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Yang-Mills and some related algebras

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Dedicated to Jacques Bros

Summary. After a short introduction on the theory of homogeneous algebras we describe the application of this theory to the analysis of the cubic Yang-Mills algebra, the quadratic self-duality algebras, their "super" versions as well as to some generalization.

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1.1 Introduction

Consider the classical Yang-Mills equations in $(s+1)$-dimensional pseudo Euclidean space $\mathbb{R}^{s+1}$ with pseudo metric denoted by $g_{\mu\nu}$ in the canonical basis of $\mathbb{R}^{s+1}$ corresponding to coordinates x^λ. For the moment the signature plays no role so $g_{\mu\nu}$ is simply a real nondegenerate symmetric matrix with inverse denoted by $g^{\mu\nu}$. In terms of the covariant derivatives $\nabla_\mu = \partial_\mu + A_\mu$ $(\partial_\mu = \partial/\partial x^\mu)$ the Yang-Mills equations read

$$g^{\lambda\mu}[\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] = 0 \tag{1.1}$$

for $\nu \in \{0, \dots, s\}$. By forgetting the detailed origin of these equations, it is natural to consider the abstract unital associative algebra $\mathcal{A}$ generated by $(s+1)$ elements ∇_λ with the $(s+1)$ cubic relations (1.1). This algebra will be refered to as the Yang-Mills algebra. It is worth noticing here that Equations (1.1) only involve the product through commutators so that, by its very definition the Yang-Mills algebra $\mathcal{A}$ is a universal enveloping algebra.

Our aim here is to present the analysis of the Yang-Mills algebra and of some related algebras based on the recent development of the theory of homogeneous algebras [2], [4]. This analysis is only partly published in [10].

In the next section we recall some basic concepts and results on homogeneous algebras which will be used in this paper.

Section 3 is devoted to the Yang-Mills algebra. In this section we recall the definitions and the results of [10]. The proofs are omitted since these are in [10] and since very similar proofs are given in Sections 4 and 6. Instead, we describe the structure of the bimodule resolution of the Yang-Mills algebra and the structure of the corresponding small bicomplexes which compute the Hochschild homology.

In Section 4 we define the super Yang-Mills algebra and we prove for this algebra results which are the counterpart of the results of [10] for the Yang-Mills algebra.

In Section 5 we define and study the super self-duality algebra. In particular, we prove for this algebra the analog of the results of [10] for the self-duality algebra and we point out a very surprising connection between the super self-duality algebra and the algebras occurring in our analysis of noncommutative 3-spheres [9], [11].

In Section 6 we describe some deformations of the Yang-Mills algebra and of the super Yang-Mills algebra.

1.2 Homogeneous algebras

Although we shall be concerned in the following with the cubic Yang-Mills algebra $\mathcal{A}$, the quadratic self-duality algebra $\mathcal{A}^{(+)}$ [10] and some related algebras, we recall in this section some constructions and some results for general N-homogeneous algebras [4], [2]. All vector spaces are over a fixed commutative field $\mathbb{K}$.

A *homogeneous algebra of degree N or N-homogeneous algebra* is an algebra of the form

$$\mathcal{A} = A(E,R) = T(E)/(R)$$

where E is a finite-dimensional vector space, R is a linear subspace of $E^{\otimes^N}$ and where (R) denotes the two-sided ideal of the tensor algebra $T(E)$ of E generated by R. The algebra $\mathcal{A}$ is naturally a connected graded algebra with graduation induced by the one of $T(E)$. To $\mathcal{A}$ is associated another N-homogeneous algebra, *its dual* $\mathcal{A}^! = A(E^*, R^\perp)$ with E^* denoting the dual vector space of E and $R^\perp \subset E^{\otimes^N *} = E^{*\otimes^N}$ being the annihilator of R, [4]. The N-complex $K(\mathcal{A})$ of left $\mathcal{A}$-modules is then defined to be

$$\cdots \stackrel{d}{\to} \mathcal{A}\otimes\mathcal{A}^{!*}_{n+1} \stackrel{d}{\to} \mathcal{A}\otimes\mathcal{A}^{!*}_{n} \stackrel{d}{\to} \cdots \stackrel{d}{\to} \mathcal{A} \to 0 \tag{1.2}$$

where $\mathcal{A}^{!*}_n$ is the dual vector space of the finite-dimensional vector space $\mathcal{A}^!_n$ of the elements of degree n of $\mathcal{A}^!$ and where $d : \mathcal{A}\otimes\mathcal{A}^{!*}_{n+1} \to \mathcal{A}\otimes\mathcal{A}^{!*}_n$ is induced by the map $a\otimes(e_1\otimes\cdots\otimes e_{n+1}) \mapsto ae_1\otimes(e_2\otimes\cdots\otimes e_{n+1})$ of $\mathcal{A}\otimes E^{\otimes^{n+1}}$ into $\mathcal{A}\otimes E^{\otimes^n}$, remembering that $\mathcal{A}^{!*}_n \subset E^{\otimes^n}$, (see [4]). In (1.2) the factors $\mathcal{A}$ are considered as left $\mathcal{A}$-modules. By considering $\mathcal{A}$ as right $\mathcal{A}$-module and by exchanging the factors one obtains the N-complex $\tilde{K}(\mathcal{A})$ of right $\mathcal{A}$-modules

$$\cdots \stackrel{\tilde{d}}{\to} \mathcal{A}^{!*}_{n+1}\otimes\mathcal{A} \stackrel{\tilde{d}}{\to} \mathcal{A}^{!*}_n\otimes\mathcal{A} \stackrel{\tilde{d}}{\to} \cdots \stackrel{\tilde{d}}{\to} \mathcal{A} \to 0 \tag{1.3}$$

where now $\tilde{d}$ is induced by $(e_1\otimes\cdots\otimes e_{n+1})\otimes a \mapsto (e_1\otimes\cdots\otimes e_n)\otimes e_{n+1}a$. Finally one defines two N-differentials $d_{\mathbf{L}}$ and $d_{\mathbf{R}}$ on the sequence of $(\mathcal{A},\mathcal{A})$-bimodules, i.e. of left $\mathcal{A}\otimes\mathcal{A}^{opp}$-modules, $(\mathcal{A}\otimes\mathcal{A}^{!*}_n\otimes\mathcal{A})_{n\geq 0}$ by setting $d_{\mathbf{L}} = d\otimes I_{\mathcal{A}}$ and $d_{\mathbf{R}} = I_{\mathcal{A}}\otimes\tilde{d}$ where $I_{\mathcal{A}}$ is the identity mapping of $\mathcal{A}$ onto itself. For each of these N-differentials $d_{\mathbf{L}}$ and $d_{\mathbf{R}}$ the sequences

$$\cdots \stackrel{d_{\mathbf{L}},d_{\mathbf{R}}}{\to} \mathcal{A}\otimes\mathcal{A}^{!*}_{n+1}\otimes\mathcal{A} \stackrel{d_{\mathbf{L}},d_{\mathbf{R}}}{\to} \mathcal{A}\otimes\mathcal{A}^{!*}_n\otimes\mathcal{A} \stackrel{d_{\mathbf{L}},d_{\mathbf{R}}}{\to} \cdots \tag{1.4}$$

are N-complexes of left $\mathcal{A}\otimes\mathcal{A}^{opp}$-modules and one has

$$d_{\mathbf{L}}d_{\mathbf{R}} = d_{\mathbf{R}}d_{\mathbf{L}} \tag{1.5}$$

which implies that

$$d^N_{\mathbf{L}} - d^N_{\mathbf{R}} = (d_{\mathbf{L}} - d_{\mathbf{R}})\left(\sum_{p=0}^{N-1} d^p_{\mathbf{L}} d^{N-p-1}_{\mathbf{R}}\right) = \left(\sum_{p=0}^{N-1} d^p_{\mathbf{L}} d^{N-p-1}_{\mathbf{R}}\right)(d_{\mathbf{L}} - d_{\mathbf{R}}) = 0 \tag{1.6}$$

in view of $d^N_{\mathbf{L}} = d^N_{\mathbf{R}} = 0$.

As for any N-complex [13] one obtains from $K(\mathcal{A})$ ordinary complexes $C_{p,r}(K(\mathcal{A}))$, *the contractions of* $K(\mathcal{A})$, by putting together alternatively p and $N-p$ arrows d of $K(\mathcal{A})$. Explicitely $C_{p,r}(K(\mathcal{A}))$ is given by

$$\cdots \stackrel{d^{N-p}}{\to} \mathcal{A}\otimes\mathcal{A}^{!*}_{Nk+r} \stackrel{d^p}{\to} \mathcal{A}\otimes\mathcal{A}^{!*}_{Nk-p+r} \stackrel{d^{N-p}}{\to} \mathcal{A}\otimes\mathcal{A}^{!*}_{N(k-1)+r} \stackrel{d^p}{\to} \cdots \tag{1.7}$$

for $0 \leq r < p \leq N-1$, [4] . These are here chain complexes of free left $\mathcal{A}$-modules. As shown in [4] the complex $C_{N-1,0}(K(\mathcal{A}))$ coincides with the *Koszul complex* of [2]; this complex will be denoted by $\mathcal{K}(\mathcal{A},\mathbb{K})$ in the sequel. That is one has

$$\mathcal{K}_{2m}(\mathcal{A},\mathbb{K}) = \mathcal{A}\otimes\mathcal{A}^{!*}_{Nm}, \quad \mathcal{K}_{2m+1}(\mathcal{A},\mathbb{K}) = \mathcal{A}\otimes\mathcal{A}^{!*}_{Nm+1} \tag{1.8}$$

for $m \geq 0$, and the differential is d^{N-1} on $\mathcal{K}_{2m}(\mathcal{A},\mathbb{K})$ and d on $\mathcal{K}_{2m+1}(\mathcal{A},\mathbb{K})$. If $\mathcal{K}(\mathcal{A},\mathbb{K})$ is acyclic in positive degrees then $\mathcal{A}$ will be said to be a *Koszul algebra*. It was shown in [2] and this was confirmed by the analysis of [4] that this is the right generalization for N-homogeneous algebra of the usual notion of Koszulity for quadratic algebras [17], [16]. One always has $H_0(\mathcal{K}(\mathcal{A},\mathbb{K}))\simeq\mathbb{K}$ and therefore if $\mathcal{A}$ is Koszul, then one has a free resolution $\mathcal{K}(\mathcal{A},\mathbb{K})\to\mathbb{K}\to 0$ of the trivial left $\mathcal{A}$-module $\mathbb{K}$, that is the exact sequence

$$\dots \stackrel{d^{N-1}}{\to} \mathcal{A}\otimes\mathcal{A}^{!*}_{N+1} \stackrel{d}{\to} \mathcal{A}\otimes R \stackrel{d^{N-1}}{\to} \mathcal{A}\otimes E \stackrel{d}{\to} \mathcal{A} \stackrel{\varepsilon}{\to} \mathbb{K}\to 0 \tag{1.9}$$

of left $\mathcal{A}$-modules where ε is the projection on degree zero. This resolution is a minimal projective resolution of $\mathcal{A}$ in the graded category [3].

One defines now the chain complex of free $\mathcal{A}\otimes\mathcal{A}^{opp}$-modules $\mathcal{K}(\mathcal{A},\mathcal{A})$ by setting

$$\mathcal{K}_{2m}(\mathcal{A},\mathcal{A})=\mathcal{A}\otimes\mathcal{A}^{!*}_{Nm}\otimes\mathcal{A},\quad \mathcal{K}_{2m+1}(\mathcal{A},\mathcal{A})=\mathcal{A}\otimes\mathcal{A}^{!*}_{Nm+1}\otimes\mathcal{A} \tag{1.10}$$

for $m\in\mathbb{N}$ with differential δ' defined by

$$\delta'=d_{\mathbf{L}}-d_{\mathbf{R}}:\mathcal{K}_{2m+1}(\mathcal{A},\mathcal{A})\to\mathcal{K}_{2m}(\mathcal{A},\mathcal{A}) \tag{1.11}$$

$$\delta'=\sum_{p=0}^{N-1} d^p_{\mathbf{L}} d^{N-p-1}_{\mathbf{R}}:\mathcal{K}_{2(m+1)}(\mathcal{A},\mathcal{A})\to\mathcal{K}_{2m+1}(\mathcal{A},\mathcal{A}) \tag{1.12}$$

the property $\delta'^2=0$ following from (1.6). *This complex is acyclic in positive degrees if and only if* $\mathcal{A}$ *is Koszul*, that is if and only if $\mathcal{K}(\mathcal{A},\mathbb{K})$ is acyclic in positive degrees, [2] and [4] . One always has the obvious exact sequence

$$\mathcal{A}\otimes E\otimes\mathcal{A} \stackrel{\delta'}{\to} \mathcal{A}\otimes\mathcal{A} \stackrel{\mu}{\to} \mathcal{A}\to 0 \tag{1.13}$$

of left $\mathcal{A}\otimes\mathcal{A}^{opp}$-modules where μ denotes the product of $\mathcal{A}$. It follows that if $\mathcal{A}$ is a Koszul algebra then $\mathcal{K}(\mathcal{A},\mathcal{A}) \stackrel{\mu}{\to} \mathcal{A}\to 0$ is a free resolution of the $\mathcal{A}\otimes\mathcal{A}^{opp}$-module $\mathcal{A}$ which will be refered to as *the Koszul resolution of* $\mathcal{A}$. This is a minimal projective resolution of $\mathcal{A}\otimes\mathcal{A}^{opp}$ in the graded category [3].

Let $\mathcal{A}$ be a Koszul algebra and let $\mathcal{M}$ be a $(\mathcal{A},\mathcal{A})$-bimodule considered as a right $\mathcal{A}\otimes\mathcal{A}^{opp}$-module. Then, by interpreting the $\mathcal{M}$-valued Hochschild homology $H(\mathcal{A},\mathcal{M})$ as $H_n(\mathcal{A},\mathcal{M})=\mathrm{Tor}^{\mathcal{A}\otimes\mathcal{A}^{opp}}_n(\mathcal{M},\mathcal{A})$ [6], the complex $\mathcal{M}\otimes_{\mathcal{A}\otimes\mathcal{A}^{opp}}\mathcal{K}(\mathcal{A},\mathcal{A})$ computes the $\mathcal{M}$-valued Hochschild homology of $\mathcal{A}$, (i.e. its homology is the ordinary $\mathcal{M}$-valued Hochschild homology of $\mathcal{A}$). We shall refer to this complex as *the small Hochschild complex of* $\mathcal{A}$ with coefficients in $\mathcal{M}$ and denote it by $\mathcal{S}(\mathcal{A},\mathcal{M})$. It reads

$$\dots \stackrel{\delta}{\to} \mathcal{M}\otimes\mathcal{A}^{!*}_{N(m+1)} \stackrel{\delta}{\to} \mathcal{M}\otimes\mathcal{A}^{!*}_{Nm+1} \stackrel{\delta}{\to} \mathcal{M}\otimes\mathcal{A}^{!*}_{Nm} \stackrel{\delta}{\to} \dots \tag{1.14}$$

where δ is obtained from δ' by applying the factors d_L to the right of $\mathcal{M}$ and the factors d_R to the left of $\mathcal{M}$.

Assume that $\mathcal{A}$ is a Koszul algebra of finite global dimension D. Then the Koszul resolution of $\mathbb{K}$ has length D, i.e. D is the largest integer such that $\mathcal{K}_D(\mathcal{A},\mathbb{K}) \neq 0$. By construction, D is also the greatest integer such that $\mathcal{K}_D(\mathcal{A},\mathcal{A}) \neq 0$ so the free $\mathcal{A}\otimes\mathcal{A}^{opp}$-module resolution of $\mathcal{A}$ has also length D. Thus *for a Koszul algebra, the global dimension is equal to the Hochschild dimension*. Applying then the functor $\mathrm{Hom}_{\mathcal{A}}(\bullet,\mathcal{A})$ to $\mathcal{K}(\mathcal{A},\mathbb{K})$ one obtains the cochain complex $\mathcal{L}(\mathcal{A},\mathbb{K})$ of free right $\mathcal{A}$-modules

$$0 \to \mathcal{L}^0(\mathcal{A},\mathbb{K}) \to \cdots \to \mathcal{L}^D(\mathcal{A},\mathbb{K}) \to 0$$

where $\mathcal{L}^n(\mathcal{A},\mathbb{K}) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{K}_n(\mathcal{A},\mathbb{K}),\mathcal{A})$. The Koszul algebra $\mathcal{A}$ is *Gorenstein* iff $H^n(\mathcal{L}(\mathcal{A},\mathbb{K})) = 0$ for $n < D$ and $H^D(\mathcal{L}(\mathcal{A},\mathbb{K})) = \mathbb{K}$ (= the trivial right $\mathcal{A}$-module). This is clearly a generalisation of the classical Poincaré duality and this implies a precise form of Poincaré duality between Hochschild homology and Hochschild cohomology [5], [20], [21]. In the case of the Yang-Mills algebra and its deformations which are Koszul Gorenstein cubic algebras of global dimension 3, this Poincaré duality gives isomorphisms

$$H_k(\mathcal{A},\mathcal{M}) = H^{3-k}(\mathcal{A},\mathcal{M}), \quad k \in \{0,1,2,3\} \tag{1.15}$$

between the Hochschild homology and the Hochschild cohomology with coefficients in a bimodule $\mathcal{M}$.

1.3 The Yang-Mills algebra

Let $(g_{\lambda\mu}) \in M_{s+1}(\mathbb{K})$ be an invertible symmetric $(s+1)\times(s+1)$-matrix with inverse $(g^{\lambda\mu})$, i.e. $g_{\lambda\mu}g^{\mu\nu} = \delta^\nu_\lambda$. The *Yang-Mills algebra* is the cubic algebra $\mathcal{A}$ generated by $s+1$ elements ∇_λ ($\lambda \in \{0,\dots,s\}$) with the $s+1$ relations

$$g^{\lambda\mu}[\nabla_\lambda,[\nabla_\mu,\nabla_\nu]] = 0, \quad \nu \in \{0,\dots,s\}$$

that is $\mathcal{A} = A(E,R)$ with $E = \oplus_\lambda \mathbb{K}\nabla_\lambda$ and $R \subset E^{\otimes^3}$ given by

$$\begin{aligned} R &= \sum_\nu \mathbb{K} g^{\lambda\mu}[\nabla_\lambda,[\nabla_\mu,\nabla_\nu]_\otimes]_\otimes \\ &= \sum_\rho \mathbb{K}(g^{\rho\lambda}g^{\mu\nu} + g^{\nu\rho}g^{\lambda\mu} - 2g^{\rho\mu}g^{\lambda\nu})\nabla_\lambda\otimes\nabla_\mu\otimes\nabla_\nu \end{aligned} \tag{1.16}$$

In [10] the following theorem was proved.

Theorem 1. *The cubic Yang-Mills algebra $\mathcal{A}$ is Koszul of global dimension 3 and is Gorenstein.*

The proof of this theorem relies on the computation of the dual cubic algebra $\mathcal{A}^!$ which we now recall.

The dual $\mathcal{A}^! = A(E^*, R^\perp)$ of the Yang-Mills algebra is the cubic algebra generated by $s+1$ elements θ^λ ($\lambda \in \{0, \dots, s\}$) with relations

$$\theta^\lambda\theta^\mu\theta^\nu = \frac{1}{s}(g^{\lambda\mu}\theta^\nu + g^{\mu\nu}\theta^\lambda - 2g^{\lambda\nu}\theta^\mu)\mathbf{g}$$

where $\mathbf{g} = g_{\alpha\beta}\theta^\alpha\theta^\beta$. These relations imply that $\mathbf{g} \in \mathcal{A}^!_2$ is central in $\mathcal{A}^!$ and that one has $\mathcal{A}^!_0 = \mathbb{K}\mathbb{1} \simeq \mathbb{K}$, $\mathcal{A}^!_1 = \oplus_\lambda \mathbb{K}\theta^\lambda \simeq \mathbb{K}^{s+1}$, $\mathcal{A}^!_2 = \oplus_{\mu\nu}\mathbb{K}\theta^\mu\theta^\nu \simeq \mathbb{K}^{(s+1)^2}$, $\mathcal{A}^!_3 = \oplus_\lambda \mathbb{K}\theta^\lambda\mathbf{g} \simeq \mathbb{K}^{s+1}$, $\mathcal{A}^!_4 = \mathbb{K}\mathbf{g}^2 \simeq \mathbb{K}$ and $\mathcal{A}^!_n = 0$ for $n \geq 5$. From this, one obtains the description of [10] of the Koszul complex $\mathcal{K}(\mathcal{A}, \mathbb{K})$ and the proof of the above theorem. It also follows that the bimodule resolution $\mathcal{K}(\mathcal{A}, \mathcal{A}) \stackrel{\mu}{\to} \mathcal{A} \to 0$ of $\mathcal{A}$ reads

$$0 \to \mathcal{A}\otimes\mathcal{A} \stackrel{\delta'_3}{\to} \mathcal{A}\otimes\mathbb{K}^{s+1}\otimes\mathcal{A} \stackrel{\delta'_2}{\to} \mathcal{A}\otimes\mathbb{K}^{s+1}\otimes\mathcal{A} \stackrel{\delta'_1}{\to} \mathcal{A}\otimes\mathcal{A} \stackrel{\mu}{\to} \mathcal{A} \to 0 \quad (1.17)$$

where the components δ'_k of δ' in the different degrees can be computed by using the description of $\mathcal{K}(\mathcal{A}, \mathbb{K}) = C_{2,0}$ given in Section 3 of [10] and are given by

$$\begin{cases} \delta'_1(a\otimes e_\lambda \otimes b) = a\nabla_\lambda \otimes b - a \otimes \nabla_\lambda b \\ \delta'_2(a\otimes e_\lambda\otimes b) = (g^{\alpha\beta}\delta^\gamma_\lambda + g^{\beta\gamma}g^\alpha_\lambda - 2g^{\gamma\alpha}\delta^\beta_\lambda)\times \\ \qquad \times(a\nabla_\alpha\nabla_\beta\otimes e_\gamma\otimes b + a\nabla_\alpha\otimes e_\gamma\otimes\nabla_\beta b + a\otimes e_\gamma\otimes\nabla_\alpha\nabla_\beta b) \\ \delta'_3(a\otimes b) = g^{\lambda\mu}(a\nabla_\mu\otimes e_\lambda\otimes b - a\otimes e_\lambda\otimes\nabla_\mu b) \end{cases} \quad (1.18)$$

where $a, b \in \mathcal{A}$, e_λ ($\lambda = 0, \dots, s$) is the canonical basis of $\mathbb{K}^{s+1}$ and ∇_λ are the corresponding generators of $\mathcal{A}$.

Let $\mathcal{M}$ be a bimodule over $\mathcal{A}$. By using the above description of the Koszul resolution of $\mathcal{A}$ one easily obtains the one of the small Hochschild complex $\mathcal{S}(\mathcal{A}, \mathcal{M})$ which reads

$$0 \to \mathcal{M} \stackrel{\delta_3}{\to} \mathcal{M}\otimes\mathbb{K}^{s+1} \stackrel{\delta_2}{\to} \mathcal{M}\otimes\mathbb{K}^{s+1} \stackrel{\delta_1}{\to} \mathcal{M} \to 0 \quad (1.19)$$

with differential δ given by

$$\begin{cases} \delta_1(m^\lambda\otimes e_\lambda) = m^\lambda\nabla_\lambda - \nabla_\lambda m^\lambda = [m^\lambda, \nabla_\lambda] \\ \delta_2(m^\lambda\otimes e_\lambda) = \\ \qquad = ([\nabla_\mu, [\nabla^\mu, m^\lambda]] + [\nabla_\mu, [m^\mu, \nabla^\lambda]] + [m^\mu, [\nabla_\mu, \nabla^\lambda]])\otimes e_\lambda \\ \delta_3(m) = g^{\lambda\mu}(m\nabla_\mu - \nabla_\mu m)\otimes e_\lambda = [m, \nabla^\lambda]\otimes e_\lambda \end{cases} \quad (1.20)$$

with obvious notations. By using (1.20) one easily verifies the duality (1.15). For instance $H_3(\mathcal{A}, \mathcal{M})$ is $\mathrm{Ker}(\delta_3)$ which is given by the $m \in \mathcal{M}$ such that

$\nabla_\lambda m = m\nabla_\lambda$ for $\lambda = 0, \dots, s$ that is such that $am = ma$, $\forall a \in \mathcal{A}$, since $\mathcal{A}$ is generated by the ∇_λ and it is well known that this coincides with $H^0(\mathcal{A}, \mathcal{M})$. Similarily $m^\lambda \otimes e_\lambda$ is in $\mathrm{Ker}(\delta_2)$ if and only if $\nabla_\lambda \mapsto D(\nabla_\lambda) = g_{\lambda\mu} m^\mu$ extends as a derivation D of $\mathcal{A}$ into $\mathcal{M}$ ($D \in \mathrm{Der}(\mathcal{A}, \mathcal{M})$) while $m^\lambda \otimes e_\lambda = \delta_3(m)$ means that this derivation is inner $D = ad(m) \in \mathrm{Int}(\mathcal{A}, \mathcal{M})$ from which $H_2(\mathcal{A}, \mathcal{M})$ identifies with $H^1(\mathcal{A}, \mathcal{M})$, and so on.

Assume now that $\mathcal{M}$ is graded in the sense that one has $\mathcal{M} = \oplus_{n\in\mathbb{Z}} \mathcal{M}_n$ with $\mathcal{A}_k \mathcal{M}_\ell \mathcal{A}_m \subset \mathcal{M}_{k+\ell+m}$. Then the small Hochschild complex $\mathcal{S}(\mathcal{A}, \mathcal{M})$ splits into subcomplexes $\mathcal{S}(\mathcal{A}, \mathcal{M}) = \oplus_n \mathcal{S}^{(n)}(\mathcal{A}, \mathcal{M})$ where $\mathcal{S}^{(n)}(\mathcal{A}, \mathcal{M})$ is the subcomplex

$$0 \to \mathcal{M}_{n-4} \overset{\delta_3}{\to} \mathcal{M}_{n-3} \otimes \mathbb{K}^{s+1} \overset{\delta_2}{\to} \mathcal{M}_{n-1} \otimes \mathbb{K}^{s+1} \overset{\delta_1}{\to} \mathcal{M}_n \to 0 \tag{1.21}$$

of (1.19). Assume furthermore that the homogeneous components $\mathcal{M}_n$ are finite-dimensional vector spaces, i.e. $\dim(\mathcal{M}_n) \in \mathbb{N}$. Then one has the following Euler-Poincaré formula

$$\begin{aligned} &\dim(H_0^{(n)}) - \dim(H_1^{(n)}) + \dim(H_2^{(n)}) - \dim(H_3^{(n)}) = \\ &\dim(\mathcal{M}_n) - (s+1)\dim(\mathcal{M}_{n-1}) + (s+1)\dim(\mathcal{M}_{n-3}) - \dim(\mathcal{M}_{n-4}) \end{aligned} \tag{1.22}$$

for the homology $H^{(n)}$ of the chain complex $\mathcal{S}^{(n)}(\mathcal{A}, \mathcal{M})$.

In the case where $\mathcal{M} = \mathcal{A}$, it follows from the Koszulity of $\mathcal{A}$ that the right hand side of (1.22) vanishes for $n \neq 0$. Denoting as usual by $HH(\mathcal{A})$ the $\mathcal{A}$-valued Hochschild homology of $\mathcal{A}$ which is here the homology of $\mathcal{S}(\mathcal{A}, \mathcal{A})$, we denote by $HH^{(n)}(\mathcal{A})$ the homology of the subcomplex $\mathcal{S}^{(n)}(\mathcal{A}, \mathcal{A})$. Since $\mathcal{A}_n = 0$ for $n < 0$, one has $HH_0^{(n)}(\mathcal{A}) = 0$ for $n < 0$, $HH_1^{(n)}(\mathcal{A}) = 0$ for $n \leq 0$, $HH_2^{(n)}(\mathcal{A}) = 0$ for $n \leq 2$ and $HH_3^{(n)}(\mathcal{A}) = 0$ for $n \leq 3$. Furthermore one has

$$HH_0^{(0)}(\mathcal{A}) = HH_3^{(4)}(\mathcal{A}) = \mathbb{K} \tag{1.23}$$

$$HH_0^{(1)}(\mathcal{A}) = HH_1^{(1)}(\mathcal{A}) = HH_2^{(3)}(\mathcal{A}) = \mathbb{K}^{s+1} \tag{1.24}$$

$$HH_0^{(2)}(\mathcal{A}) = HH_1^{(2)}(\mathcal{A}) = \mathbb{K}^{\frac{(s+1)(s+2)}{2}} \tag{1.25}$$

and the Euler Poincaré formula reads here

$$\dim(HH_0^{(n)}(\mathcal{A})) + \dim(HH_2^{(n)}(\mathcal{A})) = \dim(HH_1^{(n)}(\mathcal{A})) + \dim(HH_3^{(n)}(\mathcal{A})) \tag{1.26}$$

for $n \geq 1$ which implies

$$\dim(HH_0^{(3)}(\mathcal{A})) + (s+1) = \dim(HH_1^{(3)}(\mathcal{A})) \tag{1.27}$$

for $n = 3$ while for $n = 1$ and $n = 2$ it is already contained in (1.24) and (1.25).

The complete description of the Hochschild homology and of the cyclic homology of the Yang-Mills algebra will be given in [12].

1.4 The super Yang-Mills algebra

As pointed out in the introduction, the Yang-Mills algebra is the universal enveloping algebra of a Lie algebra which is graded by giving degree 1 to the generators ∇_λ (see in [10]). Replacing the Lie bracket by a super Lie bracket, that is replacing in the Yang-Mills equations (1.1) the commutator by the anticommutator whenever the 2 elements are of odd degrees, one obtains a super version $\tilde{\mathcal{A}}$ of the Yang-Mills algebra $\mathcal{A}$. In other words one defines the *super Yang-Mills algebra* to be the cubic algebra $\tilde{\mathcal{A}}$ generated $s+1$ elements S_λ ($\lambda \in \{0,\dots,s\}$) with the relations

$$g^{\lambda\mu}[S_\lambda, \{S_\mu, S_\nu\}] = 0, \quad \nu \in \{0,\dots,s\} \tag{1.28}$$

that is $\tilde{\mathcal{A}} = A(\tilde{E}, \tilde{R})$ with $\tilde{E} = \oplus_\lambda \mathbb{K} S_\lambda$ and $\tilde{R} \subset \tilde{E}^{\otimes^3}$ given by

$$\tilde{R} = \sum_\rho \mathbb{K}(g^{\rho\lambda} g^{\mu\nu} - g^{\nu\rho} g^{\lambda\mu}) S_\lambda \otimes S_\mu \otimes S_\nu \tag{1.29}$$

Relations (1.28) can be equivalently written as

$$[g^{\lambda\mu} S_\lambda S_\mu, S_\nu] = 0, \quad \nu \in \{0,\dots,s\} \tag{1.30}$$

which mean that $g^{\lambda\mu} S_\lambda S_\mu \in \tilde{\mathcal{A}}_2$ is central in $\tilde{\mathcal{A}}$.

It is easy to verify that the dual algebra $\tilde{\mathcal{A}}^! = A(\tilde{E}^*, \tilde{R}^\perp)$ is the cubic algebra generated by $s+1$ elements ξ^λ ($\lambda \in \{0,\dots,s\}$) with the relations

$$\xi^\lambda \xi^\mu \xi^\nu = -\frac{1}{s}(g^{\lambda\mu}\xi^\nu - g^{\mu\nu}\xi^\lambda)\mathbf{g}$$

where $\mathbf{g} = g_{\alpha\beta}\xi^\alpha\xi^\beta$. These relations imply that $\mathbf{g}\xi^\nu + \xi^\nu\mathbf{g} = 0$, i.e.

$$\{g_{\lambda\mu}\xi^\lambda\xi^\mu, \xi^\nu\} = 0, \quad \nu \in \{0,\dots,s\} \tag{1.31}$$

and that one has $\tilde{\mathcal{A}}^!_0 = \mathbb{K}\mathbb{1} \simeq \mathbb{K}$, $\tilde{\mathcal{A}}^!_1 = \oplus_\lambda \mathbb{K}\xi^\lambda \simeq \mathbb{K}^{s+1}$, $\tilde{\mathcal{A}}^!_2 = \oplus_{\mu\nu}\mathbb{K}\xi^\mu\xi^\nu \simeq \mathbb{K}^{(s+1)^2}$, $\tilde{\mathcal{A}}^!_3 = \oplus_\lambda \mathbb{K}\xi^\lambda\mathbf{g} \simeq \mathbb{K}^{s+1}$, $\tilde{\mathcal{A}}^!_4 = \mathbb{K}\mathbf{g}^2 \simeq \mathbb{K}$ and $\tilde{\mathcal{A}}^!_n = 0$ for $n \geq 5$.

The Koszul complex $\mathcal{K}(\tilde{\mathcal{A}}, \mathbb{K})$ of $\tilde{\mathcal{A}}$ then reads

$$0 \to \tilde{\mathcal{A}} \stackrel{S^t}{\to} \tilde{\mathcal{A}}^{s+1} \stackrel{N}{\to} \tilde{\mathcal{A}}^{s+1} \stackrel{S}{\to} \tilde{\mathcal{A}} \to 0$$

where S means right multiplication by the column with components S_λ, S^t means right multiplication by the row with components S_λ and N means right multiplication (matrix product) by the matrix with components

$$N^{\mu\nu} = (g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta})S_\alpha S_\beta$$

with $\lambda, \mu, \nu \in \{0, \dots, s\}$. One has the following result.

Theorem 2. *The cubic super Yang-Mills algebra $\tilde{\mathcal{A}}$ is Koszul of global dimension 3 and is Gorenstein.*

Proof. By the very definition of $\tilde{\mathcal{A}}$ by generators and relations, the sequence

$$\tilde{\mathcal{A}}^{s+1} \overset{N}{\to} \tilde{\mathcal{A}}^{s+1} \overset{S}{\to} \tilde{\mathcal{A}} \overset{\varepsilon}{\to} \mathbb{K} \to 0$$

is exact. On the other hand it is easy to see that the mapping $\tilde{\mathcal{A}} \overset{S^t}{\to} \tilde{\mathcal{A}}^{s+1}$ is injective and that the sequence

$$0 \to \tilde{\mathcal{A}} \overset{S^t}{\to} \tilde{\mathcal{A}}^{s+1} \overset{N}{\to} \tilde{\mathcal{A}}^{s+1} \overset{S}{\to} \tilde{\mathcal{A}} \overset{\varepsilon}{\to} \mathbb{K} \to 0$$

is exact which implies that $\tilde{\mathcal{A}}$ is Koszul of global dimension 3. The Gorenstein property follows from the symmetry by transposition. □

The situation is completely similar to the Yang-Mills case, in particular $\tilde{\mathcal{A}}$ has Hochschild dimension 3 and, by applying a result of [15], $\tilde{\mathcal{A}}$ has the same Poincaré series as $\mathcal{A}$ i.e. one has the formula

$$\sum_{n\in\mathbb{N}} \dim(\tilde{\mathcal{A}}_n)t^n = \frac{1}{(1-t^2)(1-(s+1)t+t^2)} \tag{1.32}$$

which, as will be shown elsewhere, can be interpreted in terms of the quantum group of the bilinear form $(g_{\mu\nu})$ [14] by noting the invariance of Relations (1.30) by this quantum group. For $s = 1$ the Yang-Mills algebra and the super Yang-Mills algebra are particular cubic Artin-Schelter algebras [1] whereas for $s \geq 2$ these algebras have exponential growth as follows from Formula (1.32).

1.5 The super self-duality algebra

There are natural quotients $\mathcal{B}$ of $\mathcal{A}$ and $\tilde{\mathcal{B}}$ of $\tilde{\mathcal{A}}$ which are connected with parastatistics and which have been investigated in [15]. The *parafermionic algebra* $\mathcal{B}$ is the cubic algebra generated by elements ∇_λ ($\lambda \in \{0, \dots, s\}$) with relations

$$[\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] = 0$$

for any $\lambda, \mu, \nu \in \{0, \dots, s\}$, while the *parabosonic algebra* $\tilde{\mathcal{B}}$ is the cubic algebra generated by elements S_λ ($\lambda \in \{0, \dots, s\}$) with relations

$$[S_\lambda, \{S_\mu, S_\nu\}] = 0$$

for any $\lambda, \mu, \nu \in \{0, \dots, s\}$. In contrast to the Yang-Mills and the super Yang-Mills algebras $\mathcal{A}$ and $\tilde{\mathcal{A}}$ which have exponential growth whenever $s \geq 2$, these algebras $\mathcal{B}$ and $\tilde{\mathcal{B}}$ have polynomial growth with Poincaré series given by

$$\sum_n \dim(\mathcal{B}_n) t^n = \sum_n \dim(\tilde{\mathcal{B}}_n) t^n = \left(\frac{1}{1-t}\right)^{s+1} \left(\frac{1}{1-t^2}\right)^{\frac{s(s+1)}{2}}$$

but they are not Koszul for $s \geq 2$, [15].

In a sense, the algebra $\mathcal{B}$ can be considered to be somehow trivial from the point of view of the classical Yang-Mills equations in dimension $s+1 \geq 3$ although the algebras $\mathcal{B}$ and $\tilde{\mathcal{B}}$ are quite interesting for other purposes [15]. It turns out that in dimension $s+1 = 4$ with $g_{\mu\nu} = \delta_{\mu\nu}$ (Euclidean case), the Yang-Mills algebra $\mathcal{A}$ has non trivial quotients $\mathcal{A}^{(+)}$ and $\mathcal{A}^{(-)}$ which are quadratic algebras refered to as the *self-duality algebra* and the *anti-self-duality algebra* respectively [10]. Let $\varepsilon = \pm$, the algebra $\mathcal{A}^{(\varepsilon)}$ is the quadratic algebra generated by the elements ∇_λ ($\lambda \in \{0, 1, 2, 3, \}$) with relations

$$[\nabla_0, \nabla_k] = \varepsilon[\nabla_\ell, \nabla_m]$$

for any cyclic permutation (k, ℓ, m) of (1,2,3). One passes from $\mathcal{A}^{(-)}$ to $\mathcal{A}^{(+)}$ by changing the orientation of $\mathbb{K}^4$ so one can restrict attention to the self-duality algebra $\mathcal{A}^{(+)}$. This algebra has been studied in [10] where it was shown in particular that it is Koszul of global dimension 2. For further details on this algebra and on the Yang-Mills algebra, we refer to [10] and to the forthcoming paper [12]. Our aim now in this section is to define and study the super version of the self-duality algebra.

Let $\varepsilon = +$ or $-$ and define $\tilde{\mathcal{A}}^{(\varepsilon)}$ to be the quadratic algebra generated by the elements S_0, S_1, S_2, S_3 with relations

$$i\{S_0, S_k\} = \varepsilon[S_\ell, S_m] \tag{1.33}$$

for any cyclic permutation (k, ℓ, m) of (1,2,3). One has the following.

Lemma 1. *Relations (1.33) imply that one has*

$$[\sum_{\mu=0}^{3} (S_\mu)^2, S_\lambda] = 0$$

for any $\lambda \in \{0, 1, 2, 3\}$. In other words, $\tilde{\mathcal{A}}^{(+)}$ and $\tilde{\mathcal{A}}^{(-)}$ are quotients of the super Yang-Mills algebra $\tilde{\mathcal{A}}$ for s+1=4 and $g_{\mu\nu} = \delta_{\mu\nu}$.

The proof which is a straightforward verification makes use of the Jacobi identity (see also in [18]). Thus $\tilde{\mathcal{A}}^{(+)}$ and $\tilde{\mathcal{A}}^{(-)}$ play the same role with respect to $\tilde{\mathcal{A}}$ as $\mathcal{A}^{(+)}$ and $\mathcal{A}^{(-)}$ with respect to $\mathcal{A}$. Accordingly they will be respectively called the *super self-duality algebra* and the *super anti-self-duality algebra*. Again $\tilde{\mathcal{A}}^{(+)}$ and $\tilde{\mathcal{A}}^{(-)}$ are exchanged by changing the orientation of $\mathbb{K}^4$ and we shall restrict attention to the super self-duality algebra in the following, i.e. to the quadratic algebra $\tilde{\mathcal{A}}^{(+)}$ generated by S_0, S_1, S_2, S_3 with relations

$$i\{S_0, S_k\} = [S_\ell, S_m] \tag{1.34}$$

for any cyclic permutation (k, ℓ, m) of (1,2,3). One has the following result.

Theorem 3. *The quadratic super self-duality algebra $\tilde{\mathcal{A}}^{(+)}$ is a Koszul algebra of global dimension 2.*

Proof. One verifies that the dual quadratic algebra $\tilde{\mathcal{A}}^{(+)!}$ is generated by elements $\xi^0, \xi^1, \xi^2, \xi^3$ with relations $(\xi^\lambda)^2 = 0$, for $\lambda = 0,1,2,3$ and $\xi^\ell\xi^m = -\xi^m\xi^\ell = i\xi^0\xi^k = i\xi^k\xi^0$ for any cyclic permutation (k, ℓ, m) of (1,2,3). So one has $\tilde{\mathcal{A}}^{(+)!}_0 = \mathbb{K}\mathbb{1} \simeq \mathbb{K}$, $\tilde{\mathcal{A}}^{(+)!}_1 = \oplus_\lambda \mathbb{K}\xi^\lambda \simeq \mathbb{K}^4$, $\tilde{\mathcal{A}}^{(+)!}_2 = \oplus_k \mathbb{K}\xi^0\xi^k \simeq \mathbb{K}^3$ and $\tilde{\mathcal{A}}^{(+)!}_n = 0$ for $n \geq 3$ since the above relations imply $\xi^\lambda\xi^\mu\xi^\nu = 0$ for any $\lambda, \mu, \nu \in \{0,1,2,3\}$. The Koszul complex $K(\tilde{\mathcal{A}}^{(+)}) = \mathcal{K}(\tilde{\mathcal{A}}^{(+)}, \mathbb{K})$ (quadratic case) then reads

$$0 \to \tilde{\mathcal{A}}^{(+)^3} \stackrel{D}{\to} \tilde{\mathcal{A}}^{(+)^4} \stackrel{S}{\to} \tilde{\mathcal{A}}^{(+)} \to 0$$

where S means right matrix product with the column with components S_λ ($\lambda \in \{0,1,2,3\}$) and D means right matrix product with

$$D = \begin{pmatrix} iS_1 & iS_0 & S_3 & -S_2 \\ iS_2 & -S_3 & iS_0 & S_1 \\ iS_3 & S_2 & -S_1 & iS_0 \end{pmatrix} \tag{1.35}$$

It follows from the definition of $\tilde{\mathcal{A}}^{(+)}$ by generators and relations that the sequence

$$\tilde{\mathcal{A}}^{(+)^3} \stackrel{D}{\to} \tilde{\mathcal{A}}^{(+)^4} \stackrel{S}{\to} \tilde{\mathcal{A}}^{(+)} \stackrel{\varepsilon}{\to} \mathbb{K} \to 0$$

is exact. On the other hand one shows easily that the mapping $\tilde{\mathcal{A}}^{(+)^3} \stackrel{D}{\to} \tilde{\mathcal{A}}^{(+)^4}$ is injective so finally the sequence

$$0 \to \tilde{\mathcal{A}}^{(+)3} \stackrel{D}{\to} \tilde{\mathcal{A}}^{(+)^4} \stackrel{S}{\to} \tilde{\mathcal{A}}^{(+)} \stackrel{\varepsilon}{\to} \mathbb{K} \to 0$$

is exact which implies the result. □

This theorem implies that the super self-duality algebra $\tilde{\mathcal{A}}^{(+)}$ has Hochschild dimension 2 and that its Poincaré series is given by

$$P_{\tilde{\mathcal{A}}^{(+)}}(t) = \frac{1}{(1-t)(1-3t)}$$

in view of the structure of its dual $\tilde{\mathcal{A}}^{(+)!}$ described in the proof. Thus everything is similar to the case of the self-duality algebra $\mathcal{A}^{(+)}$.

Let us recall that the *Sklyanin algebra*, in the presentation given by Sklyanin [18], is the quadratic algebra $\mathcal{S}(\alpha_1, \alpha_2, \alpha_3)$ generated by 4 elements S_0, S_1, S_2, S_3 with relations

$$i\{S_0, S_k\} = [S_\ell, S_m]$$

$$[S_0, S_k] = i\frac{\alpha_\ell - \alpha_m}{\alpha_k}\{S_\ell, S_m\}$$

for any cycic permutation (k, ℓ, m) of (1,2,3). One sees that the relations of the super self-duality algebra $\tilde{\mathcal{A}}^{(+)}$ are the relations of the Sklyanin algebra which are independent from the parameters α_k. Thus one has a sequence of surjective homomorphisms of connected graded algebra

$$\tilde{\mathcal{A}} \to \tilde{\mathcal{A}}^{(+)} \to \mathcal{S}(\alpha_1, \alpha_2, \alpha_3)$$

On the other hand for *generic values of the parameters* the Sklyanin algebra is Koszul Gorenstein of global dimension 4 [19] with the same Poincaré series as the polynomial algebra $\mathbb{K}[X_0, X_1, X_2, X_3]$ and corresponds to the natural ambiant noncommutative 4-dimensional Euclidean space containing the noncommutative 3-spheres described in [9], [11] (their “homogeneisation”). This gives a very surprising connection between the present study and our noncommutative 3-spheres for generic values of the parameters. It is worth noticing here that in the analysis of [11] several bridges between noncommutative differential geometry in the sense of [7], [8] and noncommutative algebraic geometry have been established.

1.6 Deformations

The aim of this section is to study deformations of the Yang-Mills algebra and of the super Yang-Mills algebra. We use the notations of Sections 3 and 4.

Let the dimension $s+1 \geq 2$ and the pseudo metric $g_{\lambda\mu}$ be fixed and let $\zeta \in P_1(\mathbb{K})$ have homogeneous coordinates $\zeta_0, \zeta_1 \in \mathbb{K}$. Define $\mathcal{A}(\zeta)$ to be the cubic algebra generated by $s+1$ elements ∇_λ ($\lambda \in \{0, \dots, s\}$) with relations

$$(\zeta_1(g^{\rho\lambda}g^{\mu\nu} + g^{\nu\rho}g^{\lambda\mu}) - 2\zeta_0 g^{\rho\mu}g^{\lambda\nu})\nabla_\lambda\nabla_\mu\nabla_\nu = 0$$

for $\rho \in \{0, \dots, s\}$. The Yang-Mills algebra corresponds to the element ζ^{YM} of $P_1(\mathbb{K})$ with homogeneous coordinates $\zeta_0 = \zeta_1$. Let ζ^{sing} be the element of $P_1(\mathbb{K})$ with homogeneous coordinates $\zeta_0 = \frac{s+2}{2}\zeta_1$; one has the following result.

Theorem 4. *For $\zeta \neq \zeta^{sing}$ the cubic algebra $\mathcal{A}(\zeta)$ is Koszul of global dimension 3 and is Gorenstein.*

Proof. The dual algebra $\mathcal{A}(\zeta)^!$ is the cubic algebra generated by elements θ^λ with relations

$$\theta^\lambda\theta^\mu\theta^\nu = \frac{1}{(s+2)\zeta_1 - 2\zeta_0}(\zeta_1(g^{\lambda\mu}\theta^\nu + g^{\mu\nu}\theta^\lambda) - 2\zeta_0 g^{\lambda\nu}\theta^\mu)\mathbf{g} \tag{1.36}$$

for $\lambda,\mu,\nu \in \{0,\dots,s\}$ with $\mathbf{g} = g_{\alpha\beta}\theta^\alpha\theta^\beta$. This again implies that $\mathbf{g}$ is in the center and that one has $\mathcal{A}^!_0 = \mathbb{K}\mathbb{1} \simeq \mathbb{K}$, $\mathcal{A}^!_1 = \oplus_\lambda \mathbb{K}\theta^\lambda \simeq \mathbb{K}^{s+1}$, $\mathcal{A}^!_2 = \oplus_{\lambda,\mu}\mathbb{K}\theta^\lambda\theta^\mu \simeq \mathbb{K}^{(s+1)^2}$, $\mathcal{A}^!_3 = \oplus_\lambda\mathbb{K}\theta^\lambda\mathbf{g} \simeq \mathbb{K}^{s+1}$, $\mathcal{A}^!_4 = \mathbb{K}\mathbf{g}^2 \simeq \mathbb{K}$ while $\mathcal{A}^!_n = 0$ for $n \geq 5$, where we have set $\mathcal{A}^!_n = \mathcal{A}(\zeta)^!_n$. Setting $\mathcal{A} = \mathcal{A}(\zeta)$, the Koszul complex $\mathcal{K}(\mathcal{A}(\zeta),\mathbb{K})$ of $\mathcal{A}(\zeta)$ reads

$$0 \to \mathcal{A} \stackrel{\nabla^t}{\to} \mathcal{A}^{s+1} \stackrel{M}{\to} \mathcal{A}^{s+1} \stackrel{\nabla}{\to} \mathcal{A} \to 0$$

with the same conventions as before and M with components

$$M^{\mu\nu} = \frac{1}{(s+2)\zeta_1 - 2\zeta_0}(\zeta_1(g^{\mu\nu}g^{\alpha\beta} + g^{\mu\alpha}g^{\nu\beta}) - 2\zeta_0 g^{\mu\beta}g^{\nu\alpha})\nabla_\alpha\nabla_\beta$$

$\mu,\nu,\in \{0,\dots,s\}$. The theorem follows then by the same arguments as before, using in particular the symmetry by transposition for the Gorenstein property. □

It follows that $\mathcal{A}(\zeta)$ has Hochschild dimension 3 and the same Poincaré series as the Yang-Mills algebra for $\zeta \neq \zeta^{sing}$.

Remark. One can show that the cubic algebra generated by elements ∇_λ with relations

$$(\zeta_1 g^{\rho\lambda}g^{\mu\nu} + \zeta_2 g^{\nu\rho}g^{\lambda\mu} - 2\zeta_0 g^{\rho\mu}g^{\lambda\nu})\nabla_\lambda\nabla_\mu\nabla_\nu = 0$$

cannot be Koszul and Gorenstein if $\zeta_1 \neq \zeta_2$ and $\zeta_0 \neq 0$ or if $(\zeta_1)^2 \neq (\zeta_2)^2$.

Let now $(B_{\lambda\mu}) \in M_{s+1}(\mathbb{K})$ be an arbitrary invertible $(s+1)\times(s+1)$-matrix with inverse $(B^{\lambda\mu})$, i.e. $B_{\lambda\mu}B^{\mu\nu} = \delta^\nu_\lambda$, and let $\varepsilon = +$ or $-$. We define $\mathfrak{A}(B,\varepsilon)$ to be the cubic algebra generated by $s+1$ elements E_λ with relations

$$(B^{\rho\lambda}B^{\mu\nu} + \varepsilon B^{\lambda\mu}B^{\nu\rho})E_\lambda E_\mu E_\nu = 0 \tag{1.37}$$

for $\rho \in \{0,\dots,s\}$. Notice that B is not assumed to be symmetric. If $B_{\lambda\mu} = g_{\lambda\mu}$ and $\varepsilon = -$ then $\mathfrak{A}(g,-)$ is the super Yang-Mills algebra $\tilde{\mathcal{A}}$ $(E_\lambda \mapsto S_\lambda)$ while if $B_{\lambda\mu} = g_{\lambda\mu}$ and $\varepsilon = +$ then $\mathfrak{A}(g,+)$ is $\mathcal{A}(\zeta^0)$ $(E_\lambda \mapsto \nabla_\lambda)$ where ζ^0 has homogeneous coordinates $\zeta_1 \neq 0$ and $\zeta_0 = 0$. Thus $\mathfrak{A}(B,+)$ and $\mathfrak{A}(B,-)$ belong to deformations of the Yang-Mills and of the super Yang-Mills algebra respectively.

Theorem 5. *Assume that* $1 + \varepsilon B^{\rho\lambda}B^{\mu\nu}B_{\mu\lambda}B_{\rho\nu} \neq 0$, *then* $\mathfrak{A}(B,\varepsilon)$ *is Koszul of global dimension 3 and is Gorenstein.*

Proof. The Koszul complex $\mathcal{K}(\mathfrak{A}(B,\varepsilon),\mathbb{K})$ can be put in the form

$$0 \to \mathfrak{A} \stackrel{E^t}{\to} \mathfrak{A}^{s+1} \stackrel{L}{\to} \mathfrak{A}^{s+1} \stackrel{E}{\to} \mathfrak{A} \to 0$$

where $\mathfrak{A} = \mathfrak{A}(B,\varepsilon)$ and with the previous conventions, the matrix L being given by

$$L^{\mu\nu} = (B^{\mu\alpha}B^{\beta\nu} + \varepsilon B^{\nu\mu}B^{\alpha\beta})E_\alpha E_\beta \tag{1.38}$$

for $\mu,\nu \in \{0,\dots,s\}$. The arrow $\mathfrak{A} \stackrel{E^t}{\to} \mathfrak{A}^{s+1}$ is always injective and the exactness of $\mathfrak{A} \stackrel{E^t}{\to} \mathfrak{A}^{s+1} \stackrel{L}{\to} \mathfrak{A}^{s+1}$ follows from the condition $1+\varepsilon B^{\rho\lambda}B^{\mu\nu}B_{\mu\lambda}B_{\rho\nu} \neq 0$. On the other hand, by definition of $\mathcal{A}$ by generators and relations, the sequence $\mathfrak{A}^{s+1} \stackrel{L}{\to} \mathfrak{A}^{s+1} \stackrel{E}{\to} \mathfrak{A} \stackrel{\varepsilon}{\to} \mathbb{K} \to 0$ is exact. This shows that $\mathfrak{A}$ is Koszul of global dimension 3. The Gorenstein property follows from (see also in [1])

$$B^{\rho\lambda}B^{\mu\nu} + \varepsilon B^{\nu\rho}B^{\lambda\mu} = \varepsilon(B^{\nu\rho}B^{\lambda\mu} + \varepsilon B^{\mu\nu}B^{\rho\lambda})$$

for $\rho,\lambda,\mu,\nu \in \{0,\dots,s\}$. □

Remark. It is worth noticing here in connection with the analysis of [5] that for all the deformations of the Yang-Mills algebra (resp. the super Yang-Mills algebra) considered here which are cubic Koszul Gorenstein algebras of global dimension 3, the dual cubic algebras are Frobenius algebras with structure automorphism equal to the identity (resp. $(-1)^{\text{degree}}\times$ identity).

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