lambda \tag{5.7}$$

and it extends to an additive map $\tilde f:\mathbb{R}\to\mathbb{R}^\ell$ which is continuous since $\tilde f$ maps the interval $[-\lambda,\lambda]$ inside $\|H\mathbb{R}\|_\mu(\ell_+)$. Thus there exist real numbers r_i, $1\leq i\leq\ell$, such that $\tilde f(x)=(r_ix)\ \forall x\in\mathbb{R}$. One has $\sum|r_i\lambda|\leq\mu$ and thus $r=(r_i)\in\|H\mathbb{R}\|_{\mu/\lambda}(\ell_+)$. Finally, using (5.6) it follows that $\phi=\eta_X(r)$. This proves that η as in (5.2) is an isomorphism. $\square$

5.2. *Pontryagin duality*

Following Weil's proof of the Riemann-Roch theorem for function fields [13] we adapt Pontryagin duality to our context. We consider, for $\lambda>0$, the $\mathbb{S}[\pm1]$-module $U(1)_\lambda=(U(1),d)_\lambda$, where $U(1)$ is the abelian group $\mathbb{R}/\mathbb{Z}$ endowed with its canonical metric d of length 1. For a metric abelian group (A,d), we denote by $\widehat{A}$ the abelian group of continuous characters, *i.e.* of continuous group homomorphisms $A\to U(1)$, where A is endowed with the topology associated to the metric d. We retain the notations of Appendix B. Next statement is motivated by Lemma 5.1.

Proposition 5.2. *Let (A,d) be an abelian group endowed with a translation invariant metric d.*

(i) *For $\lambda,\mu\in\mathbb{R}_{>0}$, the $\mathbb{S}[\pm1]$-module $\underline{\mathrm{Hom}}_{\Gamma\mathcal{T}_*}((A,d)_\lambda,U(1)_\mu)$ is isomorphic to the sub $\mathbb{S}[\pm1]$-module of $H\widehat{A}$ which, on the set k_+, is given by k-tuples (χ_j), $1\leq j\leq k$*

of continuous characters $\chi_j \in \widehat{A}$ *such that, with* $|x| := d(x,0)$, *and for all finite collections* $\{x_i\} \subset A$, *fulfill*

$$\sum_i |x_i| \le \lambda \ \Rightarrow \ \sum_{i,j} |\chi_j(x_i)| \le \mu. \tag{5.8}$$

(ii) *Let* $p : A' \to A$ *be a surjective morphism of abelian groups and let* $p^*(A,d)_\lambda$ *be the pullback as in Proposition B.4. One has the following canonical isomorphism*

$$\underline{\mathrm{Hom}}_{\Gamma\mathcal{T}_*}(p^*(A,d)_\lambda, U(1)_\mu) \simeq \underline{\mathrm{Hom}}_{\Gamma\mathcal{T}_*}((A,d)_\lambda, U(1)_\mu). \tag{5.9}$$

Proof. (i) Let $\phi \in \underline{\mathrm{Hom}}_{\Gamma\mathcal{T}_*}((A,d)_\lambda, U(1)_\mu)(1_+) = \mathrm{Hom}_{\Gamma\mathcal{T}_*}((A,d)_\lambda, U(1)_\mu)$. By applying the forgetful functor $\mathcal{F} : \mathcal{T} \longrightarrow \mathfrak{Sets}$ which associates to $(X,\mathcal{R})$ the set X (see Proposition B.2), one obtains an element $\mathcal{F}(\phi) \in \mathrm{Hom}_{\mathbb{S}}(HA, HU(1))$. Since the Eilenberg-MacLane functor H determines a full and faithful embedding of the category of abelian groups inside the category of $\mathbb{S}$-modules, there exists a unique group homomorphism $\chi : A \to U(1)$ such that $H(\chi) = \mathcal{F}(\phi)$. The condition that ϕ preserves the relation $\mathcal{R}_k$ on the set k_+ means that for any $x_i, y_i \in A$, $1 \le i \le k$ such that $\sum d(x_i,y_i) \le \lambda$ one has $\sum d(\chi(x_i),\chi(y_i)) \le \mu$. By translation invariance of the metrics this condition is equivalent to

$$\sum_i |x_i| \le \lambda \ \Rightarrow \ \sum_i |\chi(x_i)| \le \mu. \tag{5.10}$$

This shows that $\mathrm{Hom}_{\Gamma\mathcal{T}_*}((A,d)_\lambda, U(1)_\mu)$ consists exactly of the group homomorphisms $\chi : A \to U(1)$ fulfilling (5.8). Specializing (5.10) to the case where all $x_i = x$, $i = 1,\ldots,n$, one obtains the implication $|x| \le \lambda/n \Rightarrow |\chi(x)| \le \mu/n$, and hence that χ is uniformly continuous. Let $\phi \in \underline{\mathrm{Hom}}_{\Gamma\mathcal{T}_*}((A,d)_\lambda, U(1)_\mu)(k_+) = \mathrm{Hom}_{\Gamma\mathcal{T}_*}((A,d)_\lambda, U(1)_\mu(k_+ \wedge -))$. The object $U(1)_\mu(k_+ \wedge -)$ of $\Gamma\mathcal{T}_*$ is $(U(1)^{\times k}, d_k)_\mu$, where the metric d_k on the product group $(U(1)^{\times k}$ is defined by

$$d_k((x_i),(y_i)) := \sum_1^k d(x_i,y_i) \quad \forall x_i, y_i \in U(1).$$

Replacing $U(1)_\mu$ with $(U(1)^{\times k}, d_k)_\mu$ in the first part of the proof one obtains that $\mathcal{F}(\phi) \in \mathrm{Hom}_{\mathbb{S}}(HA, HU(1)^{\times k}) = H((\chi_j))$, where (χ_j), $1 \le j \le k$ is a k-tuple of characters of A fulfilling (5.8). It follows that $\chi_j \in \widehat{A}$.
(ii) Let $\phi \in \underline{\mathrm{Hom}}_{\Gamma\mathcal{T}_*}(p^*(A,d)_\lambda, U(1)_\mu)(1_+) = \mathrm{Hom}_{\Gamma\mathcal{T}_*}(p^*(A,d)_\lambda, U(1)_\mu)$. As in the proof of (i), there exists a group homomorphism $\chi' : A' \to U(1)$ such that $H(\chi') = \mathcal{F}(\phi)$. Moreover χ' preserves the relation $\mathcal{R}_k$ for any k, and this implies

$$\sum_{i=1}^k |p(x_i')| \le \lambda \ \Rightarrow \ \sum_{i=1}^k |\chi'(x_i')| \le \mu, \qquad \forall (x_i') \in (A')^{\times k}.$$

In particular, taking all $x'_i = x' \in \ker(p)$ one obtains $|\chi'(x')| \leq \mu/k \ \forall k$, and hence $\chi'(\ker(p)) = \{1\}$. This implies that there exists a group homomorphism $\chi : A \to U(1)$ such that $\chi' = \chi \circ p$. $\square$

We can now state and prove Serre's duality.

Theorem 5.3. *Let $D = \sum_j a_j\{p_j\} + a\{\infty\}$ be an Arakelov divisor on $\overline{\mathrm{Spec}\,\mathbb{Z}}$. There is a canonical isomorphism of $\mathbb{S}[\pm 1]$-modules*

$$H^0(K - D) \simeq \underline{\mathrm{Hom}}_{\Gamma\mathcal{T}_*}(H^1(D), U(1)_{\frac{1}{4}}), \tag{5.11}$$

where K is the divisor $K = -2\{2\}$.

Proof. By Proposition B.5, with $\lambda = \exp a$, one has $H^1(D) = \pi^*((\mathbb{R}/L, d)_\lambda)$ and by Proposition 5.2, (ii), one gets the isomorphism

$$\underline{\mathrm{Hom}}_{\Gamma\mathcal{T}_*}(H^1(D), U(1)_{\frac{1}{4}}) \simeq \underline{\mathrm{Hom}}_{\Gamma\mathcal{T}_*}((\mathbb{R}/L, d)_\lambda, U(1)_{\frac{1}{4}}).$$

In fact, we can assume that $D = a\{\infty\}$, with $a = \deg D$ so that $L = \mathbb{Z}$. Then we apply Proposition 5.2 (i), with $A = \mathbb{R}/\mathbb{Z}$ and $\mu = \frac{1}{4}$. One has $\widehat{A} = \mathbb{Z}$ and the characters $\chi_n \in \widehat{A}$ are given by multiplication by n, *i.e.* $\chi_n(s) := ns \in \mathbb{R}/\mathbb{Z}$, $\forall s \in \mathbb{R}/\mathbb{Z}$. Next, we need to determine the $\mathbb{S}[\pm 1]$-submodule of $H\widehat{A} = H\mathbb{Z}$ which, on the set k_+, is given by k-tuples (n_j), $1 \leq j \leq k$ of characters $n_j \in \mathbb{Z}$ such that (5.8) holds. This means, using the distance d on $\mathbb{R}/\mathbb{Z}$, that

$$\sum_i d(x_i, 0) \leq \lambda \ \Rightarrow \ \sum_{i,j} d(n_j x_i, 0) \leq \frac{1}{4}. \tag{5.12}$$

The distance $d(x, 0)$ is given, for any $x' \in \mathbb{R}$ in the class of x by the distance between x' and the closed subset $\mathbb{Z} \subset \mathbb{R}$. Thus for any integer n one has: $d(nx, 0) \leq |n| d(x, 0)$. Assume that $\sum_j |n_j| \leq \frac{1}{4\lambda}$, then (5.12) follows since

$$\sum_{i,j} d(n_j x_i, 0) \leq \sum_{i,j} |n_j| d(x_i, 0) \leq \frac{1}{4\lambda} \sum_i d(x_i, 0).$$

Conversely, assume (5.12). Then repeating m times the same x, gives

$$d(x, 0) \leq \frac{\lambda}{m} \ \Rightarrow \ \sum_j d(n_j x, 0) \leq \frac{1}{4m}.$$

Taking m large enough and $x = \frac{\lambda}{m}$ one obtains $\sum_j |n_j| x \leq \frac{1}{4m}$, and hence $\sum_j |n_j| \leq \frac{1}{4\lambda}$. This proves that the $\mathbb{S}[\pm 1]$-submodule of $H\widehat{A} = H\mathbb{Z}$ determined by (5.8) is equal to $\|H\mathbb{Z}\|_{\frac{1}{4\lambda}}$ which gives (5.11). $\square$

Declaration of competing interest

The authors declare that they have no conflict of interest.

Data availability

No data was used for the research described in the article.

Appendix A. $\mathbb{S}$-modules

One denotes with Γ^{op} the small full subcategory of the category of finite pointed sets whose objects are the pointed finite sets[6] $k_+ := \{0, \ldots, k\}$, for $k \geq 0$. In particular, 0_+ is both initial and final in Γ^{op}, making Γ^o a *pointed category.* A Γ-set (aka $\mathbb{S}$-module) is a (covariant) functor $\Gamma^o \longrightarrow \mathfrak{Sets}_*$ between pointed categories, and the morphisms in this category are natural transformations. One lets $\mathbb{S} : \Gamma^o \longrightarrow \mathfrak{Sets}_*$ be the inclusion functor. The internal hom functor is defined by

$$\underline{\mathrm{Hom}}_{\mathbb{S}}(M, N) := \{k_+ \mapsto \mathrm{Hom}_{\mathbb{S}}(M, N(k_+ \wedge -))\}.$$

This formula uniquely defines the smash product of Γ-sets by applying the adjunction

$$\underline{\mathrm{Hom}}_{\mathbb{S}}(M_1 \wedge M_2, N) = \underline{\mathrm{Hom}}_{\mathbb{S}}(M_1, \underline{\mathrm{Hom}}_{\mathbb{S}}(M_2, N)).$$

The basic construction of $\mathbb{S}$-modules associates to an abelian monoid A with a zero element, the Eilenberg-MacLane functor $M = HA$

$$HA(k_+) = A^k, \qquad Hf : HA(k_+) \to HA(n_+),$$
$$Hf(m)(j) := \sum_{f(\ell)=j} m_\ell,$$

where $m = (m_1, \ldots, m_k) \in HA(k_+)$, and the zero element of A gives meaning to the empty sum. An $\mathbb{S}$-algebra $\mathcal{A}$ is an $\mathbb{S}$-module $\mathcal{A} : \Gamma^o \longrightarrow \mathfrak{Sets}_*$ endowed with an associative multiplication $\mu : \mathcal{A} \wedge \mathcal{A} \to \mathcal{A}$ and a unit $1 : \mathbb{S} \to \mathcal{A}$. An ordinary semiring R gives rise to the $\mathbb{S}$-algebra HR, and the corresponding embedding of categories is fully faithful so that no information is lost. In contrast, the basic $\mathbb{S}$-algebra $\mathbb{S}$ now lies under HR for any semiring R.

Appendix B. Tolerance $\mathbb{S}$-modules

The construction of the category $\Gamma\mathcal{S}_*$ of Γ-spaces (see appendix C) can be broadly generalized by considering in place of the category $\mathcal{S}_*$ of simplicial pointed sets any

[6] where 0 is the base point.

pointed category $\mathcal{C}$ with initial and final object $*$. In this way, one obtains a category $\Gamma\mathcal{C}$ of pointed covariant functor $\Gamma^o \longrightarrow \mathcal{C}$. We shall apply this formal construction to the category of tolerance relations and introduce the notion of tolerant $\mathbb{S}[\pm 1]$-modules which plays a central role in the development of the absolute Riemann-Roch problem. We start with the following general fact

Lemma B.1. *Let $\mathcal{C}$ be a pointed category with initial and final object $*$. Then $\Gamma\mathcal{C}$ is naturally enriched in $\mathbb{S}$-modules. More precisely, the following formula endows the internal $\underline{\mathrm{Hom}}_{\Gamma\mathcal{C}}(A,B)$ with a structure of $\mathbb{S}$-module defined by*

$$\underline{\mathrm{Hom}}_{\Gamma\mathcal{C}}(A,B)(k_+) := \mathrm{Hom}_{\Gamma\mathcal{C}}(A, B(k_+ \wedge -)) \qquad k \in \mathbb{N}. \tag{B.1}$$

Proof. Let $\phi \in \mathrm{Hom}_{\Gamma^o}(k_+, \ell_+)$. For every object F of Γ^o the morphism $\phi \wedge \mathrm{id} \in \mathrm{Hom}_{\Gamma^o}(k_+ \wedge F, \ell_+ \wedge F)$ gives, by functoriality of $B : \Gamma^o \longrightarrow \mathcal{C}$, a morphism $B(\phi \wedge \mathrm{id}) \in \mathrm{Hom}_{\mathcal{C}}(B(k_+ \wedge F), B(\ell_+ \wedge F))$. These morphisms define a natural transformation of functors $B(\phi \wedge -) \in \mathrm{Hom}_{\Gamma\mathcal{C}}(B(k_+ \wedge -), B(\ell_+ \wedge -))$, and one obtains the functoriality on the right hand side of (B.1) using the left composition

$$\psi \in \mathrm{Hom}_{\Gamma\mathcal{C}}(A, B(k_+ \wedge -)) \mapsto B(\phi \wedge -) \circ \psi \in \mathrm{Hom}_{\Gamma\mathcal{C}}(A, B(\ell_+ \wedge -)). \quad \square$$

B.1. The category $\Gamma\mathcal{T}_*$

A tolerance relation $\mathcal{R}$ on a set X is a reflexive and symmetric relation on X. Equivalently, $\mathcal{R}$ is a subset $\mathcal{R} \subset X \times X$ which is symmetric and containing the diagonal. We shall denote by $\mathcal{T}$ the category of tolerance relations $(X, \mathcal{R})$. Morphisms in $\mathcal{T}$ are defined by

$$\mathrm{Hom}_{\mathcal{T}}((X,\mathcal{R}),(X',\mathcal{R}')) := \{\phi : X \to X',\ \phi(\mathcal{R}) \subset \mathcal{R}'\}.$$

We denote $\mathcal{T}_*$ the pointed category under the object $\{*\}$ endowed with the trivial relation. One has the following

Definition B.1. A tolerant $\mathbb{S}$-module is a pointed covariant functor $\Gamma^o \longrightarrow \mathcal{T}_*$. We denote by $\Gamma\mathcal{T}_*$ the category of tolerant $\mathbb{S}$-modules.

Next statement is an easy but useful fact

Proposition B.2.

(i) *The functor $\mathfrak{Sets} \longrightarrow \mathcal{T}$ which endows a set with the diagonal relation, embeds the category of sets as a full subcategory of $\mathcal{T}$, and consequently the category of $\mathbb{S}$-modules as a full subcategory of the category $\Gamma\mathcal{T}_*$.*

(ii) *The forgetful functor is the right adjoint of the inclusion in (i).*

B.2. The tolerant $\mathbb{S}$*-module* $(A, d)_\lambda$

A relevant example of tolerant $\mathbb{S}$-module is given by the following construction. Let A be an additive abelian group. A translation invariant metric d on A is a metric on A that satisfies $d(x, y) = d(x - y, 0)$, so that the triangle inequality can be read as $d(x + y, 0) \leq d(x, 0) + d(y, 0)\ \forall x, y \in A$. This fact implies that the inequality

$$d\Big(\sum_I x_i, \sum_I y_i\Big) \leq \sum_I d(x_i, y_i) \tag{B.2}$$

holds for any finite index set I and maps $x, y : I \to A$.

Proposition B.3. *Let* (A, d) *be an abelian group endowed with a translation invariant metric* d *and let* $\lambda \in \mathbb{R}_{>0}$*. The following relations turn the Eilenberg-MacLane* $\mathbb{S}$*-module* HA *into a tolerant* $\mathbb{S}[\pm 1]$*-module* $(A, d)_\lambda : \Gamma^o \longrightarrow \mathcal{T}_*$*:*

$$\mathcal{R}_k = \{((x_i), (y_i)) \in A^{\times k} \times A^{\times k} \mid \sum_{I = k_+} d(x_i, y_i) \leq \lambda\} \qquad \forall k_+,\ k \in \mathbb{N}. \tag{B.3}$$

Proof. For any $\phi \in \mathrm{Hom}_{\Gamma^o}(k_+, \ell_+)$, the map $HA(\phi) : HA(k_+) \to HA(\ell_+)$ fulfills $HA(\phi)^{\times 2}(\mathcal{R}_k) \subset \mathcal{R}_\ell$. Indeed, this follows from (B.2) applied to the finite sets $I_j = \{i \in k_+ \mid \phi(i) = j\}$ which label pairs of elements of A. $\square$

Let $U(1)$ be the abelian group $\mathbb{R}/\mathbb{Z}$ endowed with the canonical metric d of length 1. We shall denote by $U(1)_\lambda$ the tolerant $\mathbb{S}$-module $(U(1), d)_\lambda$, $(\lambda \in \mathbb{R}_{>0})$.

B.3. The dimension of a tolerant $\mathbb{S}[\pm 1]$*-module*

In this part we introduce a notion of dimension for a tolerant $\mathbb{S}[\pm 1]$-module that naturally generalizes, in the absolute context, the definition of dimension of a vector space.

Definition B.2. Let $(E, \mathcal{R})$ be a tolerant $\mathbb{S}[\pm 1]$-module. A subset $F \subset E(1_+)$ generates $E(1_+)$ if the following two conditions hold

1. For $x, y \in F$, with $x \neq y \implies (x, y) \notin \mathcal{R}$
2. For every $x \in E(1_+)$ there exists $\alpha_j \in \{-1, 0, 1\}$, $j \in F$ and $y \in E(1_+)$ such that $y = \sum_F \alpha_j j \in E(1_+)$ in the sense of (2.2), and $(x, y) \in \mathcal{R}$.

The dimension $\dim_{\mathbb{S}[\pm 1]}(E, \mathcal{R})$ is defined as the minimal cardinality of a generating set F.

To familiarize with this notion we prove the following

Proposition B.4. *Let $p : A \to B$ be a morphism of abelian groups and $(HB, \mathcal{R})$ a tolerant $\mathbb{S}[\pm 1]$-module.*

(i) *Consider the relation $p^*(\mathcal{R}_k) := \{(x, y) \in HA(k_+) \times HA(k_+) \mid (Hp(x), Hp(y)) \in \mathcal{R}_k\}$. Then the pair $p^*(HB, \mathcal{R}) := (HA, p^*(\mathcal{R}))$ is a tolerant $\mathbb{S}[\pm 1]$ module.*

(ii) *If p is surjective:* $\dim_{\mathbb{S}[\pm 1]}(p^*(HB, \mathcal{R})) = \dim_{\mathbb{S}[\pm 1]}(HB, \mathcal{R})$.

Proof. (i) Since $(HB, \mathcal{R})$ is a tolerant $\mathbb{S}[\pm 1]$-module, for any $\phi \in \mathrm{Hom}_{\Gamma^o}(k_+, \ell_+)$ one has $HB(\phi)^{\times 2}(\mathcal{R}_k) \subset \mathcal{R}_\ell$. Also

$$HA(\phi)^{\times 2}(p^*\mathcal{R}_k) \subset p^*\mathcal{R}_\ell \qquad \forall \phi \in \mathrm{Hom}_{\Gamma^o}(k_+, \ell_+),$$

which shows that $p^*(HB, \mathcal{R}) = (HA, p^*(\mathcal{R}))$ is a tolerant $\mathbb{S}[\pm 1]$ module.
(ii) Let $F \subset HB(1_+) = B$ be a generating set for $(HB, \mathcal{R})$, and let $F' \subset HA(1_+) = A$ be a lift of F, with $\#F' = \#F$. Let us show that F' is a generating set for $p^*(HB, \mathcal{R}) = (HA, p^*(\mathcal{R}))$. Let $a \in HA(1_+) = A$, then there exist coefficients $\alpha_j \in \{-1, 0, 1\}$, $j \in F$, such that $(p(a), \sum_F \alpha_j j) \in \mathcal{R}$. It follows that using the lifts $j' \in F'$ of $j \in F$ one has $(a, \sum_{F'} \alpha_j j') \in p^*(\mathcal{R})$, hence F' is a generating set for $p^*(HB, \mathcal{R})$. Thus $\dim_{\mathbb{S}[\pm 1]}(p^*(HB, \mathcal{R})) \leq \dim_{\mathbb{S}[\pm 1]}(HB, \mathcal{R})$. Conversely, let $F' \subset A$ be a generating set for $p^*(HB, \mathcal{R})$ and $F := p(F')$. Then condition 1. of Definition B.2 for F' implies the same condition for F, thus one has $\#F' = \#F$. Let $b \in HB(1_+) = B$ and $a \in A$ with $p(a) = b$. Then there exists coefficients $\alpha_j \in \{-1, 0, 1\}$, $j \in F'$, such that $(a, \sum_{F'} \alpha_j j') \in p^*(\mathcal{R})$. This implies $(b, \sum \alpha_j j) \in \mathcal{R}$ so that F is a generating set for $(HB, \mathcal{R})$. $\square$

Next, we apply this functorial machinery to the geometry of $\overline{\mathrm{Spec}\,\mathbb{Z}}$. We retain the notations of section 2.

Proposition B.5. *Let $D = \sum_j a_j\{p_j\} + a\{\infty\}$ be an Arakelov divisor on $\overline{\mathrm{Spec}\,\mathbb{Z}}$ and let $\pi : \mathbb{A}_\mathbb{Q} \to \mathbb{R}/L$ be the projection of the adeles on their archimedean component modulo the lattice*

$$L = H^0(\mathrm{Spec}\,\mathbb{Z}, \mathcal{O}(D)_f) := \{q \in \mathbb{Q} \mid |q|_\nu \leq \exp(D)(\nu),\ \forall \nu \neq \infty\}. \tag{B.4}$$

(i) *Let d be the metric on $\mathbb{R}/L$ induced by the standard metric on $\mathbb{R}$ and set $\lambda = \exp a$. Then one has $H^1(D) = \pi^*((\mathbb{R}/L, d)_\lambda)$.*

(ii) $\dim_{\mathbb{S}[\pm 1]} H^1(D) = \dim_{\mathbb{S}[\pm 1]}(\mathbb{R}/L, d)_\lambda$.

Proof. (i) Let $j : \mathbb{A}^f_\mathbb{Q} \to \mathbb{A}_\mathbb{Q}$, $j(x) := (x, 0)$, be the embedding of finite adeles in adeles. Using the ultrametric property of the local norms at the finite places one sees that $j(\mathcal{O}(D)_f) = \mathcal{O}(D) \cap j(\mathbb{A}^f_\mathbb{Q}) \subset \mathbb{A}_\mathbb{Q}$ is a compact subgroup. Set $G = \mathbb{Q} \times j(\mathcal{O}(D)_f)$: one has $j(\mathcal{O}(D)_f) \cap \mathbb{Q} = \{0\}$ since all the adeles in $j(\mathcal{O}(D)_f)$ have archimedean component equal to 0. Thus, the restriction of the morphism of addition, $\alpha : \mathbb{Q} \times \mathbb{A}_\mathbb{Q} \to \mathbb{A}_\mathbb{Q}$, $\alpha(q, a) = q + a$, to G determines an isomorphism of G with the subgroup $\alpha(G) = \mathbb{Q} + j(\mathcal{O}(D)_f)$ of $\mathbb{A}_\mathbb{Q}$. Note, in particular, that $\alpha(G)$ is closed in $\mathbb{A}_\mathbb{Q}$, since $\mathbb{Q}$ is discrete (hence closed) and $j(\mathcal{O}(D)_f)$ is compact. In the following, we identify (set-theoretically) $\mathbb{A}_\mathbb{Q}$ with the product $\mathbb{A}^f_\mathbb{Q} \times \mathbb{R}$ endowed with the two projection morphisms $p_f : \mathbb{A}_\mathbb{Q} \to \mathbb{A}^f_\mathbb{Q}$ and $p_\infty : \mathbb{A}_\mathbb{Q} \to \mathbb{R}$. The subgroup $p_f(\mathbb{Q}) \subset \mathbb{A}^f_\mathbb{Q}$ is dense and the subgroup $p_f(j(\mathcal{O}(D)_f)) = \mathcal{O}(D)_f \subset \mathbb{A}^f_\mathbb{Q}$ is open. Hence $p_f(\alpha(G)) = \mathbb{A}^f_\mathbb{Q}$. Thus the projection p_∞ induces the isomorphism of groups $p : \mathbb{A}_\mathbb{Q}/\alpha(G) \xrightarrow{\sim} \mathbb{R}/L$, where $L = p_\infty(\ker(p_f \circ \alpha))$. The kernel of the composite $p_f \circ \alpha : G \to \mathbb{A}^f_\mathbb{Q}$ is the group of pairs $(q, a) \in \mathbb{Q} \times j(\mathcal{O}(D)_f)$ such that $p_f(q) + p_f(a) = 0$. Such pairs are determined by the value of q and thus

$$L = p_\infty(H^0(\operatorname{Spec}\mathbb{Z}, \mathcal{O}(D)_f)) = \{q \in \mathbb{Q} \mid |q|_\nu \le \exp(D)(\nu), \quad \forall \nu \neq \infty\}. \tag{B.5}$$

By Proposition 2.2, $H^1(D)$ is the tolerant $\mathbb{S}[\pm 1]$-module $H^1(D) = (H\mathbb{A}_\mathbb{Q}, \mathcal{R})$ where the relations are given by (2.5), *i.e.*

$$\mathcal{R}_k := \{(x, y) \in H\mathbb{A}_\mathbb{Q}(k_+) \times H\mathbb{A}_\mathbb{Q}(k_+) \mid x - y \in \operatorname{Image}\psi(k_+)\},$$

and where ψ is as in (2.3). By construction one has $\mathbb{Q} \times \mathcal{O}(D) \simeq G \times [-e^a, e^a]$. After quotienting both sides of (2.3) by $G = \mathbb{Q} \times j(\mathcal{O}(D)_f)$, the map ψ becomes

$$\phi : \|H\mathbb{R}\|_{e^a} \to H(\mathbb{R}/L), \quad \phi(x) = x + L \in \mathbb{R}/L \quad \forall x \in [-e^a, e^a] \subset \mathbb{R}, \tag{B.6}$$

where $L \subset \mathbb{Q} \subset \mathbb{R}$ is the lattice (B.5). Thus one obtains that the relation $\mathcal{R}$ is equal to the inverse image by the map $\pi : \mathbb{A}_\mathbb{Q} \to \mathbb{R}/L$ of the relation (B.3).
(*ii*) Follows from Proposition B.4 (*ii*). □

Appendix C. The Γ-space H(D)

Let $\phi : A \to B$ be a morphism of abelian groups. To ϕ one associates the following (short) complex $\mathfrak{C} = \{C_n, \phi_n\}$ of abelian groups indexed in non-negative degrees

$$\mathfrak{C} = \{C_1 \stackrel{\phi_1}{\to} C_0\}; \qquad C_0 = B,\ C_1 = A, \quad \phi_1(x) := \phi(x). \tag{C.1}$$

The Dold-Kan correspondence associates to $\mathfrak{C}$ the simplicial abelian group (see [8] III.2, Proposition 2.2)

$$\mathcal{A}_n = \bigoplus_{\mathcal{F}(n,k)} C_k, \quad \mathcal{F}(n,k) := \{\sigma \in \operatorname{Hom}_\Delta([n],[k]) \mid \sigma([n]) = [k]\} \tag{C.2}$$

where Δ denotes the simplicial category. The direct sum in (C.2) repeats the term C_k of the chain complex as many times as the number of elements of the set $\mathcal{F}(n,k)$ of surjective morphisms $\sigma \in \operatorname{Hom}_\Delta([n],[k])$. For the short complex $\mathfrak{C}$, the allowed values of k are $k = 0, 1$. Therefore the set $\mathcal{F}(n,0)$ is reduced to $\operatorname{Hom}_\Delta([n],[0]) =: \Delta^0_n$, *i.e.* to a single point. A morphism $\xi \in \mathrm{Hom}_\Delta([n],[1]) =: \Delta_n^1$ is characterized by $\xi^{-1}(\{1\})$ which is an hereditary subset of $[n]$. It follows that the vertices in Δ_n^1 are labelled by the

$$\xi_j,\ 0 \le j \le n+1, \quad \xi_j(k) = 1 \iff k \ge j. \tag{C.3}$$

For each integer $n \ge 0$ the finite set $F(n) := \mathcal{F}(n,1)$ of surjective elements $\xi \in \Delta_n^1$ excludes ξ_0 and ξ_{n+1}, thus the set $F(n) = \{\xi_j,\ 1 \le j \le n\}$ has n elements. This gives the identification

$$\mathcal{A}_n = HB(1_+) \oplus HA(F(n)_+) \tag{C.4}$$

We refer to [8] (III.2, pp. 160-161) for a detailed description of the simplicial structure, namely the definition for each $\theta \in \mathrm{Hom}_\Delta([m],[n])$ of a map of sets $\mathcal{A}(\theta) : \mathcal{A}_n \to \mathcal{A}_m$. Next, we introduce some notations.

We identify the opposite Δ^o of the simplicial category with (the skeleton of) the category of finite intervals. An interval is a totally ordered set with the smallest element distinct from the largest one. A morphism of intervals is a non-decreasing map that preserves the smallest and largest elements. The canonical contravariant functor $\Delta \longrightarrow \Delta^o$ which identifies the opposite category of Δ with Δ^o (described by intervals as above), maps the finite ordinal object $[n] = \{0,1,\ldots,n\}$ in Δ to the interval $[n]^* = \{0,\ldots,n+1\}$.

We denote by $\mathfrak{Sets}_{2,*}$ the category of pairs of pointed sets (X,Y), with $X \supset Y$. The morphisms are maps of pairs of pointed sets. We let $c : \mathfrak{Sets}_{2,*} \longrightarrow \mathfrak{Sets}_*$ be the functor $(X,Y) \to X/Y$ of collapsing Y to the base point $*$.

Let $\partial : \Delta^o \longrightarrow \mathfrak{Sets}_{2,*}$ be the functor that replaces an interval I with the pair $\partial I = (X,Y)$, where X is the set I pointed by its smallest element, and $Y \subset X$ is the subset formed by the smallest and largest elements of I.

Finally, we denote by $\gamma : \mathfrak{Sets}_{2,*} \longrightarrow \Gamma\mathfrak{Sets}_{2,*}$ (see Appendix B) the functor that associates to an object (X,Y) of $\mathfrak{Sets}_{2,*}$ the covariant functor

$$\gamma(X,Y) : \Gamma^o \longrightarrow \mathfrak{Sets}_{2,*}, \quad k_+ := \{0,\ldots,k\} \mapsto (X \wedge k_+, Y \wedge k_+).$$

The following formula defines a covariant functor that associates to a pair of pointed sets an abelian group directly related to the morphism $\phi : A \to B$

$$H_\phi : \mathfrak{Sets}_{2,*} \longrightarrow \mathfrak{Ab}, \qquad H_\phi(X,Y) := HB(Y) \times HA(X/Y). \tag{C.5}$$

On morphisms $f : (X,Y) \to (X',Y')$ in $\mathfrak{Sets}_{2,*}$ with $\alpha = (\alpha_Y, \alpha_{X/Y}) \in H_\phi(X,Y)$, the functor H_ϕ acts as follows

$$\begin{aligned} H_\phi(f)(\alpha) &= \big(\alpha_{Y'}, HA(f)(\alpha_{X/Y})\big) \\ \alpha_{Y'}(y') &:= HB(f)(\alpha_Y)(y') + \sum_{x \in X\setminus Y, f(x)=y'} \phi(\alpha_{X/Y}(x)), \quad y' \ne *. \end{aligned} \tag{C.6}$$

By [4] (Proposition 4.5), the Dold-Kan correspondence for the short complex (C.1), *i.e.* the simplicial abelian group $\mathcal{A}_n$ in (C.4) is canonically isomorphic to the composite functor $H_\phi \circ \partial : \Delta^o \longrightarrow \mathfrak{Ab}$, with H_ϕ as in (C.5). By composing H_ϕ with the Eilenberg-MacLane functor H one obtains a covariant functor $HH_\phi = H \circ H_\phi : \mathfrak{Sets}_{2,*} \longrightarrow \mathbb{S} - \mathrm{Mod}$ to the category of $\mathbb{S}$-modules, which is naturally isomorphic to the functor $((U \circ H_\phi) \times \mathrm{id}) \circ \gamma$, where $U : \mathfrak{Ab} \longrightarrow \mathfrak{Sets}_*$ is the forgetful functor (see [4] Lemma 4.6). Moreover, as shown in [4] (Theorem 4.7) the Γ-space associated by the Dold-Kan correspondence to the complex $\mathfrak{C}$ is canonically isomorphic to the functor

$$((U \circ H_\phi) \times \mathrm{id}) \circ \gamma \circ \partial : \Delta^o \longrightarrow \mathbb{S} - \mathrm{Mod}. \tag{C.7}$$

Notice that the above construction involves the morphism $\phi : A \to B$ *only* through the composite functor $U \circ H_\phi : \mathfrak{Sets}_{2,*} \longrightarrow \mathfrak{Sets}_*$. This latter functor is still meaningful when one restricts ϕ to a sub-$\mathbb{S}$-module E of the $\mathbb{S}$-module HA and it is given by

$$H_\phi|_E : \mathfrak{Sets}_{2,*} \longrightarrow \mathfrak{Sets}_*, \qquad H_\phi(X, Y) := HB(Y) \times E(X/Y). \tag{C.8}$$

This provides the following construction

Definition C.1. Let $\phi : A \to B$ be a morphism of abelian groups and E a sub $\mathbb{S}$-module of the $\mathbb{S}$-module HA. We denote by $\Gamma(\phi|_E)$ the Γ-space obtained as a sub-functor of (C.7)

$$\Gamma(\phi|_E) := (H_\phi|_E \times \mathrm{id}) \circ \gamma \circ \partial : \Delta^o \longrightarrow \mathbb{S} - \mathrm{Mod}. \tag{C.9}$$

When evaluated on the object $1_+ = \{0, 1\}$ of Γ^o, the Γ-space $\Gamma(\phi|_E)$ defines a sub-simplicial set of the Kan simplicial set $\mathcal{A}_n$ in (C.4). However it is not in general a Kan simplicial set, thus one needs to exert care when considering its homotopy. We refer to [3] 2.1, for the definition of the homotopies used there. The relation $\mathcal{R}$ of homotopy between n-simplices $(x, y) \in X_n \times X_n$ is defined as follows

$$(x, y) \in \mathcal{R} \iff \partial_j x = \partial_j y \, \forall j \,\&\, \exists z \text{ s.t. } \partial_j z = s_{n-1} \partial_j x \, \forall j < n, \; \partial_n z = x, \, \partial_{n+1} z = y. \tag{C.10}$$

In general, the relation $\mathcal{R}$ in (C.10) fails to be an equivalence relation. In place of the quotient we consider pairs of sets and relations. We define $\pi_n^{\mathcal{T}}$ to be the set of spherical elements in X_n (*i.e.* of n-simplices x, with $\partial_j x = * \, \forall j$), endowed with the relation $\mathcal{R}$. Then we have the following result (we refer to Appendix B for the notion of tolerant module)

Proposition C.1. *Let $\phi : A \to B$ be a morphism of abelian groups and let E be a sub-$\mathbb{S}[\pm 1]$-module of the Eilenberg-MacLane $\mathbb{S}[\pm 1]$-module HA (where $(\pm 1)x := \pm x$).*

(i) *The homotopy* $\pi_1^{\mathcal{T}}(\Gamma(\phi|_E))$ *is the sub-*$\mathbb{S}[\pm 1]$*-module* $\ker(H\phi|_E)$ *of* E

$$\ker(H\phi|_E)(k_+) = \{x \in E(k_+) \mid H\phi(x) = *\}. \tag{C.11}$$

(ii) *The homotopy* $\pi_0^{\mathcal{T}}(\Gamma(\phi|_E))$ *is the tolerant* $\mathbb{S}[\pm 1]$*-module* HB *endowed with the relations*

$$\mathcal{R}_k = \{(x, y) \in HB(k_+) \times HB(k_+) \mid x - y \in H\phi(E(k_+))\} \quad k \in \mathbb{N}. \tag{C.12}$$

Proof. By construction, $\Gamma(\phi|_E)$ is the composite of the functors $H_\phi|_E : \mathfrak{Sets}_{2,*} \longrightarrow \mathfrak{Sets}_*$ of (C.8) and $\gamma \circ \partial : \Delta^o \longrightarrow \mathfrak{Sets}_{2,*}$. One has $\partial[0]^* = (X, Y)$ where $X = Y = \{0, 1\}$ with base point 0. Thus $\gamma \circ \partial[0]^*(k_+) = (X \wedge k_+, Y \wedge k_+) = (k_+, k_+)$ and by (C.8)

$$H_\phi|_E(\gamma \circ \partial[0]^*)(k_+) = H_\phi|_E(X \wedge k_+, Y \wedge k_+) = HB(k_+).$$

One has $\partial[1]^* = (X, Y)$, where $X = \{0, 1, 2\}$, $Y = \{0, 2\}$ with base point 0. Thus one has $\gamma \circ \partial[1]^*(k_+) = (X \wedge k_+, Y \wedge k_+)$ and $Y \wedge k_+ = k_+$, while $(X \wedge k_+)/(Y \wedge k_+) = \{0, 1\} \wedge k_+ = k_+$ and by (C.8) it follows that

$$H_\phi|_E(\gamma \circ \partial[1]^*)(k_+) = E(k_+) \times HB(k_+). \tag{C.13}$$

The boundaries $\partial_j : H_\phi|_E(\gamma \circ \partial[1]^*) \to H_\phi|_E(\gamma \circ \partial[0]^*)$ are obtained as in [4] Proposition 4.11

$$\partial_0(\psi) = \psi_2, \quad \partial_1(\psi) = H\phi(\psi_1) + \psi_2, \quad \forall \psi = (\psi_1, \psi_2) \in H_\phi|_E(\gamma \circ \partial[1]^*). \tag{C.14}$$

(i) The spherical condition $\partial_j(\psi) = *$ on $\psi \in H_\phi|_E(\gamma \circ \partial[1]^*)(k_+)$ means that $\psi_2 = 0$ and $H\phi(\psi_1) = 0$. Thus the solutions correspond to $\ker(H\phi|_E)(k_+)$ as in (C.11). One shows as in [4] Proposition 4.11 (iii), that the relation of homotopy is the identity.
(ii) The relation $\mathcal{R}_k$ on elements of $H_\phi|_E(\gamma \circ \partial[0]^*)(k_+)$ is defined as follows

$$(\alpha, \beta) \in \mathcal{R}_k \iff \exists \psi \in H_\phi|_E(\gamma \circ \partial[1]^*)(k_+) \text{ s.t. } \partial_0(\psi) = \alpha \ \& \ \partial_1(\psi) = \beta. \tag{C.15}$$

By (C.14) it coincides with (C.12).

Note that this relation is a tolerance relation *i.e.* is symmetric since E is a sub $\mathbb{S}[\pm 1]$-module of the $\mathbb{S}[\pm 1]$-module HA, so that

$$x - y \in H\phi(E(k_+)) \iff y - x \in H\phi(E(k_+)). \quad \square$$

The above general construction applies to the geometric adelic context and by implementing the action of $\mathbb{Q}^\times$ on adeles, we obtain the following variant of Proposition 4.9 in [4].

Proposition C.2. *Let D be an Arakelov divisor on $\overline{\operatorname{Spec}\mathbb{Z}}$. Let $A = \mathbb{Q} \times \mathbb{A}_{\mathbb{Q}}$, $B = \mathbb{A}_{\mathbb{Q}}$ and $\alpha : A \to B$, $\alpha(q,a) = q + a$. Let $E = H\mathbb{Q} \times H\mathcal{O}(D)$ be the sub $\mathbb{S}[\pm 1]$-module of HA as in Proposition 2.1. Then the functor*

$$\mathbf{H}(D) := \Gamma(\alpha|_E) : \Delta^o \longrightarrow \mathbb{S} - \mathrm{Mod} \tag{C.16}$$

defines a Γ-space that depends only on the linear equivalence class of D.

C.1. Proof of Proposition 2.2

With the notations of Appendix B and Proposition C.2, one has $H^0(D) := \pi_1^{\mathcal{T}}(\mathbf{H}(D)) = \pi_1^{\mathcal{T}}(\Gamma(\alpha|_E))$. Proposition C.1, (i), gives

$$H^0(D)(k_+) = \{x \in E(k_+) \mid H\alpha(x) = *\}.$$

An element $x \in E(k_+)$ is a k-tuple (q_j, a_j) with $q_j \in \mathbb{Q}$ for all j, and $(a_j) \in H\mathcal{O}(D)(k_+)$, where $H\mathcal{O}(D) := H\mathcal{O}(D)_f \times \|H\mathbb{R}\|_{e^a}$. The condition $H\alpha(x) = *$ means $q_j = -a_j$ for all j, so that x is uniquely determined by the k-tuple (q_j). Moreover, the allowed k-tuples are those for which $\sum |q_j| \le e^a$ and $q_j \in \mathcal{O}(D)_f$ for all j. One has

$$\mathbb{Q} \cap \mathcal{O}(D)_f = H^0(\operatorname{Spec}\mathbb{Z}, \mathcal{O}(D)_f).$$

This proves (i) of Proposition 2.2. The statement (ii) follows from Proposition C.1 (ii). Finally, (iii) follows from Proposition C.2.

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