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The BC-system and L-functions⋆

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Abstract. In these lectures we survey some relations between L-functions and the BC-system, including new results obtained in collaboration with C. Consani. For each prime p and embedding σ of the multiplicative group of an algebraic closure of $\mathbb{F}_p$ as complex roots of unity, we construct a p-adic indecomposable representation π_σ of the integral BC-system. This construction is done using the identification of the big Witt ring of $\bar{\mathbb{F}}_p$ and by implementing the Artin–Hasse exponentials. The obtained representations are the p-adic analogues of the complex, extremal KMS_∞ states of the BC-system. We use the theory of p-adic L-functions to determine the partition function. Together with the analogue of the Witt construction in characteristic one, these results provide further evidence towards the construction of an analogue, for the global field of rational numbers, of the curve which provides the geometric support for the arithmetic of function fields.

Keywords and phrases: BC-system, Witt rings, adèle class space

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1. Introduction

It is a great pleasure to present these lectures in honor of the founder of class field theory. The topic of my lectures establishes a deep connection between the theory founded by Teiji Takagi in arithmetic and the Tomita–Takesaki theory in analysis, due to two other famous Japanese mathematicians, which was the starting point of my work on the classification of factors. Moreover further work in progress ([15]) on this relation uses in a crucial manner the work of K. Iwasawa on p-adic L-functions.

The origin of the link between class field theory and type III factors is a system of Quantum Statistical Mechanics: the BC (Bost–Connes)-system [4], whose properties illustrate the theory of type III factors in operator algebras and the KMS (Kubo–Matrin–Schwinger) condition in physics, while the associated noncommutative space, the adèle class space, is a direct descendant of the class field idea through the filiation

$$\textit{Ideal classes} \longrightarrow \textit{Idèle class group} \longrightarrow \textit{Adèle class space}.$$

The interrelationship between the fields of noncommutative geometry and arithmetic originated in [4], where an important link was established between the operator algebra formalism of quantum statistical mechanics and arithmetic. The Riemann zeta function is the partition function of the BC-system whose phase transitions with spontaneous symmetry breaking yield the class field theory isomorphism. In §2 we recall briefly the basic properties of the BC-system and overview the presentation of the *integral* BC-system as defined in [17]. The new results presented in these notes are contained in §3, they are joint work with C. Consani, and will appear in detailed form in [15]. We show that for each prime p there is a strong relation between the integral BC-system and the universal Witt ring $\mathbb{W}_0(\bar{\mathbb{F}}_p)$ of an algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_p$. As shown in [39] the abelian part of the integral BC-system is naturally a Λ-ring whose Frobenius endomorphisms agree with the basic endomorphisms defining the system. In §3 we prove that, if one drops the p-component, the full structure of the integral BC-system is completely described as $\mathbb{W}_0(\bar{\mathbb{F}}_p)$ with a precise dictionary expressing the key ingredients σ_n and $\tilde{\rho}_n$ of the integral BC-system as respectively the Frobenius F_n and Verschiebung V_n on $\mathbb{W}_0(\bar{\mathbb{F}}_p)$. Here, the Witt functor $\mathbb{W}_0$ is a dense subfunctor of the big Witt ring functor $\mathbb{W}$ and it is defined as the Grothendieck ring of the category of endomorphisms, whose objects are pairs (E, f) for f an endomorphism of the finite projective module E over a (commutative) ring A. The completion process from $\mathbb{W}_0(\bar{\mathbb{F}}_p)$ to $\mathbb{W}(\bar{\mathbb{F}}_p)$, together with the explicit isomorphism coming from the Artin–Hasse exponential

$$\mathbb{W}(\bar{\mathbb{F}}_p) = (\mathbb{W}_{p^\infty}(\bar{\mathbb{F}}_p))^{I(p)} \tag{1}$$

(where $\mathbb{W}_{p^\infty}$ is the Witt functor using the set of powers of the prime p while $I(p) \subset \mathbb{N}$ is the set of integers which are prime to p) yield an indecomposable p-adic representation of the integral BC-system. This representation π_σ is parameterized by an embedding

$$\sigma : \bar{\mathbb{F}}_p^\times \longrightarrow \mathbb{Q}^{\text{cyc}} \tag{2}$$

of the multiplicative group $\bar{\mathbb{F}}_p^\times$ as the group $\mu^{(p)} \subset \mu \sim \mathbb{Q}/\mathbb{Z}$ of roots of unity, in the cyclotomic field $\mathbb{Q}^{\text{cyc}}$, whose order is prime to p. The representations π_σ play, in the p-adic case, the same role as the positive energy irreducible complex representations of the BC-system which intertwine the class field theory isomorphism. In §3.6 we show that the analogy with the complex case goes much further and we construct an analogue in the p-adic case of the partition function and the KMS$_\beta$ states using the theory of p-adic L-functions. We thus obtain the p-adic analogue of the results of [20] for function fields. One should also acknowledge an important nuance with the complex case, i.e., the presence of an added symmetry at non-zero temperature, due to the invariance of the states under the natural involution of $\mathbb{Q}^{\text{cyc}}$ replacing each root of unity by its inverse.

A central problem in the theory of L-functions is the generalized Riemann conjecture on the location of their zeros. So far the only successful methods are due to the implementation of algebraic geometry in positive characteristic. The lack of a geometric analogue, in characteristic zero, of the curve underlying global fields of positive characteristic has prevented to transpose the proof of A. Weil. This situation is slowly changing under the influence of noncommutative geometry and also of the quest for the "absolute arithmetic" starting from ideas of Steinberg [48], Tits [50] and Krasner [31] in the 50's, and of Kurokawa [33] and Manin [38] in the 90's.

In [8], [16], [40], it was shown that the dual of the BC-system gives a trace formula interpretation of the Riemann–Weil explicit formulas and a spectral realization of the zeros of the Riemann zeta-function and of L-functions with Grössencharakters. This interpretation has recently been refined at the conceptual level in [11] where the spectral realization is obtained as the first cohomology group of a sheaf over a very basic geometric space: the projective line $\mathbb{P}^1_{\mathbb{F}_1}$ over the "absolute point" Spec $(\mathbb{F}_1)$. In several recent research papers (*cf.* [17], [10], [11], [12], [13], [14]) it is observed that some of the main features of the rich interconnection between noncommutative geometry and arithmetic originate in a basic algebro-geometric framework that we designate as "absolute". Some combinatorial formulas, like the equation supplying the cardinality of the set of rational points of a Grassmannian over a finite field $\mathbb{F}_q$, are known to be rational expressions keeping a meaningful value also when $q = 1$. These results motivate the search for a mathematical object that is expected to be a non-trivial limit of Galois fields $\mathbb{F}_q$, for $q = 1$. The goal is to define an analogue, for number fields, of the geometry underlying the arithmetic theory of the function fields and a geometric analogue of the "curve" underlying the Weil proof of the Riemann hypothesis for function fields. In [11] we showed how to determine the counting function $N(q)$, defined as a distribution on $[1,\infty)$ which plays for $\mathbb{K} = \mathbb{Q}$ the same role as the Weil counting function does for a field $\mathbb{K}$ of functions on a curve Y over $\mathbb{F}_p$ (*cf.* [38], [47]). The Weil function determines the number of points over various extensions of $\mathbb{F}_p$, $\#Y(\mathbb{F}_q) = N(q) = q - \sum_\alpha \alpha^r + 1,\ q = p^r$, where the numbers α's are the complex roots of the characteristic polynomial of the Frobenius endomorphism acting on the étale cohomology $H^1(Y \otimes \bar{\mathbb{F}}_p, \mathbb{Q}_\ell)$ of the curve ($\ell \neq p$). As recalled in §4.1, we have shown that the distribution $N(q)$ associated to the Riemann zeta function is described by a similar formula

$$N(u) = u - \frac{d}{du}\Big(\sum_{\rho\in Z} \text{order}\,(\rho)\frac{u^{\rho+1}}{\rho+1}\Big) + 1$$

and is positive on $(1,\infty)$ while having all the expected properties such as the correct value $N(1) = -\infty$ for the Euler characteristic. Moreover, in §4.2, we explain (*cf.* [14]) how to implement the trace formula understanding of the explicit formulas in number theory, to express the counting function $N(q)$ as an *intersection number* involving the scaling action of the idèle class group on the adèle class space. This is done using a Lefschetz formula and reveals a feature which is specific to characteristic zero: in the number field case the contribution of the archimedean places cannot be understood in a naive manner by counting a number of points but involves a transversality factor measuring the transversality of the action of the idèle class group with respect to periodic orbits. The periodic orbits cannot be considered in isolation and must be immersed in the ambient adèle class space. This gives a precious hint and shows that the role of ergodic theory and noncommutative geometry is indispensable. In the earlier work on the adèle class space this space was studied as a noncommutative space but its algebraic structure has only emerged very recently. First as a monoid (*cf.* [11]) and more recently (*cf.* [13]) as a hyperring in the sense of Krasner ([31]). In [13] we have shown that the prime elements of the hyperring $\mathbb{H}_{\mathbb{K}}$ of adèle classes of a global field $\mathbb{K}$ form a groupoid. Moreover when $\mathbb{K}$ is a function field associated to a curve X over a finite field, this groupoid is canonically isomorphic to the loop groupoid of the abelian cover of X corresponding the maximal abelian extension of $\mathbb{K}$.

The basic problem now is to construct a geometric counterpart X of the adèle class space $\mathbb{H}_{\mathbb{Q}}$ (endowed with the natural action of the idèle class group $C_{\mathbb{Q}}$) in which the points $v \in X$ are concretely realized as valuations and the Galois ambiguity is completely respected, without making an artificial choice of a base point over each place. The relation between the sought for geometric space X and the adèle class space should be of the same nature as the relation between the two sides of the class field theory isomorphism.

This problem is intimately related to noncommutative geometry in the sense of foliation spaces, and the original Lefschetz formula of Guillemin [23] in that context, which motivated our original approach in [8]. As we shall explain in §4.4 it is also related to the question of A. Weil (*cf.* [54]) of finding a Galois interpretation of the connected component of the identity in the idèle class group, and that of finding a canonical construction of the algebraic closure of the finite fields $\mathbb{F}_p$ (*cf.* [35]). We shall give in §4.4 a specific way of constructing the geometric counterpart of the adèle class space $\mathbb{H}_{\mathbb{Q}}$ with the remaining problem of explicitly comparing the constructions which *a priori* depend upon the choice of a prime. In the last section of these notes we shall explain, as a modest step towards the above goals, that the Witt construction extends (*cf.* [9]) to the case of characteristic one. This involves the notion of entropy as it enters in ergodic theory and also ties in with idempotent analysis [29], [37] and the approach recently developed by O. Viro [52] for tropical geometry using hyperfields.

2. The BC-system

The BC-system [4] is a system of quantum statistical mechanics whose partition function is the Riemann zeta function and which exhibits phase transitions with spontaneous symmetry breaking while the zero temperature states implement the class field theory isomorphism for the global field $\mathbb{Q}$.

2.1. *Hecke algebra*

The origin of the BC-system comes from the extension of the functor λ from discrete groups Γ to finite von Neumann algebras N which associates to Γ the commutant $\lambda(\Gamma)$ of the right translations by Γ in the Hilbert space $\ell^2(\Gamma)$. To go beyond the type II case one extends λ to pairs (Γ, Γ_0) where $\Gamma_0 \subset \Gamma$ is an almost normal subgroup, i.e., the orbits of the left action of Γ_0 on Γ/Γ_0 are all *finite*. The Hecke algebra $\mathscr{H}_{\mathbb{Q}}(\Gamma, \Gamma_0)$ is by definition the convolution algebra of functions of finite support

$$f : \Gamma_0\backslash\Gamma \longrightarrow \mathbb{Q}, \tag{3}$$

which fulfill the Γ_0-invariance condition

$$f(\gamma\gamma_0) = f(\gamma), \quad \forall\gamma \in \Gamma,\ \forall\gamma_0 \in \Gamma_0, \tag{4}$$

so that f is really defined on $\Gamma_0\backslash\Gamma/\Gamma_0$. The convolution product is then given by

$$(f_1 * f_2)(\gamma) = \sum_{\Gamma_0\backslash\Gamma} f_1(\gamma\gamma_1^{-1})f_2(\gamma_1). \tag{5}$$

One can then construct the left representation of the Hecke algebra $\mathscr{H}_{\mathbb{C}}(\Gamma, \Gamma_0)$ in $\ell^2(\Gamma_0\backslash\Gamma)$, and its weak closure is the commutant of the action of Γ. Moreover the state given by the canonical separating vector gives a canonical time evolution σ_t. It turns out that this time evolution preserves the dense subalgebra $\mathscr{H}_{\mathbb{C}}(\Gamma, \Gamma_0)$ and has the explicit form (*cf.* [4])

$$\sigma_t(f)(\gamma) = \left(\frac{L(\gamma)}{R(\gamma)}\right)^{-it} f(\gamma), \quad \forall\gamma \in \Gamma_0\backslash\Gamma/\Gamma_0, \tag{6}$$

where the integer valued functions L and R on the double coset space are given respectively by

$$L(\gamma) = \text{cardinality of left } \Gamma_0 \text{ orbit of } \gamma \text{ in } \Gamma/\Gamma_0, \quad R(\gamma) = L(\gamma^{-1}). \tag{7}$$

2.2. *The integral BC-system*

Let us consider the "$ax + b$" algebraic group P, i.e., the functor which to any commutative ring R assigns the group P_R of 2 by 2 matrices over R of the form

$$P_R = \left\{\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix};\ a, b \in R,\ a \text{ invertible}\right\}. \tag{8}$$

By construction $P_{\mathbb{Z}}^+ \subset P_{\mathbb{Q}}^+$ is an inclusion $\Gamma_0 \subset \Gamma$ of countable groups, where $P_{\mathbb{Z}}^+$ and $P_{\mathbb{Q}}^+$ denote the restrictions to $a > 0$ and it fulfills the commensurability condition

$$\text{The orbits of the left action of } \Gamma_0 \text{ on } \Gamma/\Gamma_0 \text{ are all } \textit{finite}. \tag{9}$$

The corresponding Hecke algebra $\mathscr{H}(\Gamma, \Gamma_0)$ can in fact be defined over $\mathbb{Z}$ ([17]) and we now give its presentation by generators and relations. One considers the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ of the group $\mathbb{Q}/\mathbb{Z}$ of abstract roots of unity. One lets $e(r) \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, be the canonical generators for $r \in \mathbb{Q}/\mathbb{Z}$. For each $n \in \mathbb{N}$, one defines endomorphisms σ_n of the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ by $\sigma_n(e(\gamma)) = e(n\gamma)$ and additive maps $\tilde{\rho}_n$ by

$$\tilde{\rho}_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \longrightarrow \mathbb{Z}[\mathbb{Q}/\mathbb{Z}], \quad \tilde{\rho}_n(e(\gamma)) = \sum_{n\gamma'=\gamma} e(\gamma'). \tag{10}$$

We recall from [17], Proposition 4.4, the following:

Proposition 2.1. *The maps σ_n define endomorphisms of $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ which fulfill the following relations with the maps $\tilde{\rho}_m$:*

$$\sigma_{nm} = \sigma_n\sigma_m, \quad \tilde{\rho}_{mn} = \tilde{\rho}_m\tilde{\rho}_n, \quad \forall m, n \in \mathbb{N}, \tag{11}$$

$$\tilde{\rho}_m(\sigma_m(x)y) = x\tilde{\rho}_m(y), \quad \forall x, y \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}], \tag{12}$$

$$\sigma_c(\tilde{\rho}_b(x)) = (b,c)\tilde{\rho}_{b'}(\sigma_{c'}(x)), \quad b' = b/(b,c), \quad c' = c/(b,c), \tag{13}$$

where (b, c) denotes the greatest common divisor of b and c.

Note that taking $b = c = n$ in (13) gives

$$\sigma_n(\tilde{\rho}_n(x)) = nx, \quad \forall x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]. \tag{14}$$

On the contrary, if we take $b = n$ and $c = m$ to be relatively prime we get

$$\sigma_n \circ \tilde{\rho}_m = \tilde{\rho}_m \circ \sigma_n. \tag{15}$$

The integral BC-algebra $\mathscr{H}_{\mathbb{Z}} = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes_{\tilde{\rho}} \mathbb{N}$ is the algebra generated by the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, and by the elements $\tilde{\mu}_n$ and μ_n^*, with $n \in \mathbb{N}$, which satisfy the relations:

$$\begin{aligned} \tilde{\mu}_n x \mu_n^* &= \tilde{\rho}_n(x), \\ \mu_n^* x &= \sigma_n(x)\mu_n^*, \\ x\tilde{\mu}_n &= \tilde{\mu}_n\sigma_n(x), \end{aligned} \tag{16}$$

where $\tilde{\rho}_m,\ m \in \mathbb{N}$ is defined in (10), as well as the relations

$$\begin{aligned} \tilde{\mu}_{nm} &= \tilde{\mu}_n\tilde{\mu}_m, \quad \forall n, m, \\ \mu_{nm}^* &= \mu_n^*\mu_m^*, \quad \forall n, m, \\ \mu_n^*\tilde{\mu}_n &= n, \\ \tilde{\mu}_n\mu_m^* &= \mu_m^*\tilde{\mu}_n, \quad (n, m) = 1. \end{aligned} \tag{17}$$

After tensoring by $\mathbb{Q}$, the Hecke algebra $\mathscr{H}_{\mathbb{Q}} = \mathscr{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ has a simpler explicit presentation with generators $\mu_n, \mu_n^*,\ n \in \mathbb{N}$ and $e(r)$, for $r \in \mathbb{Q}/\mathbb{Z}$, satisfying the following relations (and their adjoints)

- $\mu_n^* \mu_n = 1$, for all $n \in \mathbb{N}$,
- $\mu_k \mu_n = \mu_{kn}$, for all $k, n \in \mathbb{N}$,
- $\mu_n \mu_m^* = \mu_m^* \mu_n$, $(n, m) = 1$,
- $e(0) = 1$, $e(r)^* = e(-r)$, and $e(r)e(s) = e(r+s)$, for all $r, s \in \mathbb{Q}/\mathbb{Z}$,
- For all $n \in \mathbb{N}$ and all $r \in \mathbb{Q}/\mathbb{Z}$,

$$\mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s). \tag{18}$$

2.3. Phase transition with spontaneous symmetry breaking

After tensoring by $\mathbb{C}$ and completion one gets a C^*-algebra $\bar{\mathscr{H}}_{\mathbb{C}}$ with a natural time evolution σ_t ([4], [19] Chapter III).

Suppose given a C^*-dynamical system $(\mathscr{A}, \sigma_t)$, that is, a C^*-algebra $\mathscr{A}$ together with a 1-parameter group of automorphisms $\sigma : \mathbb{R} \to \mathrm{Aut}\,(\mathscr{A})$.

Definition 2.2. *For a given $0 < \beta < \infty$, a state φ on the unital C^*-algebra $\mathscr{A}$ satisfies the KMS condition at inverse temperature β if for all $a, b \in \mathscr{A}$, there exists a function $F_{a,b}(z)$ which is holomorphic on the strip (Fig. 1)*

$$I_\beta = \{z \in \mathbb{C} \mid 0 < \Im(z) < \beta\}, \tag{19}$$

continuous on the boundary ∂I_β and bounded, with the property that for all $t \in \mathbb{R}$

$$F_{a,b}(t) = \varphi(a\sigma_t(b)) \quad \textit{and} \quad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a). \tag{20}$$

The set Σ_β of KMS states at $\beta < \infty$ is a compact convex set for the weak topology:

$$\varphi_n \to \varphi \iff \varphi_n(a) \to \varphi(a), \quad \forall a \in \mathscr{A}.$$

Thus, it makes sense to consider the set $\mathscr{E}_\beta$ of its extremal points. Moreover one has

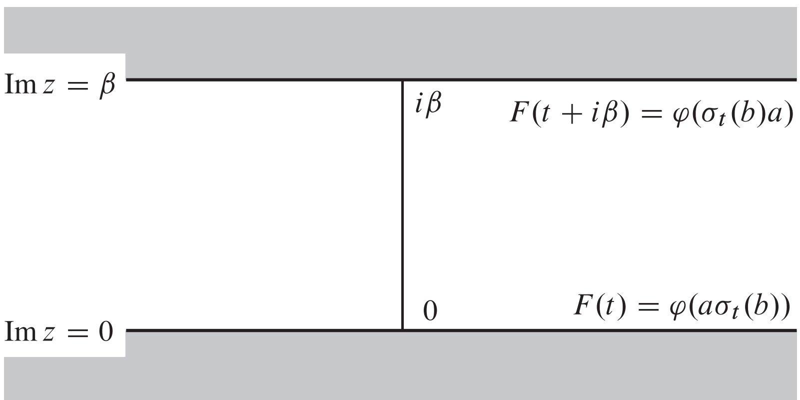

Fig. 1. The strip I_β in the KMS condition

- A KMS$_\beta$ state for $0 < \beta < \infty$ is extremal if and only if the corresponding GNS representation is factorial.
- For any $0 < \beta < \infty$, the set Σ_β of KMS$_\beta$ states is a convex compact Choquet simplex.

We apply this to the BC-system, i.e., the C^*-algebra $\bar{\mathscr{H}}_{\mathbb{C}}$, whose time evolution σ_t fixes the elements of the group ring of $\mathbb{Q}/\mathbb{Z}$ and fulfills

$$\sigma_t(\mu_n) = n^{it}\mu_n, \quad \forall n \in \mathbb{N},\ t \in \mathbb{R}. \tag{21}$$

Theorem 2.3. *The unique KMS state above critical temperature is*

$$\varphi_\beta(e(a/b)) = b^{-\beta} \prod_{p \text{ prime},\ p|b} \Big(\frac{1-p^{\beta-1}}{1-p^{-1}}\Big),$$

and the extremal KMS states below critical temperature are

$$\varphi_{\beta,\rho}(e(a/b)) = \frac{\mathrm{Tr}\,(\pi_\rho(e(a/b))e^{-\beta H})}{\mathrm{Tr}\,(e^{-\beta H})} = \frac{1}{\zeta(\beta)} \sum_{n=1}^{\infty} n^{-\beta}\rho(\zeta_{a/b}^n), \tag{22}$$

where π_ρ is the representation of the algebra $\mathscr{A}$ on the Hilbert space $\mathscr{H} = \ell^2(\mathbb{N})$ given by

$$\pi_\rho(\mu_n)\epsilon_m = \epsilon_{nm}, \quad \pi_\rho(e(a/b))\epsilon_m = \rho(\zeta_{a/b}^m)\epsilon_m, \tag{23}$$

where $\rho \in \hat{\mathbb{Z}}^*$ determines an embedding in $\mathbb{C}$ of the cyclotomic field $\mathbb{Q}^{\mathrm{cyc}}$ generated by the abstract roots of unity.

3. The Witt rings and the BC-system at p-adic places

We shall now display the relation between the Witt ring $\mathbb{W}_0(\bar{\mathbb{F}}_p)$ of an algebraic closure of the finite field $\mathbb{F}_p$ and the BC-system. We first need to recall the construction of the Witt functors $\mathbb{W}_0(A), \mathbb{W}(A), \mathbb{W}_{p^\infty}(A)$. For more details we refer to [55], [41], [5], [25], [3], [45], [26].

3.1. The Witt ring $\mathbb{W}_0(A)$

The algebraic properties of the Witt functor $A \mapsto \mathbb{W}(A)$, from rings to rings, are simple to understand in terms of a dense subring $\mathbb{W}_0(A) \subset \mathbb{W}(A)$ which is functorially defined as the Grothendieck group which classifies the endomorphisms of finite projective modules over the given commutative ring A with unit. One considers the category $\underline{\mathrm{End}}_A$ whose objects are pairs (E, f) where E is a finite projective module over A and $f \in \mathrm{End}_A(E)$ is an endomorphism of

E. The morphisms in the category commute with the endomorphisms f. The following operations of direct sum and tensor product

$$\begin{aligned}(E_1, f_1) \oplus (E_2, f_2) &= (E_1 \oplus E_2, f_1 \oplus f_2),\\ (E_1, f_1) \otimes (E_2, f_2) &= (E_1 \otimes E_2, f_1 \otimes f_2)\end{aligned} \tag{24}$$

turn the Grothendieck group $K_0(\underline{\mathrm{End}}_A)$ into a (commutative) ring. The pairs of the form $(E, f = 0)$ generate the ideal $K_0(A) \subset K_0(\underline{\mathrm{End}}_A)$. We denote the quotient ring by $\mathbb{W}_0(A)$

$$\mathbb{W}_0(A) = K_0(\underline{\mathrm{End}}_A)/K_0(A). \tag{25}$$

By construction $\mathbb{W}_0$ is a functor from the category $\mathfrak{Ring}$ of commutative rings with unit to itself. The key additional structures are given by

(1) The Teichmüller lift which is a multiplicative map $\tau : A \to \mathbb{W}_0(A)$.
(2) The Frobenius endomorphisms F_n for $n \in \mathbb{N}$.
(3) The Verschiebung (shift) additive functorial endomorphisms V_n, $n \in \mathbb{N}$.
(4) The ghost components $\mathrm{gh}_n : \mathbb{W}_0(A) \to A$ for $n \in \mathbb{N}$.

They are defined as follows:

(1) The Teichmüller lift is simply given by the map $A \ni f \mapsto (A, f)$.

(2) For $n \in \mathbb{N}$, the following operations on $\underline{\mathrm{End}}_A$ induce endomorphisms in $\mathbb{W}_0(A)$ which are the Frobenius endomorphisms

$$F_n(E, f) = (E, f^n). \tag{26}$$

(3) The Verschiebung maps V_n are described by the following operation on matrices:

$$V_n(f) = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 & f \\ 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{27}$$

(4) The ghost components are given by

$$\mathrm{gh}_n : \mathbb{W}_0(A) \longrightarrow A, \quad \mathrm{gh}_n(E, f) = \mathrm{Trace}\,(f^n). \tag{28}$$

One has

$$V_{nm} = V_n \circ V_m = V_m \circ V_n, \quad F_{nm} = F_n \circ F_m = F_m \circ F_n. \tag{29}$$

In order to get familiar with this construction one can check the following proposition:

Proposition 3.1. *Let A be a commutative ring and $x, y \in \mathbb{W}_0(A)$.*

(1) $F_n \circ V_n(x) = nx$.
(2) $V_n(F_n(x)y) = xV_n(y)$.
(3) If m is prime to n, one has $V_m \circ F_n = F_n \circ V_m$.
(4) For $n \in \mathbb{N}$ *one has* $V_n(x)V_n(y) = nV_n(xy)$.
(5) $F_n(\tau(f)) = \tau(f^n)$.
(6) $\mathrm{gh}_n(F_m(f)) = \mathrm{gh}_{nm}(f)$.
(7) $\mathrm{gh}_n(V_m(f)) = m\,\mathrm{gh}_{n/m}(f)$ *if m divides n and* $\mathrm{gh}_n(V_m(f)) = 0$ *otherwise.*

Let $\Lambda(A) := 1 + tA[[t]]$. The characteristic polynomial defines a map

$$L : \mathbb{W}_0(A) \longrightarrow \Lambda(A), \quad L(E, f) = \det(1_E - tf)^{-1}, \tag{30}$$

and by a fundamental result of Almkvist ([2] Main Theorem), the map L is always injective and its image is the subset of $\Lambda(A)$ whose elements are rational fractions

$$\text{Range}\,(L) = \{(1 + a_1 t + \cdots + a_n t^n)/(1 + b_1 t + \cdots + b_n t^n) \mid a_j, b_j \in A\}.$$

3.2. *The big Witt ring* $\mathbb{W}(A)$

Both the ring structure and the four additional structures of the functor $\mathbb{W}_0$ extend "by continuity" to $\Lambda(A) := 1 + tA[[t]]$ using the map L to view $\mathbb{W}_0$ as a subfunctor of the completion $\mathbb{W}$. As a functor to the category $\mathfrak{Sets}$ of sets, $\mathbb{W}$ coincides with $A \mapsto \Lambda(A) := 1 + tA[[t]]$. Under the map L of (30), one can easily translate (*cf.* [55], [41], [5])

- The addition in $\mathbb{W}_0(A)$ by

$$L(f \oplus g) = L(f)L(g), \quad \forall f, g \in \mathbb{W}_0(A). \tag{31}$$

- The Teichmüller lift which gives

$$L(\tau(f)) = (1 - tf)^{-1} \in \Lambda(A). \tag{32}$$

- The shift V_n

$$L(V_n(f))(t) = L(f)(t^n). \tag{33}$$

- The ghost components

$$\sum_1^\infty \mathrm{gh}_n(f)t^n = t\frac{d}{dt}\log(L(f(t))). \tag{34}$$

This makes it clear how to extend the addition, the Teichmüller lift, the shifts and the ghost components to the completion $\Lambda(A)$. The corresponding product $\star$ on $\Lambda(A)$ is uniquely determined by functoriality and requiring that the ghost components define ring homomorphisms. It is given by explicit polynomials with integral coefficients of the form

$$\begin{aligned}&\Big(1+\sum a_n t^n\Big)\star\Big(1+\sum b_n t^n\Big)\\ &= 1 + a_1 b_1 t + (a_1^2 b_1^2 - a_2 b_1^2 - a_1^2 b_2 + 2a_2 b_2)t^2 + (a_1^3 b_1^3 - 2a_1 a_2 b_1^3\\ &\quad + a_3 b_1^3 - 2a_1^3 b_1 b_2 + 5a_1 a_2 b_1 b_2 - 3a_3 b_1 b_2 + a_1^3 b_3 - 3a_1 a_2 b_3\\ &\quad + 3a_3 b_3)t^3 + \cdots .\end{aligned}$$

Finally as shown in [5] the Frobenius F_n is given by the norm map N_n from $A[[t]]$ to $A[[t^n]] \subset A[[t]]$ composed with the change of variables $t^n \mapsto t$. Thus

$$N_n[f] = F_n(f)(t^n), \quad \forall f \in \Lambda(A). \tag{35}$$

3.3. *Witt vectors and* $\mathbb{W}_{p^\infty}(A)$

Every element $f(t) \in \Lambda(A)$ can be written uniquely as an infinite product

$$f(t) = \prod (1 - z_n t^n), \quad z_n \in A, \quad \forall n.$$

Passing to the inverse this shows that the following map is a bijection from the set $\mathbb{W}(A)$ of sequences (x_n) of elements of A to $\Lambda(A)$

$$\begin{aligned}&\varphi_A : \mathbb{W}(A) \longrightarrow \Lambda(A) := 1 + tA[[t]],\\ &x = (x_n)_{n\in\mathbb{N}} \longmapsto f_x(t) = \prod_{\mathbb{N}} (1 - x_n t^n)^{-1}.\end{aligned} \tag{36}$$

In other words any element of $\Lambda(A)$ can be uniquely written as

$$f = \sum V_n(\tau(x_n)). \tag{37}$$

The "Witt vector coordinates" allow one to express the ghost components quite simply as

$$\mathrm{gh}_n((x_n)) = \sum_{d|n} d x_d^{n/d}. \tag{38}$$

Using this formula one can compute the universal polynomials which express the addition and multiplication in terms of the Witt vector coordinates. These operations are uniquely defined by functoriality and the requirement that the ghost coordinates define ring homomorphisms. Formula (38) shows that the n-th addition and multiplication polynomials $\mu_{S,n}$, $\mu_{P,n}$ are polynomials that only involve the x_d and y_d with d a divisor of n. Thus, for a truncation set $N \subseteq \mathbb{N}$, i.e., a subset of $\mathbb{N}$ which contains every positive divisor of each of its elements, one associates the truncated functor $\mathbb{W}_N : \mathfrak{Ring} \to \mathfrak{Ring}$, where one only retains the Witt components with an index $n \in N$. This applies in particular for $N = \{p^n \mid n \geq 0\}$ where p is a prime number, so the p-adic Witt vectors $\mathbb{W}_{p^\infty}(A)$ are a functorial quotient of the big Witt vectors (one similarly obtains $\mathbb{W}_{p^n}(A)$ as the p-adic Witt vectors of length $n+1$ and $\mathbb{W}_n(A)$ as the Witt vectors of length n).

For $N \subset \mathbb{N}$ a truncation set, and $n \in N$ such that $nN \subset N$, the shift is the additive map given by

$$\begin{aligned} &V_n : \mathbb{W}_N(A) \longrightarrow \mathbb{W}_N(A), \\ &V_n((a_d | d \in N)) = (a'_m | m \in N); \quad a'_m = \begin{cases} a_d & \text{if } m = nd \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

At the level of the Witt vector components x_j the Frobenius F_n is given by polynomials with integral coefficients and for instance the first 5 components of $F_3(x)$ are

$$\begin{aligned} F_3(x)_1 &= x_1^3 + 3x_3, \\ F_3(x)_2 &= x_2^3 - 3x_1^3 x_3 - 3x_3^2 + 3x_6, \\ F_3(x)_3 &= -3x_1^6 x_3 - 9x_1^3 x_3^2 - 8x_3^3 + 3x_9, \\ F_3(x)_4 &= -3x_1^9 x_3 + 3x_1^3 x_2^3 x_3 - 18x_1^6 x_3^2 + 3x_2^3 x_3^2 - 36x_1^3 x_3^3 \\ &\quad - 24x_3^4 + x_4^3 - 3x_2^3 x_6 + 9x_1^3 x_3 x_6 + 9x_3^2 x_6 - 3x_6^2 + 3x_{12}, \\ F_3(x)_5 &= -3x_1^{12} x_3 - 18x_1^9 x_3^2 - 54x_1^6 x_3^3 - 81x_1^3 x_3^4 - 48x_3^5 + x_5^3 + 3x_{15}. \end{aligned}$$

One has $\mathrm{gh}_m(F_n(x)) = \mathrm{gh}_{mn}(x)$ and the Witt vector component $(F_n(x))_m$ only depends on the Witt vector components x_d where d is a divisor of mn. Note that when p is a rational prime one has (*cf.* [42] Proposition 5.12)

$$F_p(x)_m \equiv x_m^p \pmod{pA}. \tag{39}$$

3.4. The BC-system and $\mathbb{W}_0(\bar{\mathbb{F}}_p)$

The abelian part of the integral BC-system is naturally (*cf.* [39]) a Λ-ring whose Frobenius endomorphisms agree with the basic endomorphisms σ_n. Our first goal is to relate this Λ-ring with the canonical Λ-ring $\mathbb{W}_0(\bar{\mathbb{F}}_p)$ for each prime p. Thus we let p be a prime, and let X_p be the space of embeddings

$$\sigma : \bar{\mathbb{F}}_p^\times \longrightarrow \mathbb{C}^\times \tag{40}$$

of the multiplicative group $\bar{\mathbb{F}}_p^\times$ as the group $\mu^{(p)} \subset \mu \sim \mathbb{Q}/\mathbb{Z}$ of roots of unity in $\mathbb{C}$ whose order is prime to p. Let r be the retraction

$$r : \mathbb{Z}[\mu] \xrightarrow{id \otimes \epsilon} \mathbb{Z}[\mu^{(p)}]$$

associated to the augmentation ϵ of $\mathbb{Z}[\mu_{p^\infty}]$ in the decomposition

$$\mathbb{Z}[\mu] = \mathbb{Z}[\mu^{(p)}] \otimes \mathbb{Z}[\mu_{p^\infty}],$$

where μ_{p^∞} is the group of roots of unity whose order is a power of p. The relation between $\mathbb{W}_0(\bar{\mathbb{F}}_p)$ and $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ is given by the following:

Theorem 3.2. *To each σ as in* (40)*, corresponds a canonical isomorphism $\tilde{\sigma}$*

$$\mathbb{W}_0(\bar{\mathbb{F}}_p) \xrightarrow{\tilde{\sigma}} \mathbb{Z}[\mu^{(p)}] \subset \mathbb{Z}[\mu].$$

The Frobenius F_n and Verschiebung maps V_n of $\mathbb{W}_0(\bar{\mathbb{F}}_p)$ are obtained by restriction of the endomorphisms σ_n and maps $\tilde{\rho}_n$ of $\mathbb{Z}[\mu] = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ by the formulas

$$\tilde{\sigma} \circ F_n = \sigma_n \circ \tilde{\sigma}, \quad \tilde{\sigma} \circ V_n = r \circ \tilde{\rho}_n \circ \tilde{\sigma}. \tag{41}$$

This theorem shows that the integral BC-system with its full structure is, if one drops the p-component, completely described as $\mathbb{W}_0(\bar{\mathbb{F}}_p)$. As a corollary of the above Theorem one gets:

Theorem 3.3. *Let $\sigma \in X_p$. The following formulas define a representation π_σ of the integral BC-system $\mathscr{H}_\mathbb{Z}$ on $\mathbb{W}_0(\bar{\mathbb{F}}_p)$,*

$$\pi_\sigma(x)\xi = \tilde{\sigma}^{-1}(r(x))\xi, \quad \pi_\sigma(\mu_n^*) = F_n, \quad \pi_\sigma(\tilde{\mu}_n) = V_n \tag{42}$$

for all $\xi \in \mathbb{W}_0(\bar{\mathbb{F}}_p)$, $x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and $n \in \mathbb{N}$.

3.5. $\mathbb{W}(\bar{\mathbb{F}}_p)$ and p-adic representations of the BC-system

Now since $\mathbb{W}_0(\bar{\mathbb{F}}_p)$ is only a dense subring of the ring $\mathbb{W}(\bar{\mathbb{F}}_p)$ of big Witt vectors on $\bar{\mathbb{F}}_p$, one needs to understand the latter and how the completion process works. The structure of the ring $\mathbb{W}(\bar{\mathbb{F}}_p)$ follows from [5], [45] where a canonical isomorphism is constructed

$$\mathbb{W}(\bar{\mathbb{F}}_p) = (\mathbb{W}_{p^\infty}(\bar{\mathbb{F}}_p))^{I(p)}, \tag{43}$$

where $\mathbb{W}_{p^\infty}$ is the Witt functor using the set of powers of the prime p while $I(p) \subset \mathbb{N}$ is the set of integers which are prime to p. Thus, by construction, $\mathbb{W}_{p^\infty}(\bar{\mathbb{F}}_p)$ is the completion of the maximal unramified extension of the p-adic integers $\mathbb{Z}_p$. It sits naturally as a subring $\mathscr{O}_{\widehat{\mathbb{Q}_p^{ur}}}$ of the field $\mathbb{C}_p$ which is the completion for the p-adic norm of an algebraic closure of $\mathbb{Q}_p$ (*cf.* [44]).

The isomorphism (43) holds for commutative $\mathbb{F}_p$-algebras A. On $\Lambda(A)$, a central role is played by the Artin–Hasse exponential

$$E_p(t) = \text{hexp}\,(t) = \exp\Big(t + \frac{t^p}{p} + \frac{t^{p^2}}{p^2} + \cdots\Big) \in \Lambda(\mathbb{F}_p). \tag{44}$$

One has (*cf.* [3], [45]):

Proposition 3.4. *(1) $E_p(t) \in \Lambda(\mathbb{F}_p)$ is an idempotent of the ring $\Lambda(\mathbb{F}_p)$.*
(2) The $E_p(n) = \frac{1}{n}V_n(E_p) \in \Lambda(\mathbb{F}_p)$, $n \in I(p)$, determine a partition of unity by idempotents.
(3) For $n \notin p^{\mathbb{N}}$, $F_n(E_p) = 1 (= 0_\Lambda)$ and $F_{p^k}(E_p) = E_p$ for all $k \in \mathbb{N}$.

Note that any $n \in I(p)$ is invertible in $\Lambda(\mathbb{F}_p)$. This corresponds to the extraction of the n-th root of the power series $f = 1 + g$ and it is given by the binomial formula

$$(1+g)^{\frac{1}{n}} = 1 + \frac{1}{n}g + \cdots + \frac{\frac{1}{n}(\frac{1}{n}-1)\cdots(\frac{1}{n}-k+1)}{k!}g^k + \cdots . \tag{45}$$

The p-adic valuation of the rational coefficient of g^k is positive because $\frac{1}{n}$ is a p-adic integer, thus this coefficient can be approximated arbitrarily by a binomial coefficient. By Proposition 3.1, (4), for $n \in I(p)$, $\frac{1}{n}V_n$ is an endomorphism of $\Lambda(\bar{\mathbb{F}}_p)$ and is a right inverse of F_n. One has (*cf.* [5], [3], [45]):

Proposition 3.5. *Let A be an $\mathbb{F}_p$-algebra.*

(a) The following map is an isomorphism of $\mathbb{W}_{p^\infty}(A)$ with the reduced ring $\Lambda(A)_{E_p}$

$$\begin{aligned} \psi_A : \mathbb{W}_{p^\infty}(A) &\longrightarrow \Lambda(A)_{E_p}, \\ x = (x_{p^n})_{n\in\mathbb{N}} &\longmapsto h_x(t) = \prod_{\mathbb{N}} E_p(x_{p^n} t^{p^n}). \end{aligned} \tag{46}$$

(b) For $n \in I(p)$, the composite $\psi_A^{-1} \circ F_n$ is an isomorphism of the reduced algebra $\Lambda(A)_{E_p(n)}$ with $\mathbb{W}_{p^\infty}(A)$.
(c) The following is a canonical isomorphism of $\Lambda(A)$ with the Cartesian product $\mathbb{W}_{p^\infty}(A)^{I(p)}$

$$\theta_A(x)_n = \psi_A^{-1} \circ F_n(x \star E_p(n)), \quad \forall n \in I(p). \tag{47}$$

(d) The composite isomorphism $\Theta_A = \theta_A \circ \varphi_A : \mathbb{W}(A) \to \mathbb{W}_{p^\infty}(A)^{I(p)}$ is given explicitly on the components by

$$(\Theta_A(x)_n)_{p^k} = F_n(x)_{p^k}, \quad \forall x \in \mathbb{W}(A),\ n \in I(p). \tag{48}$$

We now apply these results to an algebraic closure $A = \bar{\mathbb{F}}_p$. We identify $\mathbb{W}_{p^\infty}(A)$ with a subring of $\mathbb{C}_p$ (the p-adic completion of an algebraic closure of the p-adic numbers $\mathbb{Q}_p$). Let $\widehat{\mathbb{Q}_p^{\mathrm{ur}}} \subset \mathbb{C}_p$ be the completion of the maximal unramified extension $\mathbb{Q}_p^{\mathrm{ur}}$ of p-adic numbers. Then $\mathbb{W}_{p^\infty}(\bar{\mathbb{F}}_p)$ is the completion $\mathscr{O}_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}}} \subset \widehat{\mathbb{Q}_p^{\mathrm{ur}}}$ of the subring generated by roots of unity. With Θ the isomorphism of (48), we have

$$\Theta : \mathbb{W}(\bar{\mathbb{F}}_p) \longrightarrow (\mathscr{O}_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}}})^{I(p)}, \quad (\Theta(x)_n)_{p^k} = F_n(x)_{p^k}. \tag{49}$$

Θ makes $\mathbb{W}(\bar{\mathbb{F}}_p)$ a module over $\mathscr{O}_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}}}$. For $m \in I(p)$, we let ϵ_m be the vector with only one non-zero component: $\epsilon_m(m) = 1$. To the Frobenius automorphism of $\bar{\mathbb{F}}_p$ corresponds, by functoriality, a canonical automorphism Fr of $\mathscr{O}_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}}}$ which extends to a continuous automorphism

$$\mathrm{Fr} \in \mathrm{Aut}\,(\widehat{\mathbb{Q}_p^{\mathrm{ur}}}). \tag{50}$$

We can now describe the p-adic analogues of the complex irreducible representations π_ρ of the BC-system of (23). Recall that X_p denotes the space of all injective group homomorphisms $\sigma : \bar{\mathbb{F}}_p^\times \to \mathbb{C}^\times$. The choice of $\sigma \in X_p$ determines an embedding $\rho : \mathbb{Q}^{\mathrm{cyc},p} \to \mathbb{C}_p$ of the cyclotomic field generated by the abstract roots of unity of order prime to p.

Theorem 3.6. *Let $\sigma \in X_p$ and $\rho : \mathbb{Q}^{\mathrm{cyc},p} \to \mathbb{C}_p$ the associated embedding. The representation π_σ of Theorem 3.3 extends by continuity to a representation of the integral BC-system $\mathscr{H}_{\mathbb{Z}}$ on $\mathbb{W}(\bar{\mathbb{F}}_p)$. For $n \in I(p)$, the $\pi_\sigma(\mu_n)$ and for $x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, the $\pi_\sigma(x)$ are $\mathscr{O}_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}}}$-linear operators such that*

$$\pi_\sigma(\mu_n)\epsilon_m = \epsilon_{nm}, \quad \pi_\sigma(e(a/b))\epsilon_m = \rho(\zeta_{a/b}^m)\epsilon_m, \quad \forall n \in \mathbb{N},\ m, b \in I(p). \tag{51}$$

Moreover

$$\pi_\sigma(\mu_p) = \mathrm{Fr}^{-1} \tag{52}$$

is the inverse of the Frobenius automorphism, acting componentwise as a skew linear operator.

Note that $\pi_\sigma(\mu_p) = \mathrm{Fr}^{-1}$ is $\mathbb{Z}_p$-linear but not $\mathscr{O}_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}}}$-linear. The presence of this operator allows one to show that the representation π_σ is indecomposable when viewed as a $\mathbb{Z}_p$-representation of the integral BC-system $\mathscr{H}_{\mathbb{Z}}$. It is important however to also view π_σ as a $\mathscr{O}_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}}}$-skew-linear representation. The natural parameter for the representation is then the embedding

$$\rho : \mathbb{Q}^{\mathrm{cyc},p} \longrightarrow \mathbb{C}_p. \tag{53}$$

We shall now investigate the analogue, in this p-adic context, of the KMS states given in the complex case by (22).

3.6. Partition function, p-adic L-functions and KMS at a prime p

We let p be a prime, $I(p)$ the set of all positive integers not divisible by p and we take an embedding $\rho : \mathbb{Q}^{\mathrm{cyc},p} \to \mathbb{C}_p$ of the cyclotomic field generated by the abstract roots of unity of order prime to p to the field $\mathbb{C}_p$ which is the completion of an algebraic closure of $\mathbb{Q}_p$. We consider an expression of the form

$$Z\Big(\frac{a}{b}, \beta\Big) = \sum_{m \in I(p)} \rho(\zeta_{a/b}^m) m^{-\beta}. \tag{54}$$

Here $b \in I(p)$ and as a function of $m \in I(p)$ with values in $\mathbb{C}_p$ the function $\rho(\zeta_{a/b}^m)$ only depends on the residue of m modulo b. We let $f = bp$ and decompose the sum (54) according to the residue α of m modulo f. Since $b \in I(p)$ is prime to p one has $\mathbb{Z}/f\mathbb{Z} = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}$. Only values of $\alpha = (\alpha_1, \alpha_2)$ whose component in $\alpha_1 \in \mathbb{Z}/p\mathbb{Z}$ is non-zero come from $I(p)$ and this characterizes $I(p)$. For each $\alpha \in \mathbb{Z}/p\mathbb{Z}^\times \times \mathbb{Z}/b\mathbb{Z}$ we let $\tilde{\alpha} \in \mathbb{N}$ be the smallest integer with residue modulo f equal to α. We thus get

$$Z\Big(\frac{a}{b}, \beta\Big) = \sum_{\alpha} \rho(\zeta_{a/b}^\alpha) \sum_{n \in \mathbb{N}} (\tilde{\alpha} + fn)^{-\beta}. \tag{55}$$

The first sum involves $\alpha \in \mathbb{Z}/p\mathbb{Z}^\times \times \mathbb{Z}/b\mathbb{Z}$ and is finite. Each infinite sum is of the form

$$\sum_{n \in \mathbb{N}} (\tilde{\alpha} + fn)^{-\beta} = f^{-\beta} \sum_{n \in \mathbb{N}} (z + n)^{-\beta}$$

where $z = \tilde{\alpha}/f$ has p-adic norm $|z|_p > 1$. We are using formally the rule

$$(xy)^\beta = x^\beta y^\beta. \tag{56}$$

To understand how to make sense of the above expression in the p-adic case, we recall the Euler–Maclaurin formula:

$$\begin{aligned} \sum_{k=a}^{b} f(k) &= \int_a^b f(t)\, dt + \frac{f(a) + f(b)}{2} \\ &\quad + \sum_{j=2}^{m} \frac{B_j}{j!} (f^{(j-1)}(b) - f^{(j-1)}(a)) - R_m, \end{aligned} \tag{57}$$

where $B_j = B_j(0)$, the $B_j(x)$ are the Bernoulli polynomials, and the remainder R_m is expressed as

$$R_m = \frac{(-1)^m}{m!} \int_a^b f^{(m)}(x) B_m(x - [x])\, dx. \tag{58}$$

One applies this formula to the function $f(x) = (z + x)^{-s}$. One gets

$$\sum_{k=0}^{\infty}(z+k)^{-s} \sim \frac{z^{1-s}}{-1+s}\sum_{0}^{\infty}\binom{1-s}{j}B_j z^{-j}. \tag{59}$$

This formula gives a much better numerical approximation than the original series (and was used by Euler to get 20 decimal places for $\sum n^{-2}$ which he then proved equal to $\pi^2/6$). It is not a convergent series but an asymptotic series. In the p-adic case we write

$$\sum_{n=0}^{\infty}(z+n)^{-\beta} := \frac{z^{1-\beta}}{-1+\beta}\sum_{0}^{\infty}\binom{1-\beta}{j}B_j z^{-j}$$

and thus

$$f^{-\beta}\sum_{n=0}^{\infty}(z+n)^{-\beta} := \frac{1}{f}\frac{\tilde{\alpha}^{1-\beta}}{-1+\beta}\sum_{0}^{\infty}\binom{1-\beta}{j}B_j z^{-j}. \tag{60}$$

By [53] Chapter V, Theorem 5.9, this formula defines a p-adic meromorphic function on

$$D=\{\beta\in\mathbb{C}_p \mid |\beta|_p < qp^{-1/(p-1)}(>1)\}, \quad q=4, \text{ if } p=2, \; q=p, \text{ if } p\neq 2. \tag{61}$$

We can now write the p-adic formula for (54)

$$Z\Big(\frac{a}{b},\beta\Big) := \frac{1}{bp}\sum_{1\le c<bp,\; c\notin p\mathbb{N}}\rho(\zeta_{a/b}^{c})\frac{c^{1-\beta}}{-1+\beta}\sum_{0}^{\infty}\binom{1-\beta}{j}\Big(\frac{c}{bp}\Big)^{-j}B_j. \tag{62}$$

The term $c^{1-\beta}$ requires a precise definition and one first decomposes c as

$$c = \omega(c)\langle c\rangle \tag{63}$$

where $\omega(c)$ is the Teichmüller lift of the residue of c modulo p, and where $\langle c\rangle$ is in $1 + p\mathbb{Z}_p$. Since c is not a multiple of p, $\omega(c)$ is a $(p-1)$-root of unity and one has

$$\langle c\rangle^{p-1} = c^{p-1}.$$

We assumed $p \neq 2$. For $p = 2$ the torsion in $\mathbb{Z}_2^{\times}$ is given by the two elements $\pm 1 \in \mathbb{Z}_2^{\times}$. To know the torsion one needs to take the residue of c modulo 4 and lift it to $\omega(c) = \pm 1 \in \mathbb{Z}_2^{\times}$ so that $c \cong \omega(c)$ modulo 4. One then uses (63) to define $\langle c\rangle$ in this case $p = 2$.

One now defines in general

$$\langle c\rangle^{s} := \sum_{0}^{\infty}\binom{s}{j}(\langle c\rangle - 1)^{j} \tag{64}$$

which is convergent for $s \in D$ by [53] Chapter V, §1. This is also equal to $\exp(s\log(c))$ where $\log : \mathbb{C}_p^\times \to \mathbb{C}_p$ is the Iwasawa logarithm and exp the analogue of the exponential (*cf.* [53]) given by the usual series which converges in the domain

$$\{u \in \mathbb{C}_p \mid |u|_p < p^{-1/(p-1)}\}. \tag{65}$$

Thus the final form of the definition is

Definition 3.7. *With q as in* (61)*,*

$$Z\Big(\frac{a}{b},\beta\Big) := \frac{1}{bq}\sum_{1\le c<bq,\ c\notin p\mathbb{N}} \rho(\zeta_{a/b}^c)\frac{\langle c\rangle^{1-\beta}}{\beta-1}\sum_0^\infty \binom{1-\beta}{j}\Big(\frac{bq}{c}\Big)^j B_j. \tag{66}$$

Let us first look at the normalization factor (partition function) which is

$$Z(\beta) := \frac{1}{bq}\sum_{1\le c<bq,\ c\notin p\mathbb{N}} \frac{\langle c\rangle^{1-\beta}}{\beta-1}\sum_0^\infty \binom{1-\beta}{j}\Big(\frac{bq}{c}\Big)^j B_j. \tag{67}$$

The first point is that $Z(\beta)$ has a pole at $\beta = 1$ and the residue is given by

$$\frac{1}{bq}\sum_{1\le c<bq,\ c\notin p\mathbb{N}} 1 = \frac{\varphi(q)}{q} = \frac{p-1}{p}.$$

Proposition 3.8. *When $\beta \to 1$ one has*

$$Z(\beta)^{-1}Z\Big(\frac{a}{b},\beta\Big) \longrightarrow \begin{cases} 1, \ \textit{if}\ \dfrac{a}{b} \in \mathbb{Z} \\ 0, \ \textit{otherwise.} \end{cases} \tag{68}$$

Proof. Assume first that $\frac{a}{b} \notin \mathbb{Z}$. Then $\xi = \rho(\zeta_{a/b})$ is a non-trivial root of unity, whose order $m > 1$ divides b which is prime to p and hence to q. Thus

$$\sum_{1\le c<bq,\ c\notin p\mathbb{N}} \xi^c = \varphi(q)\sum_{n\in\mathbb{Z}/b\mathbb{Z}} \xi^n = 0$$

using $\mathbb{Z}/bq\mathbb{Z} = \mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}$. For $\frac{a}{b} \in \mathbb{Z}$ the result follows from the above discussion. □

In particular the limit for $\beta \to 1$ of the functional values $Z(\beta)^{-1}Z(\frac{a}{b},\beta)$ is independent of the value of ρ (i.e., of $\sigma \in X_p$). In the classical case of the BC-system the value for $\beta > 1$ is given by (22). In that case, let us check directly that for $\beta > 1$ these functional values determine ρ embedding in $\mathbb{C}$ of the cyclotomic field $\mathbb{Q}^{\mathrm{cyc}}$ generated by the abstract roots of unity. One lets $\lambda \in \hat{\mathbb{Z}}^*$ and one has

Lemma 3.10. *Assume that $\lambda \in \hat{\mathbb{Z}}^*$ is not equal to ± 1, then the graph of the multiplication by λ in $\mathbb{Q}/\mathbb{Z}$ is a dense subset of $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$.*

Proof. The subset

$$G = \{(\alpha, \lambda\alpha) \mid \alpha \in \mathbb{Q}/\mathbb{Z}\}$$

is a subgroup of $\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}$ and so is its closure $\bar{G}$. If G is not dense there exists a non-trivial character χ of the compact group $\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}$ whose kernel contains G. Thus there exists a non-zero pair $(n, m) \in \mathbb{Z}^2$ such that $n\alpha + m\lambda\alpha \in \mathbb{Z}$ for all $\alpha \in \mathbb{Q}/\mathbb{Z}$. This implies that the multiplication by $\lambda \in \hat{\mathbb{Z}}^*$ in the group $\mathbb{Q}/\mathbb{Z} = \mathbb{A}_{\mathbb{Q},f}/\hat{\mathbb{Z}}$ fulfills

$$n\alpha + m\lambda\alpha \in \hat{\mathbb{Z}}, \quad \forall \alpha \in \mathbb{A}_{\mathbb{Q},f}.$$

If $n/m \notin \{\pm 1\}$ there exists a prime p which divides only one of them, say $p|n$. Let then $k = v_p(n)$, one has p-adic units u and v such that

$$up^k\alpha + v\lambda_p\alpha \in \mathbb{Z}_p, \quad \forall \alpha \in \mathbb{Q}_p$$

which is a contradiction. Hence $n/m \in \{\pm 1\}$ and $\lambda \in \{\pm 1\}$. □

It follows from Lemma 3.10 that, if $f : \{z \in \mathbb{C} \mid |z| = 1\} \to \mathbb{C}$ is a continuous non-constant function, an equality of the form

$$f(\rho_1(\zeta_{a/b})) = f(\rho_2(\zeta_{a/b})), \quad \forall a/b \in \mathbb{Q}/\mathbb{Z} \tag{69}$$

implies that $\rho_2 = \rho_1$ or $\rho_2 = \bar{\rho}_1$. Moreover in the latter case one gets

$$f(\bar{z}) = f(z), \quad \forall z,\ |z| = 1.$$

This case does not happen for the function $f(z) = \sum_{n=1}^{\infty} n^{-\beta} z^n$ by uniqueness of the Fourier decomposition.

We now return to the p-adic case and consider the values of the functions for $\beta = 1-(p-1)k$ where k is a positive integer. We assume $p > 2$ for simplicity. In that case one has

$$\langle c\rangle^{1-\beta} = c^{k(p-1)} \tag{70}$$

from the definition of $\langle c\rangle$. Also the binomial coefficients $\binom{1-\beta}{j}$ all vanish for $j > k(p-1)$ and the sum (66) defining $Z(\frac{a}{b}, \beta)$ is finite. One has

$$\frac{\langle c\rangle^{1-\beta}}{\beta-1}\sum_0^{\infty}\binom{1-\beta}{j}\Big(\frac{bp}{c}\Big)^j B_j = -\frac{c^{k(p-1)}}{(p-1)k}\sum_0^{k(p-1)}\binom{k(p-1)}{j}\Big(\frac{bp}{c}\Big)^j B_j$$

and for any integer $m > 0$

$$\sum_0^{m}\binom{m}{j}\Big(\frac{bp}{c}\Big)^j B_j = \Big(\frac{bp}{c}\Big)^m B_m\Big(\frac{c}{bp}\Big).$$

Thus one gets, with $m = k(p-1)$, $1 \le c \le bp$,

$$T(c) = \frac{1}{bp}\frac{\langle c\rangle^{1-\beta}}{\beta - 1}\sum_0^\infty \binom{1-\beta}{j}\Big(\frac{bp}{c}\Big)^j B_j = -\frac{(bp)^{m-1}}{m} B_m\Big(\frac{c}{bp}\Big). \quad (71)$$

One now uses [53] Chapter IV, Proposition 4.1, which gives

$$g^{n-1}\sum_{j=0}^{g-1} B_n\Big(\frac{x+j}{g}\Big) = B_n(x). \quad (72)$$

Together with (71) this gives (with $\beta = 1 - m = 1 - k(p-1)$)

$$\sum_{i=0}^{p-1} T(c_0 + ib) = -\frac{(bp)^{m-1}}{m}\sum_{i=0}^{p-1} B_m\Big(\frac{c_0 + ib}{bp}\Big) = -\frac{b^{m-1}}{m} B_m\Big(\frac{c_0}{b}\Big).$$

Next one has

$$\begin{aligned} Z\Big(\frac{a}{b},\beta\Big) &= \sum_{1\le c<bp,\ c\notin p\mathbb{N}} T(c)\rho(\zeta_{a/b}^c) \\ &= \sum_{1\le c\le bp} T(c)\rho(\zeta_{a/b}^c) - \sum_{c=jp,\ 1\le j\le b} T(c)\rho(\zeta_{a/b}^c) \end{aligned}$$

and

$$\sum_{1\le c\le bp} T(c)\rho(\zeta_{a/b}^c) = -\frac{b^{m-1}}{m}\sum_{1\le c\le b}\rho(\zeta_{a/b}^c) B_m\Big(\frac{c}{b}\Big)$$

while

$$\sum_{c=jp,\ 1\le j\le b} T(c)\rho(\zeta_{a/b}^c) = -\frac{(bp)^{m-1}}{m}\sum_{1\le j\le b}\rho(\zeta_{a/b}^{pj}) B_m\Big(\frac{j}{b}\Big).$$

Thus one gets (with $\beta = 1 - m = 1 - k(p-1)$)

$$Z\Big(\frac{a}{b},\beta\Big) = -\frac{b^{m-1}}{m}(1 - p^{-\beta}\mathrm{Fr})\sum_{1\le c\le b}\rho(\zeta_{a/b}^c) B_m\Big(\frac{c}{b}\Big). \quad (73)$$

We thus obtain:

Theorem 3.10. *For β a negative odd integer of the form $\beta = 1 - m = 1 - k(p-1)$, one has*

$$(1 - p^{-\beta}\mathrm{Fr})^{-1} Z\Big(\frac{a}{b},\beta\Big) = -\frac{b^{m-1}}{m}\sum_{1\le c\le b}\rho(\zeta_{a/b}^c) B_m\Big(\frac{c}{b}\Big) \in \mathbb{Q}^{\mathrm{cyc}} \quad (74)$$

which is formally independent of the prime p.

We now fix $\beta = 1 - m = 1 - k(p-1)$ and investigate the dependence upon the choice of ρ, i.e., of $\sigma \in X_p$. Thus we take ρ and ρ' and assume that $Z_\rho(\frac{a}{b}, \beta) = Z_{\rho'}(\frac{a}{b}, \beta)$ holds for all $a/b \in \mu^{(p)}$, i.e., all fractions with denominator b prime to p. By (74) we then have

$$\sum_{1 \le c \le b} \rho(\zeta_{a/b}^c) B_m\Big(\frac{c}{b}\Big) = \sum_{1 \le c \le b} \rho'(\zeta_{a/b}^c) B_m\Big(\frac{c}{b}\Big), \quad \forall a/b \in \mu^{(p)}. \tag{75}$$

Since both ρ and ρ' are isomorphisms with the group of roots of unity in $\mathbb{C}_p$ of order prime to p, there exists an automorphism $\theta \in \mathrm{Aut}\,(\mu^{(p)})$ such that $\rho' = \rho \circ \theta$. One has

$$\begin{aligned}\sum_{1 \le c \le b} \rho'(\zeta_{a/b}^c) B_m\Big(\frac{c}{b}\Big) &= \sum_{1 \le c \le b} \rho(\zeta_{\theta(c/b)}^a) B_m\Big(\frac{c}{b}\Big) \\ &= \sum_{1 \le c \le b} \rho(\zeta_{c/b}^a) B_m\Big(\theta^{-1}\Big(\frac{c}{b}\Big)\Big).\end{aligned}$$

The uniqueness of the Fourier transform for the finite group $\mathbb{Z}/b\mathbb{Z}$ then shows that (75) implies the equality

$$B_m\Big(\theta^{-1}\Big(\frac{c}{b}\Big)\Big) = B_m\Big(\frac{c}{b}\Big), \quad \forall c/b \in \mu^{(p)}. \tag{76}$$

Lemma 3.11. *Let $\theta \in \mathrm{Aut}\,(\mu^{(p)})$. Then if $\theta \in \{\pm 1\}$ one has*

$$Z_\rho\Big(\frac{a}{b}, \beta\Big) = Z_{\rho\circ\theta}\Big(\frac{a}{b}, \beta\Big), \quad \forall a/b \in \mu^{(p)},\ \beta \in D. \tag{77}$$

If $\theta \notin \{\pm 1\}$ and $\beta = 1 - m = 1 - k(p-1)$, the functionals $Z_\rho(\cdot, \beta)$ and $Z_{\rho\circ\theta}(\cdot, \beta)$ are distinct.

Proof. To prove (77) we can assume that $\theta = -1$, i.e., that $\theta(\zeta_{a/b}) = \zeta_{a/b}^{-1}$ for all $a/b \in \mu^{(p)}$. We then have, with $\rho' = \rho \circ \theta$,

$$\rho'(\zeta_{a/b}^c) = \rho(\zeta_{a/b}^{b-c})$$

Let first $\beta = 1 - m = 1 - k(p-1)$. One has

$$\sum_{1 \le c \le b} \rho'(\zeta_{a/b}^c) B_m\Big(\frac{c}{b}\Big) = \sum_{0 \le c \le b-1} \rho(\zeta_{a/b}^c) B_m\Big(\frac{b-c}{b}\Big).$$

Since $m = k(p-1)$ is even, the Bernoulli polynomial B_m fulfills

$$B_m(1-x) = B_m(x). \tag{78}$$

Thus the equality (77) follows for all values $\beta = 1 - m = 1 - k(p-1)$. Since these values admit 0 as an accumulation point this implies the equality of the analytic functions on their domain D.

Now assume that $\theta \notin \{\pm p^{\mathbb{Z}}\}$. Then by the proof of Lemma 3.10, the graph of θ is dense in $\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}$. Thus an equality (76) implies that $B_m(x)$ is constant which is a contradiction. It remains to show that for non-zero powers $q = p^f$ of p one cannot have an equality of the form

$$B_m(x) = B_m(qx - [qx]), \quad \forall x \in [0,1],$$

where $[qx]$ is the integral part of qx. But this would imply that $B_m(x) - B_m(qx)$ has infinitely many zeros. Thus $B_m(x) = B_m(qx)$ which is a contradiction. $\square$

In fact we can now improve Theorem 3.10 and for that we first note that at the archimedean place the extremal KMS states below critical temperature are given by (22),

$$\varphi_{\beta,\rho}(e(a/b)) = \frac{1}{\zeta(\beta)} \sum_{n=1}^{\infty} n^{-\beta} \rho(\zeta_{a/b}^n).$$

The formula

$$\ell_\beta(z) = \sum_{n=1}^{\infty} n^{-\beta} z^n \tag{79}$$

defines the multiple logarithm and fulfills

$$z\partial_z \ell_\beta(z) = \ell_{\beta-1}(z). \tag{80}$$

For $\beta = 0$ the sum gives the rational fraction

$$\ell_0(z) = \frac{z}{1-z} \tag{81}$$

and this shows that when $\beta \in -\mathbb{N}$ is a negative integer $\ell_\beta(z)$ is a rational fraction. Thus it makes sense over any field. We now show:

Theorem 3.12. *For β a negative odd integer of the form $\beta = 1 - m = 1 - k(p-1)$, one has*

$$(1 - p^{-\beta}\mathrm{Fr})^{-1} Z\Big(\frac{a}{b}, \beta\Big) = \ell_\beta(\rho(\zeta_{a/b})) \in \mathbb{Q}^{\mathrm{cyc}} \tag{82}$$

which is formally independent of the prime p.

Proof. One lets

$$F(u,t) = \frac{te^{ut}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(u) \frac{t^n}{n!}.$$

Let first $z \neq 1$ be a complex number. Then for any $b \in \mathbb{N}$

$$\sum_{j=0}^{b-1} z^j e^{\frac{j}{b}t} = \frac{z^b e^t - 1}{ze^{\frac{t}{b}} - 1}$$

so that, when $z^b = 1$,

$$\sum_{n=0}^{\infty}\Big(\sum_{j=0}^{b-1} z^j B_n\Big(\frac{j}{b}\Big)\Big)\frac{t^n}{n!} = \sum_{j=0}^{b-1} z^j F\Big(\frac{j}{b}, t\Big) = \frac{t}{ze^{\frac{t}{b}} - 1}$$

which gives the equality

$$\sum_{j=0}^{b-1} z^j B_n\Big(\frac{j}{b}\Big) = -\frac{n}{b^{n-1}}\ell_{1-n}(z), \quad \forall n > 1. \tag{83}$$

This identity holds in $\mathbb{Q}^{\mathrm{cyc}}$ and combined with (74) gives (82). □

Since when $\beta \in -\mathbb{N}$ is a negative integer $\ell_\beta(z)$ is a rational fraction, one can prove identities for arbitrary $\beta \in D$ by checking them on these special values.

Lemma 3.13. *For any $m \in \mathbb{N}$ and ζ_m a primitive m-th root of unity, one has*

$$\frac{1}{m}\sum_{j=0}^{m-1} \ell_\beta(\zeta_m^j x) = m^{-\beta}\ell_\beta(x^m). \tag{84}$$

Proof. This follows from (79). One can check it directly for $\beta = 0$ and it is stable under $z\partial_z$ so that it holds algebraically for all $\beta \in -\mathbb{N}$. □

Using (84) one can show the analogue of the KMS_β condition for the $\mathbb{C}_p$-valued functionals on the integral BC-system $\mathscr{H}_{\mathbb{Z}}$ such that, in particular,

$$\varphi_{\beta,\rho}(e(a/b)) = \frac{1}{Z(\beta)} Z_\rho\Big(\frac{a}{b}, \beta\Big). \tag{85}$$

We refer the reader to [15] where the theory is fully developed using Iwasawa's construction of p-adic L-functions. In particular it is shown in [15] that the natural group which serves as the parameter space for the modular automorphisms σ_β is the open unit disk $M = D(1, 1^-)$ in $\mathbb{C}_p$ with radius 1, viewed as a multiplicative group. It is a covering of $\mathbb{C}_p$ using the following group homomorphism

$$M = D(1, 1^-) \ni \lambda \longmapsto \beta = \ell(\lambda) = \frac{\log_p \lambda}{\log_p(1+q)} \in \mathbb{C}_p, \tag{86}$$

where $\log_p$ is the Iwasawa logarithm. The Iwasawa construction of p-adic L-functions allows one to extend the KMS theory to this parameter space [15].

4. The mysterious curve

C. Soulé has associated a zeta function to any sufficiently regular counting-type function $N(q)$, by considering the following limit

$$\zeta_N(s) := \lim_{q\to 1} Z(q, q^{-s})(q-1)^{N(1)}, \quad s \in \mathbb{R}. \tag{87}$$

Here, $Z(q, q^{-s})$ denotes the evaluation, at $T = q^{-s}$, of the Hasse–Weil zeta function

$$Z(q, T) = \exp\Big(\sum_{r\geq 1} N(q^r)\frac{T^r}{r}\Big). \tag{88}$$

For the formula (87) to make sense one requires that the counting function $N(q)$ is defined for all real numbers $q \geq 1$ and not only for prime integers powers as for the counting function in (88). For many simple examples of algebraic varieties, like the projective spaces, the function $N(q)$ is known to extend unambiguously to all real positive numbers. The associated zeta function $\zeta_N(s)$ is easy to compute and it produces the expected outcome. For a projective line, for example, one finds $\zeta_N(s) = \frac{1}{s(s-1)}$. Another case which is easy to handle and also carries a richer geometric structure is provided by a Chevalley group scheme. The study of these varieties has shown that these schemes are endowed with a basic (combinatorial) skeleton structure and, in [10], we proved that they are varieties over $\mathbb{F}_1$, as defined by Soulé in [47].

4.1. *The counting function of* $C = \overline{\operatorname{Spec}\mathbb{Z}}$

To proceed further, it is natural to wonder on the existence of a suitably defined curve $C = \overline{\operatorname{Spec}\mathbb{Z}}$ over $\mathbb{F}_1$, whose zeta function $\zeta_C(s)$ agrees with the *complete* Riemann zeta function $\zeta_{\mathbb{Q}}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ (*cf.* [33] and [38]). To bypass the difficulty inherent to the definition (87), when $N(1) = -\infty$, one works with the logarithmic derivative

$$\frac{\partial_s \zeta_N(s)}{\zeta_N(s)} = -\lim_{q\to 1} F(q, s), \tag{89}$$

where

$$F(q, s) = -\partial_s \sum_{r\geq 1} N(q^r)\frac{q^{-rs}}{r}. \tag{90}$$

Then one finds (*cf.* [11] Lemma 2.1), under suitable regularity conditions on $N(u)$, that for $\Re e(s)$ large enough, one has

$$\lim_{q\to 1} F(q, s) = \int_1^\infty N(u)u^{-s}\, d^*u, \quad d^*u = du/u \tag{91}$$

and

$$\frac{\partial_s \zeta_N(s)}{\zeta_N(s)} = -\int_1^\infty N(u) u^{-s} \, d^* u. \tag{92}$$

Using this integral equation (92) one obtains a precise description of the counting function $N_C(q) = N(q)$ associated to C. One gets

$$\frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = -\int_1^\infty N(u) u^{-s} \, d^* u. \tag{93}$$

and one uses the Euler product for $\zeta_{\mathbb{Q}}(s)$. When $\Re e(s) > 1$, one derives

$$-\frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s} + \int_1^\infty \kappa(u) u^{-s} \, d^* u, \tag{94}$$

where $\Lambda(n)$ is the von Mangoldt function taking the value $\log p$ at prime powers p^ℓ and zero otherwise. Also $\kappa(u)$ is the distribution defined, for any test function f, as

$$\begin{aligned} &\int_1^\infty \kappa(u) f(u) \, d^* u = \int_1^\infty \frac{u^2 f(u) - f(1)}{u^2 - 1} \, d^* u + c f(1), \\ &c = \frac{1}{2}(\log \pi + \gamma), \end{aligned} \tag{95}$$

where $\gamma = -\Gamma'(1)$ is the Euler constant. The distribution $\kappa(u)$ is positive on $(1, \infty)$ where, by construction, it is given by $\kappa(u) = \frac{u^2}{u^2-1}$. Hence, we deduce that the counting function $N(q)$ of the hypothetical curve C over $\mathbb{F}_1$, is the *distribution* defined by the sum of $\kappa(q)$ and a discrete term given by the derivative taken in the sense of distributions, of the function

$$\varphi(u) = \sum_{n<u} n \Lambda(n). \tag{96}$$

One derives the following formula for $N(u)$

$$N(u) = \frac{d}{du} \varphi(u) + \kappa(u). \tag{97}$$

One can then use the explicit formulas to express $\varphi(u)$ in terms of the set Z of non-trivial zeros of the Riemann zeta function. One has the formula (*cf.* [27], Chapter IV, Theorems 28 and 29) valid for $u > 1$ (and not a prime power)

$$\varphi(u) = \frac{u^2}{2} - \sum_{\rho \in Z} \operatorname{order}(\rho) \frac{u^{\rho+1}}{\rho + 1} + a(u), \tag{98}$$

where the sum over Z in (98) has to be taken in a symmetric manner to ensure the convergence, and

$$a(u) = \frac{1}{2}\log\Big(\frac{u+1}{u-1}\Big) - \frac{\zeta'(-1)}{\zeta(-1)}. \tag{99}$$

By taking the principal values into account, one obtains the following more precise result (for the proof we refer to [11], Theorem 2.2)

Theorem 4.1. *The tempered distribution $N(u)$ satisfying the equation*

$$-\frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = \int_1^\infty N(u)u^{-s}\, d^*u$$

is positive on $(1,\infty)$ and on $[1,\infty)$ is given by

$$N(u) = u - \frac{d}{du}\Big(\sum_{\rho\in Z} \operatorname{order}(\rho)\frac{u^{\rho+1}}{\rho+1}\Big) + 1, \tag{100}$$

where the derivative is taken in the sense of distributions, and the value at $u = 1$ of the term $\omega(u) = \sum_{\rho\in Z} \operatorname{order}(\rho)\frac{u^{\rho+1}}{\rho+1}$ is given by

$$\omega(1) = \sum_{\rho\in Z} \operatorname{order}(\rho)\frac{1}{\rho+1} = \frac{1}{2} + \frac{\gamma}{2} + \frac{\log 4\pi}{2} - \frac{\zeta'(-1)}{\zeta(-1)}. \tag{101}$$

This result supplies a strong indication on the coherence of the quest for an arithmetic theory over $\mathbb{F}_1$. For an irreducible, smooth and projective algebraic curve X over a prime field $\mathbb{F}_p$, the counting function is of the form

$$\#X(\mathbb{F}_q) = N(q) = q - \sum_\alpha \alpha^r + 1, \quad q = p^r,$$

where the numbers α's are the complex roots of the characteristic polynomial of the Frobenius endomorphism acting on the étale cohomology $H^1(X \otimes \bar{\mathbb{F}}_p, \mathbb{Q}_\ell)$ of the curve ($\ell \neq p$). By writing these roots in the form $\alpha = p^\rho$, for ρ a zero of the Hasse–Weil zeta function of X, the above equality reads as

$$\#X(\mathbb{F}_q) = N(q) = q - \sum_\rho \operatorname{order}(\rho) q^\rho + 1. \tag{102}$$

The equations (100) and (102) are now completely identical, except for the fact that in (102) the values of q are restricted to the discrete set of powers of p and that (102) involves only a finite sum, which allows one to differentiate term by term.

4.2. Explicit formulas and the adèle class space

Equation (98) is a typical application of the Riemann–Weil explicit formulas. These formulas become natural when lifted to the idèle class group. The counterpart of the hypothetical curve C through the application of the class-field theory isomorphism, can be realized by a space of adelic nature the adèle class space $\mathbb{H}_{\mathbb{Q}}$ in agreement with the earlier constructions in [16], [18] and [17].

We start by considering the explicit formulas in the following form.

$$\hat{h}(0) + \hat{h}(1) - \sum_{\rho\in Z} \text{order}\,(\rho)\hat{h}(\rho)$$
$$= \sum_p \sum_{m=1}^{\infty} \log p h(p^m) + \Big(\frac{\gamma}{2} + \frac{\log\pi}{2}\Big)h(1) + \int_1^{\infty} \frac{t^2 h(t) - h(1)}{t^2-1}\, d^*t. \tag{103}$$

Here $h(u)$ is a function defined on $[1,\infty)$ and such that $h(u) = O(u^{-1-\epsilon})$, and one sets

$$\hat{h}(s) = \int_1^{\infty} h(u)u^s\, d^*u. \tag{104}$$

We apply this formula with the function h_x determined by the conditions

$$h_x(u) = u \text{ for } u \in [1,x], \quad h_x(u) = 0 \text{ for } u > x. \tag{105}$$

Then, we obtain

$$\hat{h}_x(s) = \int_1^{\infty} h_x(u)u^{s-1}\, du = \int_1^{x} uu^{s-1}\, du = \frac{x^{1+s}}{1+s} - \frac{1}{1+s}. \tag{106}$$

Thus, it follows that

$$\hat{h}_x(1) = \frac{x^2}{2} - \frac{1}{2}, \quad \hat{h}_x(0) = x-1, \quad \hat{h}_x(\rho) = \frac{x^{1+\rho}}{1+\rho} - \frac{1}{1+\rho}. \tag{107}$$

The left-hand side of the explicit formula (103) gives, up to a constant

$$J(x) = \frac{x^2}{2} + x - \sum_{\rho\in Z} \text{order}\,(\rho)\frac{x^{1+\rho}}{1+\rho}. \tag{108}$$

The first term on the right-hand side of (103) gives

$$\varphi(x) = \sum_{n<x} n\Lambda(n) \tag{109}$$

while the integral on the right-hand side of (103) gives

$$\int_1^{\infty} \frac{t^2 h_x(t) - h_x(1)}{t(t^2-1)}\, dt = x - \frac{1}{2}\log\Big(\frac{x+1}{x-1}\Big) + \text{constant}. \tag{110}$$

Thus the explicit formula (103) is transformed into the equality

$$\sum_{n<x} n\Lambda(n) = \frac{x^2}{2} - \sum_{\rho\in Z} \operatorname{order}(\rho)\frac{x^{1+\rho}}{1+\rho} + \frac{1}{2}\log\Big(\frac{x+1}{x-1}\Big) + \text{constant.} \quad (111)$$

This formula is the same as (98). We refer to [27] for a precise justification of the analytic steps. It follows that the left-hand side (108) of the explicit formula gives a natural primitive $J(x)$ of the counting function $N(x)$. It is thus natural to differentiate formally the family of functions h_x with respect to x and see what the right-hand side of the explicit formula is transformed into. By construction, one has, for $u \geq 1$

$$h_x(u) = uY(u-x),$$

where Y is the characteristic function of the interval $(-\infty, 0]$. The derivative of $Y(s)$ is $-\delta(s)$. Thus, at the formal level, one derives

$$\partial_x h_x = u\delta(u-x).$$

The function $g_x(u)$ corresponding to $\partial_x h_x$ is thus $g_x(u) = u\delta_x(u)$ and it is characterized, as a distribution, by its evaluation on test functions $b(x)$. This gives

$$\int b(u)g_x(u)\, d^*u = b(x). \quad (112)$$

We now show how to implement the trace formula interpretation of the explicit formulas to describe the counting function $N(u)$ as an intersection number. First the above explicit formula is a special case of the Weil explicit formulas which we briefly describe in the general context of global fields. One lets $\mathbb{K}$ be a global field, α a non-trivial character of $\mathbb{A}_{\mathbb{K}}/\mathbb{K}$ and $\alpha = \prod \alpha_v$ its local factors. Let $h \in \mathscr{S}(C_{\mathbb{K}})$ have compact support. Then the Weil explicit formula is

$$\hat{h}(0) + \hat{h}(1) - \sum_{\chi\in\widehat{C_{\mathbb{K},1}}} \sum_{Z_{\tilde{\chi}}} \hat{h}(\tilde{\chi},\rho) = \sum_v \int'_{\mathbb{K}_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u, \quad (113)$$

where $\int'$ is normalized by α_v and $\hat{h}(\chi, z) = \int h(u)\chi(u)|u|^z\, d^*u$. This formula becomes a trace formula whose geometric side is of the form

$$\operatorname{Tr}_{\text{distr}}\Big(\int h(u)\vartheta(u)\, d^*u\Big) = \sum_v \int'_{\mathbb{K}_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u. \quad (114)$$

Here $\vartheta(u)\xi(x) = \xi(u^{-1}x)$ is the scaling action of the idèle class group $C_{\mathbb{K}}$ on the adèle class space $\mathbb{H}_{\mathbb{K}} = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$. We refer to [8], [40] and [19] for the detailed treatment. The subgroups $\mathbb{K}_v^* \subset C_{\mathbb{K}}$ appear as isotropy groups. One can understand why the terms $\frac{h(u^{-1})}{|1-u|}$ occur in the trace formula by computing formally as follows the trace of the scaling operator $T = \theta(u^{-1})$

$$T\xi(x) = \xi(ux) = \int k(x,y)\xi(y)\,dy,$$

given by the distribution kernel $k(x,y) = \delta(ux - y)$,

$$\mathrm{Tr}_{\,\mathrm{distr}}(T) = \int k(x,x)\,dx = \int \delta(ux-x)\,dx = \frac{1}{|u-1|}\int \delta(z)\,dz = \frac{1}{|u-1|}.$$

We apply (114) by taking $\mathbb{K} = \mathbb{Q}$ and the function h of the form $h(u) = g(|u|)$ where the support of the function g is contained in $(1,\infty)$. On the left-hand side of (114) one first performs the integration in the kernel $C_{\mathbb{Q},1}$ of the module $C_{\mathbb{Q}} \to \mathbb{R}_+^*$. At the geometric level, this corresponds to taking the quotient of M by the action of $C_{\mathbb{Q},1}$. We denote by ϑ_u the scaling action on this quotient. By construction this action only depends upon $|u| \in \mathbb{R}_+^*$. The equality (112) means that when we consider the distributional trace of an expression of the form $\int g_x(u)\vartheta_u\,d^*u$ we are in fact just taking the distributional trace of ϑ_x since

$$\int g_x(u)\vartheta_u\,d^*u = \vartheta_x, \tag{115}$$

thus we are simply considering an intersection number. We now look at the right-hand side of (114), i.e., at the terms

$$\int'_{\mathbb{K}_v^*} \frac{h(u^{-1})}{|1-u|}\,d^*u. \tag{116}$$

Since $h(u) = g(|u|)$ and the support of the function g is contained in $(1,\infty)$, one sees that the integral (116) can be restricted in all cases to the unit ball $\{u;\ |u| < 1\}$ of the local field $\mathbb{K}_v$. In particular, for the finite places one has $|1-u| = 1$, thus for each finite prime $p \in \mathbb{Z}$ one has

$$\int'_{\mathbb{Q}_p^*} \frac{h(u^{-1})}{|1-u|}\,d^*u = \sum_{m=1}^{\infty} \log p\, g(p^m). \tag{117}$$

At the archimedean place one has instead

$$\frac{1}{2}\Big(\frac{1}{1-\frac{1}{u}} + \frac{1}{1+\frac{1}{u}}\Big) = \frac{u^2}{u^2-1}.$$

The above equation is applied for $u > 1$, in which case one can write equivalently

$$\frac{1}{2}\Big(\frac{1}{|1-u^{-1}|} + \frac{1}{|1+u^{-1}|}\Big) = \frac{u^2}{u^2-1}. \tag{118}$$

Thus, the term corresponding to (116) yields the distribution $\kappa(u)$ of (95).

4.3. Algebraic structure of the adèle class space

In [8], as well as in subsequent papers, the adèle class space $\mathbb{A}_{\mathbb{K}}/\mathbb{K}^\times$ of a global field $\mathbb{K}$ has been studied as a noncommutative space. Only very recently (*cf.* [13]), we have been successful to associate an algebraic structure to $\mathbb{A}_{\mathbb{K}}/\mathbb{K}^\times$ using which this space finally reveals its deeper nature of a *hyperring of functions*. The hyperring structure of $\mathbb{A}_{\mathbb{K}}/\mathbb{K}^\times$ has emerged gradually by combining the following two properties:

- $\mathbb{A}_{\mathbb{K}}/\mathbb{K}^\times$ is a commutative monoid, so that one may apply to this space the geometry of monoids of K. Kato [28] and A. Deitmar [21].

The natural monoid structure on the adèle class space, when combined with one of the simplest examples of monoidal schemes, i.e., the projective line $\mathbb{P}^1_{\mathbb{F}_1}$, provides a geometric framework to better understand conceptually the spectral realization of the zeros of L-functions of [8], [40] and [19]. It appears as the cohomology of a natural sheaf Ω of functions on the set of rational points of the monoidal scheme $\mathbb{P}^1_{\mathbb{F}_1}$ on the monoid $M = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^\times$ of adèle classes. The sheaf Ω is a sheaf of complex vector spaces over the geometric realization of the functor associated to the projective line. It is a striking fact that despite the apparent simplicity of the construction of $\mathbb{P}^1_{\mathbb{F}_1}$ the computation of $H^0(\mathbb{P}^1_{\mathbb{F}_1}, \Omega)$, already yields the graph of the Fourier transform. The cohomology $H^0(\mathbb{P}^1_{\mathbb{F}_1}, \Omega)$ is given, up to a finite dimensional space, by the graph of the Fourier transform acting on the co-invariants for the action of $\mathbb{K}^\times$ on the Bruhat–Schwartz space $\mathscr{S}(\mathbb{A}_{\mathbb{K}})$. Moreover, the spectrum of the natural action of the idèle class group $C_{\mathbb{K}}$ on the cohomology $H^1(\mathbb{P}^1_{\mathbb{F}_1}, \Omega)$ provides the spectral realization of the zeros of Hecke L-functions.

- $\mathbb{A}_{\mathbb{K}}/\mathbb{K}^\times$ is a hyperring over the Krasner hyperfield $\mathbf{K} = \{0, 1\}$.

In [13], we proved that the adèle class space possesses a rich additive structure which provides the correct arithmetic setup on this space. It is an interesting coincidence that the first summary on hyperring theory, due to M. Krasner (*cf.* [31]), was presented in the same conference and appeared in the same proceeding volume together with the seminal paper of J. Tits [50] where he introduced "le corps de caractéristique un". The distinction between the algebraic structure that Tits proposed as the degenerate case of $\mathbb{F}_q$ for $q = 1$, i.e., "le corps formé du seul élément $1 = 0$", and the Krasner hyperfield $\mathbf{K} = \{0, 1\}$ is simply that in $\mathbf{K}$ one keeps the distinction $1 \neq 0$, while recording the degeneracy by allowing the sum $1 + 1$ to be maximally ambiguous. Thus the algebraic rules of the hyperfield $\mathbf{K} = \{0, 1\}$ correspond to the property of being zero or non-zero in the same way as the algebraic rules of the field $\mathbb{F}_2$ correspond to the properties of being even or odd for relative integers. We have shown in [13], that the adèle class space $\mathbb{H}_{\mathbb{K}}$ over an arbitrary global field $\mathbb{K}$ is a hyperring extension of $\mathbf{K}$.

In general, as shown by Krasner, one obtains a hyperring by taking the quotient A/G of a ring A by a subgroup $G \subset A^{\times}$ of the group of units. We have shown that A/G is an extension of $\mathbf{K}$ if and only if $G \cup \{0\}$ is a *subfield* of A which is exactly what happens for the adèle class space $\mathbb{A}_{\mathbb{K}}/\mathbb{K}^{\times}$ and allows one to use the Poisson summation formula. Moreover, when $\mathbb{K}$ has positive characteristic, i.e., is a function field over a curve X, we construct in [13] a canonical identification of the groupoid of prime elements of the hyperring $\mathbb{H}_{\mathbb{K}}$ with the loop groupoid of the abelian cover of X associated to the maximal abelian extension of the function field $\mathbb{K}$. The nuance between the loop groupoid and the abelian cover amounts to the choice of a base point in the fiber over each place of $\mathbb{K}$ and this nuance is essential in algebraic geometry.

4.4. First hints towards the "curve" for the global field $\mathbb{Q}$

In the case of number fields, the groupoid of prime elements of the hyperring $\mathbb{H}_{\mathbb{K}}$ still makes sense and the issue is to construct, in characteristic zero, a geometric model similar to the curve and its abelian cover in the function field case. Let $\mathbb{K}$ be a function field, i.e., a global field of characteristic $p > 1$ and let $\mathbb{F}_q \subset \mathbb{K}$ be the field of constants. The abelian cover of the curve is obtained by the following steps (*cf.* [24]):

(1) One considers the maximal abelian extension $\mathbb{K}^{\text{ab}} \supset \mathbb{K}$ of $\mathbb{K}$.
(2) One considers inside $\mathbb{K}^{\text{ab}}$ the finite extensions $E \supset \bar{\mathbb{F}}_p \otimes_{\mathbb{F}_q} \mathbb{K}$ of $\bar{\mathbb{F}}_p \otimes_{\mathbb{F}_q} \mathbb{K}$.
(3) For each such extension E the space of (discrete) valuations is turned into a scheme with non-empty open sets given by complements of finite subsets and structure sheaf given by the intersection of the valuation rings inside E.

Let us now turn to the global field $\mathbb{K} = \mathbb{Q}$. One can first try to ignore step (2) and consider the maximal abelian extension of $\mathbb{Q}$, i.e., the cyclotomic field $\mathbb{Q}^{\text{cyc}}$ which we view as an abstract field obtained as the quotient of the group ring $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ by the ideal generated by the $e_n = \frac{1}{n}\sum_{n\gamma=0} e(\gamma)$ for $n > 1$. One can then consider directly, for each finite prime p, the space $\text{Val}_p(\mathbb{Q}^{\text{cyc}})$, of valuations on $\mathbb{Q}^{\text{cyc}}$ extending the p-adic valuation of $\mathbb{Q}$. This space $\text{Val}_p(\mathbb{Q}^{\text{cyc}})$ is canonically isomorphic to the quotient Σ_p of the space X_p (of (40)) of isomorphisms $\sigma : \bar{\mathbb{F}}_p^{\times} \to \mu^{(p)}$, by the action of the Galois group $\text{Gal}\,(\bar{\mathbb{F}}_p : \mathbb{F}_p)$ acting by composition on the right

$$\sigma \longmapsto \sigma \circ \alpha : \bar{\mathbb{F}}_p^{\times} \longrightarrow \mu^{(p)}, \quad \forall \alpha \in \text{Gal}\,(\bar{\mathbb{F}}_p : \mathbb{F}_p),\ \sigma \in X_p. \tag{119}$$

One can describe Σ_p concretely by looking at the corresponding addition on $\tilde{\mu}^{(p)} = \{0\} \cup \mu^{(p)}$ which is inherited from the given isomorphism with $\bar{\mathbb{F}}_p$. It suffices to specify the addition with 1 and this shows that Σ_p is the set of bijections s of $\tilde{\mu}^{(p)} = \{0\} \cup \mu^{(p)}$ which commute with all their conjugates under rotations by elements of $\mu^{(p)}$, and fulfill $s(0) = 1$, $s \circ s \circ \cdots \circ s = id$ with p factors.

But the comparison with the adèle class space shows that the space Σ_p is not what one wants as a fiber over p and one expects a finer space, which is the mapping torus Y_p of the action of the Frobenius on X_p. More concretely Y_p is the quotient

$$Y_p = (X_p \times (0,1))/\theta^{\mathbb{Z}} \tag{120}$$

of the product $X_p \times (0,1)$ of X_p by the open interval $(0,1) \subset \mathbb{R}$ by θ where

$$\theta(\sigma, \rho) = (\sigma \circ \mathrm{Fr}\,, \rho^p), \quad \forall \sigma \in X_p,\ \rho \in (0,1). \tag{121}$$

It follows from the isomorphism $\mathrm{Gal}\,(\bar{\mathbb{F}}_p : \mathbb{F}_p) \sim \hat{\mathbb{Z}}$, with the Frobenius Fr as topological generator, that the finer space Y_p is the total space of a principal bundle over Σ_p with structure group the connected compact solenoid

$$S_p = (\hat{\mathbb{Z}} \times \mathbb{R})/\mathbb{Z}, \tag{122}$$

where the subgroup $\mathbb{Z}$ is generated by the element $(1, \log p) \in \hat{\mathbb{Z}} \times \mathbb{R}$.

In [54], A. Weil showed how to construct the Weil group which compensates, at the Galois level, the absence of the connected component of identity in the idèle class group. We face here a similar problem at the level of the "curve". One should then perform the gluing of the fibers Y_p for different primes since as explained in the introduction, one needs to embed all these fibers in the same noncommutative space to account for the transversality factors in the explicit formulas.

In [17], following a proposal of Soulé for $\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z}$, we noted that

$$\mathbb{F}_{1^\infty} \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \tag{123}$$

is the abelian part of the BC-system. What matters is that, with the description given in [17] of the BC-system as an *endomotive* $\mathscr{E}$, one can consider its points over any ring A and this is just

$$\mathscr{E}(A) = \mathrm{Hom}\,(\mathbb{Z}[\mathbb{Q}/\mathbb{Z}], A). \tag{124}$$

Now, replacing $\sigma \in X_p$ by the associated homomorphism ρ of Theorem 3.6 one can describe equivalently the space X_p as

$$X_p = \mathrm{Hom}\,(\mathbb{Q}^{\mathrm{cyc},p}, \widehat{\mathbb{Q}_p^{\mathrm{ur}}}) = \mathrm{Hom}\,(\mathbb{Q}^{\mathrm{cyc},p}, \mathbb{C}_p). \tag{125}$$

One obtains in this way, for each p the canonical inclusions

$$X_p \subset \mathrm{Hom}\,(\mathbb{Z}[\mathbb{Q}/\mathbb{Z}], \widehat{\mathbb{Q}_p^{\mathrm{ur}}}) = \mathscr{E}(\widehat{\mathbb{Q}_p^{\mathrm{ur}}}) \subset \mathrm{Hom}\,(\mathbb{Z}[\mathbb{Q}/\mathbb{Z}], \mathbb{C}_p) = \mathscr{E}(\mathbb{C}_p). \tag{126}$$

By comparison with the adèle class space $\mathbb{H}_{\mathbb{Q}}$, one finds that the natural noncommutative space in which the above fiber Y_p embeds naturally is the quotient

$$(\mathscr{E}(\mathbb{C}_p) \times (0,\infty))/(\mathbb{Z} \setminus \{0\}), \tag{127}$$

where $\mathbb{Z} \setminus \{0\}$ is viewed as a multiplicative semi-group and the action of $\pm n \in \mathbb{Z} \setminus \{0\}$ is the product of the action of the n-th Frobenius of the endomotive by the map $x \mapsto x^n$ in $(0,\infty)$. One uses the sign ± 1 to act on the endomotive by the complex conjugation on roots of unity.

In order to match together these spaces for different primes, it then becomes necessary to uniformize the embedding of $\mathbb{Q}^{\mathrm{cyc}}$ in $\mathbb{C}_p$ for different primes. It is at this point that appears the need for a canonical construction of $\bar{\mathbb{F}}_p$ as developed in [35].

We were careful in [17] to distinguish $\mathbb{F}_{1^\infty}$ from the unknown $\bar{\mathbb{F}}_1$ and the extension

$$\bar{\mathbb{F}}_1 \otimes_{\mathbb{F}_1} \mathbb{Q} \supset \mathbb{F}_{1^\infty} \otimes_{\mathbb{F}_1} \mathbb{Q} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \tag{128}$$

remains elusive. It could well involve the "dual system" of the BC-system. We shall explain in the final section of this paper why the analogue of the Witt construction in characteristic 1 provides a possible interpretation (see (151)) of the role of the interval $(0,\infty)$ in (127) and of the corresponding action of $\mathbb{N}$.

5. Characteristic one and the Witt construction

Besides the above unveiling of the algebraic structure of the adèle class space, the quest for an analogue of algebraic geometry in "characteristic one" has strong relations with tropical geometry and idempotent analysis [29], [37].

We shall explain in this section the results of [9], which show that there is an analogue of the Witt construction $\mathbb{W}_{p^\infty}$ in characteristic one. When this construction is applied to the semi-field $\mathbb{R}_+^{\max}$ of idempotent analysis [29], [37], as defined by (139) below, it gives in that case the inverse operation of the "dequantization".

5.1. $\mathbb{W}_{p^\infty}$ *revisited*

Our starting point is a formula which goes back to Teichmüller and which gives an explicit expression for the sum of the multiplicative lifts in the context of strict p-rings. A ring R is a strict p-ring when R is complete and Hausdorff with respect to the p-adic topology, p is not a zero-divisor in R, and the residue ring $K = R/pR$ is perfect. The ring R is uniquely determined by K up to canonical isomorphism and there exists a unique multiplicative section $\tau : K \to R$ of the residue morphism $\epsilon : R \to K = R/pR$

$$\tau : K \longrightarrow R, \quad \epsilon \circ \tau = id, \quad \tau(xy) = \tau(x)\tau(y), \quad \forall x, y \in K. \tag{129}$$

Every element x of R can be written uniquely in the form

$$x = \sum \tau(x_n)p^n, \quad x_n \in K \tag{130}$$

which gives a canonical bijection $\tilde{\tau} : K[[T]] \to R$ such that

$$\tilde{\tau}\Big(\sum x_n T^n\Big) = \sum \tau(x_n)p^n. \tag{131}$$

Let then

$$I_p = \{\alpha \in \mathbb{Q} \cap [0,1],\ \exists n,\ p^n\alpha \in \mathbb{Z}\}. \tag{132}$$

The formula which goes back to Teichmüller [49] allows one to express the sum of two (or more) multiplicative lifts in the form

Theorem 5.1. *There exists a map $w_p : I_p \to \mathbb{F}_p[[T]]$ such that for all $x, y \in K$ one has*

$$\tau(x) + \tau(y) = \tilde{\tau}\Big(\sum_{\alpha \in I_p} w_p(\alpha, T)x^\alpha y^{1-\alpha}\Big). \tag{133}$$

In this equation the sum inside the parenthesis in the right-hand side takes place in $K[[T]]$, and, since K is perfect, the terms $x^\alpha y^{1-\alpha}$ make sense. Finally the terms

$$w_p(\alpha, T) \in \mathbb{F}_p[[T]], \quad \forall \alpha \in I_p \tag{134}$$

only depend on the prime p and tend to zero at infinity in $\mathbb{F}_p[[T]]$, for the discrete topology in I_p, so that the sum (133) is convergent. The formula (133) easily extends to express the sum of n multiplicative lifts as

$$\sum \tau(x_j) = \tilde{\tau}\Big(\sum_{\alpha_j \in I_p,\ \sum \alpha_j = 1} w_p(\alpha_1, \ldots, \alpha_n, T) \prod x_j^{\alpha_j}\Big). \tag{135}$$

As is well known the algebraic structure of R was functorially reconstructed from that of K by Witt who showed that the algebraic rules in R are polynomial in terms of the components X_n

$$x = \sum \tau(X_n^{p^{-n}})p^n, \quad X_n = x_n^{p^n} \tag{136}$$

which makes sense since K is perfect. One can in fact, as already noted by Teichmüller in [49], also reconstruct the full algebraic structure of R as a deformation of K depending upon the parameter T using the above formula (135) but the corresponding algebraic relations are not so simple to handle, mostly because the map $x \to x^\alpha$ from K to K is an automorphism only when α is a power of p. As we shall now explain, this difficulty disappears in the limit case of characteristic one.

5.2. Perfect semi-rings of characteristic 1

We refer to [22] for the general theory of semi-rings. In a semi-ring R the additive structure $(R, +)$ is no longer that of an abelian group but is a commutative monoid with neutral element 0. The multiplicative structure $(R, \cdot)$ is a monoid with identity $1 \neq 0$ and distributivity holds while $r \cdot 0 = 0 \cdot r = 0$ for all $r \in R$.

Definition 5.2. *A semi-ring R is said to have characteristic* 1 *when*

$$1 + 1 = 1 \tag{137}$$

in R.

A semi-ring R is called a *semi-field* when every non-zero element in R has a multiplicative inverse. We let $\mathbb{B} = \{0, 1\}$ be the only finite semi-field which is not a field. One has $1 + 1 = 1$ in $\mathbb{B}$ ([22], [36]). By definition, a semi-ring R is *multiplicatively cancellative* when

$$x \neq 0,\ xy = xz \Longrightarrow y = z. \tag{138}$$

We recall, from [22] Propositions 4.43 and 4.44 the following result which describes the analogue of the Frobenius endomorphism in characteristic p.

Proposition 5.3. *Let R be a multiplicatively cancellative semi-ring of characteristic one. Then the map $\vartheta_n(x) = x^n$ is an injective endomorphism of R for any $n \in \mathbb{N}$.*

We shall say that A is *perfect* when ϑ_n is surjective for all n. One then gets a one parameter group of automorphisms $\vartheta_\lambda \in \mathrm{Aut}\,(A)$, $\lambda \in \mathbb{Q}_+^*$ such that

- $\vartheta_n(x) = x^n$ for all $n \in \mathbb{N}$ and $x \in A$.
- $\vartheta_\lambda \circ \vartheta_\mu = \vartheta_{\lambda\mu}$ for $\lambda, \mu \in \mathbb{Q}_+^*$.
- $\vartheta_\lambda(x)\vartheta_\mu(x) = \vartheta_{\lambda+\mu}(x)$ for $\lambda, \mu \in \mathbb{Q}_+^*$ and $x \in A$.

We denote $\vartheta_\lambda(x) = x^\lambda$. Of course $\mathbb{B}$ is perfect, another important example is $\mathbb{R}_+^{\max}$, i.e., the set $[0, \infty)$ with ordinary multiplication and the new addition given by

$$x +' y := \max(x, y), \quad \forall x, y \in \mathbb{R}_+. \tag{139}$$

To obtain the analogue of the Witt construction for multiplicatively cancellative perfect semi-ring of characteristic one, one looks for functions $w(\alpha)$ defined for $\alpha \in I = \mathbb{Q} \cap [0, 1]$ which make the following operation commutative and associative

$$x +_w y = \sum_{\alpha \in I} w(\alpha) x^\alpha y^{1-\alpha}. \tag{140}$$

Besides the symmetry

$$w(1 - \alpha) = w(\alpha), \tag{141}$$

one obtains the functional equation

$$w(\alpha)w(\beta)^{\alpha} = w(\alpha\beta)w(\gamma)^{(1-\alpha\beta)}, \quad \gamma = \frac{\alpha(1-\beta)}{1-\alpha\beta}. \tag{142}$$

The general solution of this equation is given by

Proposition 5.4. *Let G be a uniquely divisible abelian group and $w : I \to G$, with $w(0) = w(1) = 1 \in G$. Then w fulfills* (141) *and* (142)*, if and only if there exists a homomorphism $\chi : \mathbb{Q}_+^\times \to G$ such that*

$$w(\alpha) = \chi(\alpha)^{\alpha}\chi(1-\alpha)^{1-\alpha}, \quad \forall \alpha \in (0,1) \cap \mathbb{Q}. \tag{143}$$

This homomorphism χ is determined by the $\chi(p) \in G$ for all primes p.

5.3. Entropy and the $\mathbb{W}(R, \rho)$

We let, as above, R be a multiplicatively cancellative perfect semi-ring of characteristic one. The uniquely divisible group $G = R^\times$ is a vector space over $\mathbb{Q}$ using the action $\theta_\alpha(x) = x^\alpha$ and is partially ordered by the relation $x + y = y$ noted $x \leq y$. We shall now make the stronger assumption that it is a partially ordered vector space over $\mathbb{R}$. Thus G is a partially ordered group endowed with a one parameter group of automorphisms $\theta_\lambda \in \mathrm{Aut}\,(G)$, $\lambda \in \mathbb{R}^\times$ such that, with $\theta_0(x) = 1$ for all x by convention,

$$\theta_{\lambda\lambda'} = \theta_\lambda \circ \theta_{\lambda'}, \quad \theta_\lambda(x)\theta_{\lambda'}(x) = \theta_{\lambda+\lambda'}(x). \tag{144}$$

We assume the following compatibility (closedness) of the partial order with the vector space structure

$$\lambda_n \longrightarrow \lambda, \quad \theta_{\lambda_n}(x) \geq y \Longrightarrow \theta_\lambda(x) \geq y. \tag{145}$$

Theorem 5.5. *Let $w : I \to G$ fulfill* (141) *and* (142) *and*

$$w(\alpha) \geq 1, \quad \forall \alpha \in I. \tag{146}$$

Then there exists $\rho \in G$, $\rho \geq 1$ such that

$$w(\alpha) = \rho^{S(\alpha)}, \quad S(\alpha) = -\alpha\log(\alpha) - (1-\alpha)\log(1-\alpha), \quad \forall \alpha \in I. \tag{147}$$

In the above framework of multiplicatively cancellative perfect semi-rings of characteristic we associate to $\rho \in A$, $\rho \geq 1$, a metric $d(x, y)$ such that

$$d(x, y) = \inf\{\alpha \mid x \leq y\rho^\alpha,\ y \leq x\rho^\alpha\}. \tag{148}$$

It is finite on the intervals $[\rho^{-n}, \rho^n]$. We let A_ρ be the union of $\{0\}$ with the separated completions of the intervals $[\rho^{-n}, \rho^n]$.

Theorem 5.6. *Let* $\rho \in A$, $\rho > 1$ *be invertible. Then the formula*

$$x +_\rho y := \sum_{\alpha \in [0,1] \cap \mathbb{Q}} \rho^{S(\alpha)} x^\alpha y^{1-\alpha}, \quad \forall x, y \in A_\rho$$

defines an associative commutative composition law on A_ρ *with* 0 *as neutral element. Multiplication is distributive and the Grothendieck group of the monoid* $(A_\rho, +_\rho)$ *is a normed algebra* $W(A, \rho)$ *over* $\mathbb{R}$ *depending functorially upon* (A, ρ).

It follows from Gelfand's theory that the characters of the complex Banach algebra $\overline{W}(R, \rho)_{\mathbb{C}}$ form a non-empty compact space

$$X = \text{Spec}\,(\overline{W}(R, \rho)_{\mathbb{C}}) \neq \emptyset. \tag{149}$$

5.4. The $w(\alpha, T)$ *and* $\mathbb{R}^{\text{un}}$

In the case of the Witt construction, the functoriality allows one to apply the functor $\mathbb{W}_{p^\infty}$ to an algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_p$ which yields the following diagram

$$\begin{array}{ccc} \bar{\mathbb{F}}_p & \overset{\mathbb{W}_{p^\infty}}{\longmapsto} & \mathbb{W}_{p^\infty}(\bar{\mathbb{F}}_p) = \mathcal{O}_{\widehat{\mathbb{Q}_p^{\text{ur}}}} \\ \cup & & \cup \\ \mathbb{F}_p & \overset{\mathbb{W}_{p^\infty}}{\longmapsto} & \mathbb{Z}_p = \mathbb{W}_{p^\infty}(\mathbb{F}_p). \end{array} \tag{150}$$

In our case, the analogue of the extension $\mathbb{F}_p \subset \bar{\mathbb{F}}_p$ is the extension of semi-rings $\mathbb{B} \subset \mathbb{R}_+^{\max}$ and the one-parameter group of automorphisms $\theta_\lambda \in \text{Aut}\,(\mathbb{R}_+^{\max})$, $\theta_\lambda(x) = x^\lambda$, plays the role of the Frobenius. But since our construction of $W(\mathbb{R}_+^{\max}, \rho)$ depends upon the choice of ρ, one first needs to eliminate the choice of ρ by considering simultaneously all possible choices. To do this we introduce a parameter $T \geq 0$,

$$\rho = e^T \in \mathbb{R}_+^{\max}, \quad \rho \geq 1. \tag{151}$$

With this notation, $w(\alpha)$ depends on T as the $w_p(\alpha, T)$ of (134) of the Witt case, one has explicitly

$$w(\alpha, T) = \alpha^{-T\alpha}(1-\alpha)^{-T(1-\alpha)}. \tag{152}$$

The presence in $w(\alpha, T)$ of the parameter $T \geq 0$ implies that even if one adds terms which are independent of T the result will depend on T. Thus one works with functions $f(T) \in \mathbb{R}_+^{\max}$ with the usual pointwise product and the new addition

$$(f_1 +_w f_2)(T) = \sum_{\alpha \in \bar{I}} w(\alpha, T) f_1(T)^\alpha f_2(T)^{1-\alpha}. \tag{153}$$

Proposition 5.7. *The addition* (153) *is given by*

$$(f_1 +_w f_2)(T) = (f_1(T)^{1/T} + f_2(T)^{1/T})^T \tag{154}$$

for $T > 0$ *and for* $T = 0$ *by*

$$(f_1 +_w f_2)(0) = \sup(f_1(0), f_2(0)). \tag{155}$$

Thus, the sum of n terms x_j independent of T is given by

$$x_1 +_w \cdots +_w x_n = \left(\sum x_j^{1/T}\right)^T. \tag{156}$$

In particular one can compute the sum of n terms all equal to 1 which will necessarily be fixed under any automorphism of the obtained structure. One gets

$$1 +_w 1 +_w \cdots +_w 1 = n^T. \tag{157}$$

This suggests more generally that the functions of the form

$$f(T) = x^T, \quad \forall T \geq 0$$

will be fixed by the lift of the $\theta_\lambda \in \mathrm{Aut}\,(\mathbb{R}_+^{\max})$. We now review the analogy with the Witt case. The constant functions $T \mapsto x$ are the analogue of the Teichmüller representatives

$$\tau(x)(T) = x, \quad \forall T. \tag{158}$$

One has

$$\tau(x) + \tau(y) = \sum_{\alpha \in \bar{I}} w(\alpha, T) x^\alpha y^{1-\alpha}, \tag{159}$$

where the sum in the right-hand side is computed in $\mathbb{R}_+^{\max}$. The evaluation at $T = 0$ is by construction a morphism

$$\epsilon : f \longmapsto f(0) \in \mathbb{R}_+^{\max}. \tag{160}$$

We view this morphism as the analogue of the canonical map which exists for any strict p-ring

$$\epsilon_p : \mathbb{W}_{p^\infty}(K) \longrightarrow K = \mathbb{W}_{p^\infty}(K)/p\mathbb{W}_{p^\infty}(K). \tag{161}$$

One has a natural one parameter group of automorphisms α_λ of our structure, which corresponds to the $\theta_\lambda \in \mathrm{Aut}\,(\mathbb{R}_+^{\max})$. It is given by

Proposition 5.8. *The following defines a one parameter group of automorphisms*

$$\alpha_\lambda(f)(T) = f(T/\lambda)^\lambda, \quad \forall \lambda \in \mathbb{R}_+^\times. \tag{162}$$

One has

$$\epsilon \circ \alpha_\lambda = \theta_\lambda \circ \epsilon, \quad \alpha_\lambda \circ \tau = \tau \circ \theta_\lambda, \quad \forall \lambda. \tag{163}$$

The fixed points of α_λ *are of the form*

$$f(T) = a^T \tag{164}$$

and they form the semi-field $\mathbb{R}_+$ *which is the positive part of the field* $\mathbb{R}$ *of real numbers, endowed with the ordinary addition and multiplication.*

For each $T > 0$ Proposition 5.7 shows that the algebraic operations on the value $f(T)$ are the same as in the semi-field $\mathbb{R}_+$ using the character χ_T which is such that

$$\chi_T(f) = f(T)^{1/T}. \tag{165}$$

We can thus use the characters χ_T to represent the elements of the extension $\mathbb{R}^{\mathrm{un}}$ as functions of T with the ordinary operations of pointwise sum and product.

Proposition 5.9. *The following map* χ *is a homomorphism of semi-rings to the algebra of functions from* $(0, \infty)$ *to* $\mathbb{R}_+$ *with pointwise sum and product,*

$$\chi(f)(T) = f(T)^{1/T}, \quad \forall T > 0. \tag{166}$$

One has

$$\chi(\tau(x))(T) = x^{1/T}, \quad \forall T > 0 \tag{167}$$

and

$$\chi(\alpha_\lambda(f))(T) = \chi(f)(T/\lambda), \quad \forall T > 0. \tag{168}$$

These properties are straightforward consequences of (165).

In this representation χ, the residue morphism is given, under suitable continuity assumptions by

$$\epsilon(f) = \lim_{T \to 0} \chi(f)(T)^T. \tag{169}$$

The algebraic operations are very simple and this suggests to represent elements of $\mathbb{R}^{\mathrm{un}}$ as functions $\chi(f)(T)$. Among them one should have the fixed points (164) which give $\chi(f) = a$ and the Teichmüller lifts which give (167). We parameterize the latter in the form

$$e_\xi(T) = e^{-\xi/T}, \quad \forall T > 0. \tag{170}$$

Table 1.

Characteristic p	*Characteristic* 1
$\mathbb{F}_p$	$\mathbb{B} = \mathbf{S}_+$
$\bar{\mathbb{F}}_p$	$\mathbb{R}_+^{\max} = \mathscr{T}\mathbb{R}_+$
$\mathbb{W}_{p^\infty}(A)$	$W(R, \rho)$
$\tau(x) + \tau(y)$ $= \tilde{\tau}(\sum_{\alpha \in I_p} w_p(\alpha) x^\alpha y^{1-\alpha})$	$x +_w y$ $= \sum_{\alpha \in \bar{I}} w(\alpha) x^\alpha y^{1-\alpha}$
Teichmüller lift $x \longmapsto \tau(x)$	$\chi(\tau(x))(T) = x^{1/T}, \quad \forall T > 0$
$\epsilon_p : \mathbb{W}_{p^\infty}(\bar{\mathbb{F}}_p) \longrightarrow \bar{\mathbb{F}}_p$	$\epsilon(f) = \lim_{T \to 0} \chi(f)(T)^T$
Frobenius automorphism	$\chi(\alpha_\lambda(f))(T) = \chi(f)(T/\lambda)$
$\mathbb{Q}_p \subset \widehat{\mathbb{Q}_p^{\mathrm{ur}}}$	$\mathbb{R} \subset \mathbb{R}^{\mathrm{un}}$
Fixed points $= \mathbb{Q}_p$	Fixed points $= \mathbb{R}$

After symmetrization and passing to the field of quotients, the fixed points (164) and the Teichmüller lifts (158) generate the field of fractions of the form (in the χ representation)

$$\chi(f)(T) = \Big(\sum a_j e^{-\xi_j/T}\Big) \Big/ \Big(\sum b_j e^{-\eta_j/T}\Big), \tag{171}$$

where the coefficients a_j, b_j are real numbers and the exponents $\xi_j, \eta_j \in \mathbb{R}$.

It is quite remarkable that, independently of our work [13] on hyperfields, O. Viro introduced in [52] the hyperfield $\mathscr{T}\mathbb{R}$ of tropical reals for the needs of tropical geometry. The hyperfield $\mathscr{T}\mathbb{R}$ is $\mathbb{R}$ with ordinary multiplication and the following hyperaddition of real numbers:

$$a \smile b = \begin{cases} a, & \text{if } |a| > |b| \text{ or } a = b; \\ b, & \text{if } |a| < |b| \text{ or } a = b; \\ [-a, a], & \text{if } b = -a. \end{cases} \tag{172}$$

It is always useful, when at all possible, to present a hyperfield as the quotient of an ordinary field by a subgroup of its multiplicative group. Thus it comes as a pleasant fact that the hyperfield $\mathscr{T}\mathbb{R}$ of tropical reals is indeed the following quotient

Theorem 5.10. *Let E be the field of rational fractions of the form*

$$h(T) = \Big(\sum a_j e^{-\xi_j/T}\Big) \Big/ \Big(\sum b_j e^{-\eta_j/T}\Big), \tag{173}$$

where all a_j, b_j, ξ_j, η_j are real numbers. Then

$$G = \{h \in E \mid \exists a > 0,\ h(T) \longrightarrow a \textit{ for } T \longrightarrow 0\} \tag{174}$$

is a multiplicative subgroup of E and the quotient E/G is canonically isomorphic to the hyperfield $\mathscr{T}\mathbb{R}$.

The corresponding homomorphism $\tilde{\epsilon} : E \to \mathscr{T}\mathbb{R}$ is given by $\tilde{\epsilon}(0) = 0$ and for a reduced fraction, i.e., no repetition in the ξ_j or η_j, of the form (173), by

$$\tilde{\epsilon}(f) = \operatorname{sign}\left(\frac{a_{j_0}}{b_{k_0}}\right) e^{-\xi_{j_0}+\eta_{k_0}}, \quad \xi_{j_0} = \inf_j(\xi_j), \quad \eta_{k_0} = \inf_k(\eta_k). \tag{175}$$

While the field E gives a first hint towards $\mathbb{R}^{\mathrm{un}}$ one should not be satisfied yet since natural examples coming from quantum physics use expressions of the same type but involving more elaborate sums. In all these examples, including those coming from the functional integral, it turns out that not only $\lim_{T\to 0} \chi(f)(T)^T$ exists as in (169) but in fact the function $f(T) = \chi(f)(T)^T$ admits an asymptotic expansion for $T \to 0$ of the form

$$f(T) = \chi(f)(T)^T \sim \sum a_n T^n. \tag{176}$$

For functions of the form (171), this expansion only uses the terms with the lowest values of ξ_j and η_j and is thus only a crude information on the element f. But as soon as one uses integrals instead of finite sums in (171) one obtains general asymptotic expansions (176). We refer to [9] for more details and for the relation between T and $\hbar$. This suggests more generally to use the theory of divergent series (*cf.* [43]) in the construction of $\mathbb{R}^{\mathrm{un}}$. The simple reason for "series" and not just numbers is that the action of $\mathbb{R}_+^*$ given by the α_λ of (162), gives a grading which admits the f_n, $\chi(f_n)(T) = T^n$ as eigenvectors. Thus we end these notes by urging the patient reader who followed us up to this point, to ponder about the very notion of number in the light of

- The power of asymptotic series (such as (59) which Euler used for $\sum n^{-2}$) which also make sense p-adically, while being ubiquitous in quantum field theory computations.
- The unknown extension (128) in relation with $\mathbb{R}^{\mathrm{un}}$.
- The operator formalism of the quantized calculus (*cf.* [7]) where asymptotic expansions of the form (59) play a crucial role through the pseudo differential calculus.

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