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Gravity coupled with matter and the foundation of non commutative geometry

Alain CONNES

I.H.E.S., 91440 Bures-sur-Yvette, France

Abstract. We first exhibit in the commutative case the simple algebraic relations between the algebra of functions on a manifold and its infinitesimal length element ds. Its unitary representations correspond to Riemannian metrics and Spin structure while ds is the Dirac propagator $ds = \times\!\!-\!\!\times = D^{-1}$ where D is the Dirac operator. We extend these simple relations to the non commutative case using Tomita's involution J. We then write a spectral action, the trace of a function of the length element in Planck units, which when applied to the non commutative geometry of the Standard Model will be shown (in a joint work with Ali Chamseddine) to give the SM Lagrangian coupled to gravity. The internal fluctuations of the non commutative geometry are trivial in the commutative case but yield the full bosonic sector of SM with all correct quantum numbers in the slightly non commutative case. The group of local gauge transformations appears spontaneously as a normal subgroup of the diffeomorphism group.

Riemann's concept of a geometric space is based on the notion of a manifold M whose points $x \in M$ are locally labelled by a finite number of real coordinates $x^\mu \in \mathbb{R}$. The metric data is given by the infinitesimal length element,

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \tag{1}$$

which allows to measure distances between two points x, y as the infimum of the length of arcs $x(t)$ from x to y,

$$d(x,y) = \text{Inf} \int_x^y ds\,. \tag{2}$$

In this paper we shall build our notion of geometry, in a very similar but somehow dual manner, on the pair $(\mathcal{A}, ds)$ of the algebra $\mathcal{A}$ of coordinates and the infinitesimal length element ds. For the start we only consider ds as a symbol, which together with $\mathcal{A}$ generates an algebra $(\mathcal{A}, ds)$. The length element ds does not commute with the coordinates, i.e. with the functions f on our space, $f \in \mathcal{A}$. But it does satisfy non trivial relations. Thus in the simplest case where $\mathcal{A}$ is commutative we shall have,

$$\big[[f, ds^{-1}], g\big] = 0 \qquad \forall\, f, g \in \mathcal{A}\,. \tag{3}$$

The only other relation between ds and $\mathcal{A}$ is of degree n in $ds^{-1} = D$ and expresses the homological nature of the volume form (see axiom 4 below). When $\mathcal{A}$ is commutative it has a spectrum, namely the space of algebra preserving maps from $\mathcal{A}$ to the complex numbers,

$$\begin{gathered} \chi : \mathcal{A} \to \mathbb{C}\ ;\ \chi(a+b) = \chi(a) + \chi(b)\ ,\ \chi(ab) = \chi(a)\,\chi(b) \quad \forall\, a, b \in \mathcal{A}\ , \\ \chi(\lambda a) = \lambda\,\chi(a) \quad \forall\, \lambda \in \mathbb{C}\ ,\ \forall\, a \in \mathcal{A}\,. \end{gathered} \tag{4}$$

For instance, when $\mathcal{A}$ is the algebra of functions on a space M the space of such maps, called *characters* of $\mathcal{A}$ identifies with M. To each point $x \in M$ corresponds the character χ_x,

$$\chi_x(f) = f(x) \qquad \forall\, f \in \mathcal{A}\,. \tag{5}$$

While relations such as (3) between the algebra $\mathcal{A}$ and the length element ds hold at the universal level, a specific geometry will be specified as a *unitary representation* of the algebra generated by $\mathcal{A}$ and ds. In general we shall deal with complex valued functions f so that $\mathcal{A}$ will be endowed with an involution,

$$f \to f^* \tag{6}$$

which is just complex conjugation of functions in the usual case,

$$f^*(x) = \overline{f(x)} \qquad \forall\, x \in M\,. \tag{7}$$

The length element ds will be selfadjoint,

$$ds^* = ds \tag{8}$$

so that $(ds)^2$ will be automatically positive.

The unitarity of the representation just means that the operator $\pi(a^*)$ corresponding to a^* is the adjoint of the operator $\pi(a)$,

$$\pi(a^*) = \pi(a)^* \qquad \forall\, a \in (\mathcal{A}, ds)\,. \tag{9}$$

Given a unitary representation π of $(\mathcal{A}, ds)$ we measure the distance between two points x, y of our space by,

$$d(x,y) = \text{Sup}\left\{|f(x) - f(y)|\ ;\ f \in \mathcal{A}\ ,\ \|[f, ds^{-1}]\| \leq 1\right\} \tag{10}$$

where we dropped the letter π but where the representation π has been used in a crucial way to define the norm of $[f, ds^{-1}]$.

Before we proceed let us first explain what representation of $(\mathcal{A}, ds)$ corresponds to an ordinary Riemannian geometry, and check that (10) gives us back exactly the Riemannian geodesic distance of formula (2). The algebra $\mathcal{A}$ is the algebra of smooth complex valued functions on M, $\mathcal{A} = C^\infty(M)$ and we choose to represent ds as the propagator for fermions which physicists write in a suggestive ways as $\times\!\!-\!\!\times$. This means that we represent $\mathcal{A}$ in the Hilbert space $L^2(M, S)$ of square integrable sections of the spinor bundle on M by the following formula,

$$(f\,\xi)(x) = f(x)\,\xi(x) \qquad \forall\, f \in \mathcal{A} = C^\infty(M)\ ,\ \forall\, \xi \in \mathcal{H} = L^2(M, S) \tag{11}$$

and we represent ds by the formula,

$$ds = D^{-1}\ ,\ D = \frac{1}{\sqrt{-1}}\,\gamma^\mu\,\nabla_\mu \tag{12}$$

where D is the Dirac operator. We ignore the ambiguity of (12) on the kernel of D.

One checks immediately that the commutator $[D, f]$, for $f \in \mathcal{A} = C^\infty(M)$ is the Clifford multiplication by the gradient ∇f so that its operator norm, in $\mathcal{H} = L^2(M, S)$, is given by

$$\|[D, f]\| = \underset{x \in M}{\text{Sup}}\, \|\nabla f\|\,. \tag{13}$$

It then follows by integration along the path from x to y that $|f(x) - f(y)| \leq$ length of path, provided (13) is bounded by 1. Thus the equality between (10) and (2).

Note that while ds has the dimension of a length, $(ds)^{-1}$ which is represented by D has the dimension of a mass. The formula (10) is dual to formula (2). In the usual Riemannian case it gives the same answer but being dual it does not use arcs connecting x with y but rather functions from M to $\mathbb{C}$. As we shall see this will allow to treat spaces with a finite number of points on the same footing as the continuum. Another virtue of formula (10) is that it will continue to make sense when the algebra $\mathcal{A}$ is no longer commutative.

In this paper we shall write down the axioms of geometry as the presentation of the algebraic relations between $\mathcal{A}$ and ds and the representation of those relations in Hilbert space.

In order to compare different geometries, i.e. different representations of the algebra $(\mathcal{A}, ds)$ generated by $\mathcal{A}$ and ds, we shall use the following action functional,

$$\text{Trace}\left(\varphi\left(\frac{ds}{\ell_p}\right)\right) \tag{14}$$

where ℓ_p is the Planck length and φ is a suitable cutoff function which will cut off all eigenvalues of ds larger than ℓ_p.

We shall show in [CC] that for a suitable choice of the algebra $\mathcal{A}$, the above action will give Einstein gravity coupled with the Lagrangian of the standard $U(1) \times SU(2) \times SU(3)$ model of Glashow Weinberg Salam. The algebra will not be $C^\infty(M)$ with M a (compact) 4-manifold but a non commutative refinement of it which has to do with the quantum group refinement of the Spin covering of $SO(4)$,

$$1 \to \mathbb{Z}/2 \to \text{Spin}(4) \to SO(4) \to 1\,. \tag{15}$$

Amazingly, in this description the group of gauge transformations of the matter fields arises spontaneously as a normal subgroup of the generalized diffeomorphism group $\text{Aut}(\mathcal{A})$.

It is the *non commutativity* of the algebra $\mathcal{A}$ which gives for free the group of gauge transformations of matter fields as a (normal) subgroup of the group of diffeomorphisms. Indeed, when $\mathcal{A} = C^\infty(M)$ is the commutative algebra of smooth functions on M one easily checks that the following defines a one to one correspondence between *diffeomorphisms* φ of M and the *automorphisms* $\alpha \in \mathrm{Aut}(\mathcal{A})$ of the algebra $\mathcal{A}$ (preserving the $*$),

$$\alpha(f)\,(x) = f(\varphi^{-1}(x)) \qquad \forall\, x \in M \ , \ f \in C^\infty(M)\,. \tag{16}$$

In the non commutative case an algebra always has automorphisms, the *inner automorphisms* given by

$$\alpha_u(f) = u\,f\,u^* \qquad \forall\, f \in \mathcal{A} \tag{17}$$

for any element of the unitary group $\mathcal{U}$ of $\mathcal{A}$,

$$\mathcal{U} = \{u \in \mathcal{A} \ ; \ u\,u^* = u^*\,u = 1\}\,. \tag{18}$$

The subgroup $\mathrm{Int}(\mathcal{A}) \subset \mathrm{Aut}(\mathcal{A})$ of inner automorphisms is a normal subgroup and it will provide us with our group of internal gauge transformations. It is a happy coincidence that the two terminologies : inner automorphisms and internal symmetries actually match. Both groups will have to be lifted to the spinors but we shall see that later. We shall see that the action of inner automorphisms on the metric gives rise to *internal fluctuations* of the latter which replace D by $D + A + J\,A\,J^{-1}$ (see below) and give exactly the gauge bosons of the standard model, with its symmetry breaking Higgs sector, when we apply it to the above finite geometry. We had realized in [Co2] that the fermions were naturally in the adjoint representation of the unitary group $\mathcal{U}$ in the case of the standard model, i.e. that the action of $\mathcal{U}$ on $\mathcal{H}$ was,

$$\xi \to u\,J\,u\,J^{-1}\,\xi = u\,\xi\,u^*\,.$$

But we had not understood the significance of the inner automorphisms as internal diffeomorphisms of the non commutative space. What the present paper shows is that one should consider the internal gauge symmetries as part of the diffeomorphism group of the non commutative geometry and the gauge bosons as the internal fluctuations of the metric. We should thus expect that the action functional should be of purely gravitational nature. The above spectral action when restricted to these special metrics will give the interaction Lagrangian of the bosons. The interaction with Fermions, which has the right hypercharge assignement is obtained directly from $\langle \psi, D\psi \rangle$ (with $D + A + J\,A\,J^{-1}$ instead of D), and is thus also of spectral nature being invariant under the full unitary group of operators in Hilbert space. Thus the only distinction that remains between matter and gravity is the distinction $\mathrm{Int}\,\mathcal{A} \neq \mathrm{Aut}\,\mathcal{A}$ which vanishes for a number of highly non commutative algebras.

Quantized calculus.

In order to ease the presentation of our axioms let us recall the first few lines of the dictionary of the quantized calculus which makes use of the formalism of quantum mechanics to formulate a new theory of infinitesimals. The first two lines of this dictionary are just the traditional way of interpreting the observables in quantum mechanics,

Classical	**Quantum**
Complex variable	Operator in Hilbert space
Real variable	Selfadjoint operator, $T = T^*$
Infinitesimal variable	Compact operator
Infinitesimal of order α	$\mu_n(T) = 0(n^{-\alpha})$
Integral	$⨍ T = \log$ divergence of the trace of T

We recall briefly that an operator T in Hilbert space is *compact* iff for any $\varepsilon > 0$ one has $\Vert T \Vert < \varepsilon$ except on a finite dimensional subspace of $\mathcal{H}$. More precisely

(19) $$\forall\,\varepsilon \text{ there exists a finite dimensional subspace } E \text{ of } \mathcal{H}$$
$$\text{such that } \Vert T/E^{\perp} \Vert < \varepsilon \quad \text{where } E^{\perp} \text{ is the orthogonal of } E\,.$$

The size of a compact operator T is measured by the decreasing sequence μ_n of eigenvalues of $|T| = \sqrt{T^*\,T}$. The order of such an "infinitesimal" is measured by the rate at which these *characteristic values* $\mu_n(T)$ converge to 0 when $n \to \infty$.

One can show that all the intuitive rules of calculus are valid, e.g. the order of $T_1 + T_2$ or of $T_1\,T_2$ are as they should be ($\leq \alpha_1 \vee \alpha_2$ and $\alpha_1\,\alpha_2$). Moreover the trace (i.e. the sum of the eigenvalues) is logarithmically divergent for infinitesimals of order 1, since $\mu_n(T) = 0\left(\frac{1}{n}\right)$. It is a quite remarkable fact that the coefficient of the logarithmic divergency does yield an *additive trace* which in essence evaluates the "classical part" of such infinitesimals. This trace, denoted $⨍$ vanishes on infinitesimals of order $\alpha > 1$.

The only rule of the naive calculus of infinitesimals which is not fulfilled is commutativity but this lack of commutativity is crucial to allow the coexistence of variables with continuous range with infinitesimals which have discrete spectrum.

We refer to [Co] for more details on this calculus and its applications.

Axioms for commutative geometry.

Let us now proceed and write down the axioms for *commutative* geometry. The small modifications required for the non commutative case will only be handled later.

Thus $\mathcal{H}$ is a Hilbert space, $\mathcal{A}$ an involutive algebra represented in $\mathcal{H}$ and $D = ds^{-1}$ is a selfadjoint operator in $\mathcal{H}$.

We are given an integer n which controls the dimension of our space by the condition,

1) *$ds = D^{-1}$ is an infinitesimal of order $\frac{1}{n}$.*

The universal commutation relation (3) is represented by

2) $[[D, f], g] = 0 \qquad \forall\, f, g \in \mathcal{A}$.

We assume that the simple commutation with $|D|, \delta(T) = [|D|, T]$ will only yield bounded operators when we start with any $f \in \mathcal{A}$. More precisely we assume that:

3) *(Smoothness) For any $a \in \mathcal{A}$ both a and $[D, a]$ belong to the domain of δ^m, for any integer m.*

This axiom 3 is the algebraic formulation of smoothness of the coordinates.

The next axiom is depending upon the parity of n, thus let us state it first when n is even. It yields the γ_5 matrix which is abstracted here as a $\mathbb{Z}/2$ grading of the Hilbert space $\mathcal{H}$:

4) *(Orientability) (n even) There exists a Hochschild cycle $c \in Z_n(\mathcal{A}, \mathcal{A})$ such that $\pi(c) = \gamma$ satisfies*

$$\gamma = \gamma^* \ , \ \gamma^2 = 1 \ , \ \gamma D = -D\gamma \, .$$

In the odd case (n odd) one just asks that $1 = \pi(c)$ for some n-dimensional Hochschild cycle $c \in Z_n(\mathcal{A}, \mathcal{A})$.

We need to explain briefly what is a Hochschild cycle. Conceptually it is the algebraic formulation of a differential form, so that axiom 4 is really providing us with the volume form. Concretely an n-dimensional Hochschild cycle is a finite sum of elements of $\mathcal{A} \otimes \mathcal{A} \otimes \ldots \otimes \mathcal{A}$ (with $n+1$ times $\mathcal{A}$),

$$c = \sum_j a_j^0 \otimes a_j^1 \otimes a_j^2 \ldots \otimes a_j^n$$

such that the contraction $bc = 0$ where by definition b is linear and satisfies

$$b(a^0 \otimes a^1 \otimes \ldots \otimes a^n) = a^0 a^1 \otimes a^2 \otimes \ldots \otimes a^n - a^0 \otimes a^1 a^2 \otimes \ldots \otimes a^n$$
$$+ \ldots + (-1)^k a^0 \otimes \ldots \otimes a^k a^{k+1} \otimes \ldots \otimes a^n + \ldots + (-1)^n a^n a^0 \otimes \ldots \otimes a^{n-1} .$$

When $\mathcal{A}$ is commutative it is easy to construct a Hochschild cycle, it suffices to take any a^j and consider

$$c = \sum \varepsilon(\sigma)\, a^0 \otimes a^{\sigma(1)} \otimes a^{\sigma(2)} \otimes \ldots \otimes a^{\sigma(n)}$$

where the sum runs over all the permutations σ of $\{1, \ldots, n\}$.

This special construction corresponds to the familiar differential form $a^0 da^1 \wedge da^2 \wedge \ldots \wedge da^n$ but does not require the previous knowledge of the tangent bundle.

Finally $\pi(c)$ is the representation of c on $\mathcal{H}$ induced by

$$\pi(a^0 \otimes a^1 \otimes \ldots \otimes a^n) = a^0 [D, a^1] \ldots [D, a^n] .$$

To understand the meaning of axiom 4 let us take its simplest instance, with $n = 1$ and $\mathcal{A} = C^\infty(S^1)$ generated by a single unitary element $U \in \mathcal{A}$, $U\,U^* = U^*\,U = 1$. Let $c = U^{-1} \otimes U$ one checks that $bc = U^{-1}\,U - U\,U^{-1} = 0$ so that c is a Hochschild cycle, $c \in Z^1(\mathcal{A}, \mathcal{A})$. Then the condition $\pi(c) = 1$ reads:

$$[D, U] = U .$$

The reader will check at this point that this relation alone completely describes the geometry of the circle, by computing explicitly the distance between points using formula (10). One finds the usual metric on the circle with length 2π.

The next axiom will be the axiom of *finiteness*,

5) *(Finiteness and absolute continuity.) Viewed as an $\mathcal{A}$-module the space* $\mathcal{H}_\infty = \bigcap_m \text{Domain}\, D^m$ *is* finite and projective. *Moreover the following equality defines a hermitian structure (,) on this module,*

$$\langle a\,\xi, \eta \rangle = {\textstyle\int\!\!\!\!-}\, a(\xi, \eta)\, ds^n \qquad \forall\, a \in \mathcal{A} \;,\; \forall\, \xi, \eta \in \mathcal{H}_\infty .$$

We recall from [Co] that a hermitian structure on a finite projective $\mathcal{A}$-module is given by an $\mathcal{A}$-valued inner product. The prototype of such a module with inner product is the following, one lets $e \in \text{Proj}\, M_q(\mathcal{A})$ be a selfadjoint idempotent,

$$e = e^* \;,\; e^2 = e$$

and one lets $\mathcal{E}$ be the left module $\mathcal{A}^n\, e = \{(\xi_j)_{j=1,\ldots,n}\,;\, \xi_j \in \mathcal{A}\,,\, \xi_j\, e_{jk} = \xi_k\}$ where the action of $\mathcal{A}$ is given by left multiplication,

$$(a\,\xi)_j = a\,\xi_j \qquad \forall\, j = 1, \ldots, n\,.$$

The $\mathcal{A}$-valued inner product is then given by:

$$(\xi, \eta) = \Sigma\, \xi_i\, \eta_i^*\,.$$

It follows from axiom 4) and from a general theorem ([Co]) that the operators $a\, ds^n$, $a \in \mathcal{A}$ are *measurable* (cf. [Co]) so that the coefficient $⨍\, a\, ds^n$ of the logarithmic divergence of their trace is unambiguously defined.

It follows from axiom 5) that the algebra $\mathcal{A}$ is uniquely determined inside its weak closure $\mathcal{A}''$ (which is also the bicommutant of $\mathcal{A}$ in $\mathcal{H}$) by the equality

$$\mathcal{A} = \{T \in \mathcal{A}''\ ;\ T \in \bigcap_{m>0} \operatorname{Dom} \delta^m\}\,.$$

This shows that the whole geometric data $(\mathcal{A}, \mathcal{H}, D)$ is in fact uniquely determined by the triple $(\mathcal{A}'', \mathcal{H}, D)$ where $\mathcal{A}''$ is a commutative von Neumann algebra.

This also shows that $\mathcal{A}$ is a pre C^* algebra, i.e. that it is stable under the C^∞ functional calculus in the C^* algebra norm closure of $\mathcal{A}$, $A = \overline{\mathcal{A}}$. Since we assumed that $\mathcal{A}$ was commutative, so is A and by Gelfand's theorem $A = C(X)$ is the algebra of continuous complex valued functions on $X = \text{Spectrum}\,(A)$. We note finally that characters χ of $\mathcal{A}$ extend automatically to A by continuity so that

$$\text{Spectrum}\, A = \text{Spectrum}\, \mathcal{A}\,.$$

We let $K_i(\mathcal{A})$, $i = 0, 1$ be the K-groups of $\mathcal{A}$ (or equivalently of A or of X). Thus $K_0(\mathcal{A})$ classifies finite projective modules over $\mathcal{A}$ (or equivalently vector bundles over X). Similarly $K_1(\mathcal{A})$ is the group of connected components of $GL_\infty(\mathcal{A})$, i.e. $K_1(\mathcal{A}) = \pi_0\, GL_\infty(\mathcal{A})$, while by the Bott periodicity theorem one has,

$$\pi_n(GL_\infty(\mathcal{A})) \cong K_{n+1}(\mathcal{A})$$

where $n+1$ only matters modulo 2.

One easily defines the index pairing of the operator D with $K_n(\mathcal{A})$ where again n only matters modulo 2. In the even case one uses γ to decompose D as $D = D^+ + D^-$ where $D^+ = (1-p)\, Dp$, $p = \frac{1+\gamma}{2}$. Then for any selfadjoint idempotent

$e \in \mathcal{A}$ the operator $e\,D^+\,e$ is a Fredholm operator from the subspace $e\,p\,\mathcal{H}$ of $\mathcal{H}$ to the subspace $e(1-p)\mathcal{H}$. This extends immediately to projections $e \in M_q(\mathcal{A})$ for some integer q, and gives an additive map from $K_0(\mathcal{A})$ to $\mathbb{Z}$ denoted $\langle \mathrm{Ind}\, D, e\rangle$. A similar discussion applies in the odd case ([]).

Since $\mathcal{A}$ is commutative we have the diagonal map,

$$\mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A}\,,\ m(x \otimes y) = xy \qquad \forall\, x, y \in \mathcal{A}$$

which yields a corresponding map $m_* : K_*(\mathcal{A}) \times K_*(\mathcal{A}) \to K_*(\mathcal{A})$. Composing this map with $\mathrm{Ind}\, D$ we obtain the intersection form,

$$K_*(\mathcal{A}) \times K_*(\mathcal{A}) \to \mathbb{Z}\,,$$

$$(e, f) \to \langle \mathrm{Ind}\, D\ ,\ m_*(e \otimes f)\rangle\,.$$

It clearly only depends upon the stable homotopy class of the representation π and thus gives a very rough information on π. We shall assume, (Poincaré duality)

6) *The intersection form $K_*(\mathcal{A}) \times K_*(\mathcal{A}) \to \mathbb{Z}$ is invertible.*

If one wants to take in account the possible presence of torsion in the K-groups one should formulate Poincaré duality as the isomorphism,

$$K_*(\mathcal{A}) \xrightarrow{\cap\mu} K^*(\mathcal{A})$$

given by the Kasparov intersection product with the class μ of the Fredholm module $(\mathcal{H}, D, \gamma)$ over $\mathcal{A} \otimes \mathcal{A}$. We refer to [Co] [C] for the detailed formulation, but 6) will suffice for theorem 1 below.

Thanks to the work of D. Sullivan [S] one knows that the above Poincaré duality isomorphism in K theory is (in the simply connected case and ignoring 2-torsion) a characterization of the homotopy type of spaces which possess a structure of smooth manifold. This requires the use of real K theory instead of the above complex K theory and our last axiom will be precisely the existence of such a real structure on our cycle.

7) **Reality.** *There exists an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ such that $J\,a\,J^{-1} = a^* \quad \forall\, a \in \mathcal{A}$ and $J^2 = \varepsilon$, $J\,D = \varepsilon'\,D\,J$ and $J\,\gamma = \varepsilon''\,\gamma\,J$ where $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, +1\}$ are given by the following table from the value of n modulo 8,*

$n =$	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

We can now state our first result,

Theorem. *Let $\mathcal{A} = C^\infty(M)$ where M is a smooth compact manifold of dimension n.* a) *Let π be a unitary representation of $(\mathcal{A}, ds)$ satisfying the above seven axioms, then there exists a unique Riemannian metric g on M such that the geodesic distance between any two points $x, y \in M$ is given by*

$$d(x,y) = \mathrm{Sup}\,\{|a(x) - a(y)| \ ; \ a \in \mathcal{A} \ , \ \|[D,a]\| \leq 1\}\,.$$

b) *The metric $g = g(\pi)$ only depends upon the unitary equivalence class of π and the fibers of the map : $\pi \to g(\pi)$ from unitary equivalence classes to metrics form a finite collection of affine space $\mathcal{A}_\sigma$ parametrized by the Spin structures σ on M.*

c) *The action functional $⨍ ds^{n-2}$ is a positive quadratic form on each $\mathcal{A}_\sigma$ with a unique minimum π_σ.*

d) *π_σ is the representation of $(\mathcal{A}, ds)$ in $L^2(M, S_\sigma)$ given by multiplication operators and the Dirac operator of the Levi Civita Spin connection.*

e) *The value of $⨍ ds^{n-2}$ on π_σ is given by the Einstein Hilbert action,*

$$-c_n \int R\,\sqrt{g}\, d^n x \quad , \quad c_n = (n-2)/12 \times (4\pi)^{-n/2}\,\Gamma\left(\frac{n}{2}+1\right)^{-1}\, 2^{[n/2]}\,.$$

Let us make a few remarks about this theorem.

1) Note first that none of the axioms uses the fact that $\mathcal{A}$ is the algebra of smooth functions on a manifold. In fact one should deduce from the axioms that the spectrum X of $\mathcal{A}$ is a smooth manifold and that the map $X \to \mathbb{R}^N$, given by the finite collection a_j^i of elements of $\mathcal{A}$ involved in the Hochschild cycle c of axiom 4, is actually a smooth embedding of X as a submanifold of $\mathbb{R}^N$. To prove this one should use [Co], proposition 15, p.312.

2) The two axioms 2 and 3 should be considered as the presentation of the algebra $(\mathcal{A}, ds)$. Thus this presentation involves an explicit Hochschild n-cycle $c \in Z_n(\mathcal{A}, \mathcal{A})$ and for n odd gives the relation,

$$\Sigma\, a_0^j [D, a_1^j] \ldots [D, a_n^j] = 1 \,.$$

In the even case the relations are $D\,\gamma = -\gamma\, D$ and $\gamma^2 = 1$, $\gamma = \gamma^*$, with

$$\gamma = \Sigma\, a_0^j [D, a_1^j] \ldots [D, a_u^j] \,.$$

It is thus natural to fix c as part of the data. The differential form on $M =$ Spectrum $\mathcal{A}$ associated to the Hochschild n-cycle c is equal to the volume form of the metric g. Thus fixing c determines the volume form of the metric.

3) The sign in front of the Einstein Hilbert action given in (a) is the correct one for the Euclidean formulation of gravity and for instance in the $n = 4$ dimensional case the Einstein Hilbert action becomes the *area*

$$\ell_p^{-2} \textstyle\fint ds^2$$

of our space, in units of Planck length ℓ_p. Thus the only negative sign in the second derivative of $\fint ds^2$ around flat space comes from the Weyl factor which is determined by the choice of c. We refer to [K] and [KW] for the detailed calculation.

4) When M is a Spin manifold the map $\pi \to g(\pi)$ is surjective and if we fix c it surjects to the metrics g with fixed volume form. (This amounts to checking that all axioms are fulfilled.)

5) If we drop axiom 7 there is a completely similar result as theorem 1 where Spin is replaced by Spin^c (cf. [ML]) and where uniqueness is lost in c) and the minimum of the action $\int ds^{n-2}$ is reached on a linear subspace of $\mathcal{A}_\sigma$ with σ a fixed Spin^c structure. The elements of this subspace correspond exactly to the $U(1)$ gauge potentials involved in the Spin^c Dirac operator and d) e) continue to hold.

6) A commutative geometry $(\mathcal{A}, \mathcal{H}, D)$ is connected iff the only operators commuting with $\mathcal{A}$ and D are the scalars, i.e. iff the representation π of $(\mathcal{A}, ds)$ in $\mathcal{H}$ is irreducible.

The axioms for non commutative geometry.

We are now ready to proceed to the non commutative case, i.e. where we no longer assume that the algebra $\mathcal{A}$ is commutative. Among the axioms we wrote 1) 3) 5) will remain unchanged while the others will be slightly modified. One of the most significant results of the theory of operator algebras is Tomita's theorem [Ta] which asserts that for any weakly closed $*$ algebra of operators M in Hilbert space $\mathcal{H}$ which admits a cyclic and separating vector, there exists a canonical antilinear isometric involution J from $\mathcal{H}$ to $\mathcal{H}$ such that

$$J\,M\,J^{-1} = M'$$

where $M' = \{T\ ;\ T\,a = a\,T \quad \forall\, a \in M\}$ is the commutant of M. It follows then that M is antiisomorphic to its commutant, the antiisomorphism being given by the $\mathbb{C}$-linear map,

$$a \in M \to J\,a^*\,J^{-1} \in M'\,.$$

Now axiom 7 already involves an antilinear isometry J in $\mathcal{H}$, which in the usual geometric case is the charge conjugation. Since we assumed that $\mathcal{A}$ was commutative the equality $J\,a^*\,J^{-1} = a$ of axiom 7 is compatible with Tomita's antiisomorphism. In general we shall replace the requirement $J\,a^*\,J^{-1} = a \quad \forall\, a \in \mathcal{A}$ of axiom 7 by the following,

7′) One has $[a, b^0] = 0 \quad \forall\, a, b \in \mathcal{A}$ where $b^0 = J\,b^*\,J^{-1}$ (and $[a, b^0]$ is the commutator $ab^0 - b^0 a$).

Otherwise we leave 7) unchanged. This immediately turns the Hilbert space $\mathcal{H}$ into a module over the algebra $\mathcal{A} \otimes \mathcal{A}^0$ which is the tensor product of $\mathcal{A}$ by its opposite algebra $\mathcal{A}^0$. One lets,

$$(a \otimes b^0)\,\xi = a\,J\,b^*\,J^{-1}\,\xi \qquad \forall\, a, b \in \mathcal{A}\,.$$

(One can use equivalently the terminology of bimodules or correspondences (cf. [Co]).)

Thus axiom (7′) now gives a KR-homology class for the algebra $\mathcal{A} \otimes \mathcal{A}^0$ endowed with the antilinear automorphism,

$$\tau(x \otimes y^0) = y^* \otimes x^{*0} \qquad \forall\, x, y \in \mathcal{A}$$

and we do not need the diagonal map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ (which is an algebra homomorphism only in the commutative case) to formulate Poincaré duality,

$(6')$ *The cup product by* $\mu \in KR^n(\mathcal{A} \otimes \mathcal{A}^0)$ *is an isomorphism*

$$K_*(\mathcal{A}) \xrightarrow{\cap \mu} K^*(\mathcal{A}) .$$

We also note that the intersection form on $K_*(\mathcal{A})$ continues to be well defined. Given $e, f \in K_*(\mathcal{A})$ one considers $e \otimes f^0$ as an element of $K_*(\mathcal{A} \otimes \mathcal{A}^0)$ and evaluates

$$\langle e, f \rangle = \langle \text{Index } D \ , \ e \otimes f^0 \rangle .$$

From the table of commutation of axiom 7 one gets for instance that this intersection form is symplectic for $n = 2$ or 6 and quadratic for $n = 0$ or 4 modulo 8 as in the usual case.

We shall modify axiom 2 in the following way,

$$(2') \qquad [[D, a], b^0] = 0 \qquad \forall \, a, b \in \mathcal{A} .$$

Note that by $(7')$ a and b^0 commute so that the formulation of $(2')$ is symmetric, i.e. it is equivalent to

$$[[D, b^0], a] = 0 \qquad \forall \, a, b \in \mathcal{A} .$$

Finally we shall slightly modify (4). Of course Hochschild homology continues to make sense in the non commutative case.

$(4')$ *There exists a Hochschild cycle* $c \in Z_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^0)$ *such that* $\gamma = \pi(c)$ *satisfies* $\gamma = \gamma^*$, $\gamma^2 = 1$, $\gamma \, a = a \, \gamma \quad \forall \, a \in \mathcal{A}$, $\gamma \, D = - D \, \gamma$. *(In the odd case we simply require* $\pi(c) = 1$.*)*

We view $\mathcal{A} \otimes \mathcal{A}^0$ as a bimodule over $\mathcal{A}$ by restricting to the subalgebra $\mathcal{A} \otimes 1 \subset \mathcal{A} \otimes \mathcal{A}^0$ the natural structure of $\mathcal{A} \otimes \mathcal{A}^0$ bimodules on $\mathcal{A} \otimes \mathcal{A}^0$. This gives

$$a (b \otimes c^0) \, d = a \, b \, d \otimes c^0 \qquad \forall \, a, b, c, d \in \mathcal{A} .$$

Note that the Hochschild homology makes sense with coefficients in any bimodule. Since we have a representation of $\mathcal{A} \otimes \mathcal{A}^0$ in $\mathcal{H}$, $\pi(c)$ continues to make sense.

The axioms (3) and (5) are unchanged in the non commutative case, and the proof of the measurability of the operators $a \, ds^n$ for any $a \in \mathcal{A}$ continues to hold in that generality.

Thus a *non commutative geometry* of dimension n is given by a triple $(\mathcal{A}, \mathcal{H}, D)$ with real structure J satisfying (1) $(2')$ (3) $(4')$ (5) $(6')$ $(7')$.

Examples and internal diffeomorphisms.

Let us give some simple examples. Let us first assume that $\mathcal{A}$ is the finite dimensional commutative algebra $\mathcal{A} = \mathbb{C}^N$. Even though $\mathcal{A}$ is commutative it admits non trivial geometries and these are classified, up to homotopy of the operator D, by integral quadratic forms $q = (q_{ij})_{i,j\in 1,\ldots,N}$ which are invertible over $\mathbb{Z}$. The geometry associated to a quadratic form q is defined as follows. One lets $\mathcal{H} = \bigoplus_{i,j} \mathcal{H}_{ij}$ where $\mathcal{H}_{ij} = \mathbb{C}^{n_{ij}}$, $n_{ij} = |q_{ij}|$. The left action of $\mathcal{A}$ in $\mathcal{H}$ is given, with obvious notations by

$$(a\,\xi)_{ij} = a_i\,\xi_{ij} \qquad \forall\, a \in \mathcal{A}\ ,\ \xi \in \mathcal{H}\,.$$

The $\mathbb{Z}/2$ grading γ is given by

$$(\gamma\,\xi)_{ij} = \gamma_{ij}\,\xi_{ij} \qquad \gamma_{ij} = \text{Sign}\ q_{ij}$$

and the isometric antilinear involution J is given by,

$$(J\,\xi)_{ij} = \overline{\xi}_{ji}\,.$$

(It makes sense since $q_{ij} = q_{ji}$.)

One has $K_0(\mathcal{A}) = \mathbb{Z}^n$ and the intersection pairing is given by the quadratic form q.

A choice of D is determined by a function m on the edges of the graph Γ defined as follows. The vertices of Γ are the (i,j) with $q_{ij} \neq 0$, the edges of Γ are the (i,j), (k,ℓ) for which $\gamma_{ij}\,\gamma_{k\ell} = -1$ and $i = k$ or $j = \ell$. The function m on $\Gamma^{(1)}$ has to satisfy $m_{ij,k\ell} = \overline{m}_{k\ell,ij}$ and $m_{ij,k\ell} = \overline{m}_{ji,\ell k}$. One can find D so that the obtained geometry is connected iff the above graph is connected.

As a next simple example let $\mathcal{A} = C^\infty(M, M_k(\mathbb{C}))$ be the algebra of $k \times k$ matrix valued functions on a smooth compact Spin manifold M. Let g be a Riemannian metric on M and $\mathcal{H} = L^2(M, S \otimes M_k(\mathbb{C}))$ be the Hilbert space of L^2 sections of the tensor product $S \otimes M_k(\mathbb{C})$ of the Spinor bundle S by $M_k(\mathbb{C})$. Then $\mathcal{A}$ acts on $\mathcal{H}$ by left multiplication,

$$(a\,\xi)(x) = a(x)\,\xi(x) \qquad \forall\, x \in M\,.$$

The real structure J is given by

$$J = C \otimes *$$

where C is the charge conjugation on spinors and $*$ is the adjoint operation $T \to T^*$ on matrices. This operation transforms the left multiplication operators $\xi \to a\,\xi$ on matrices into right multiplication operators $\xi \to \xi\, a^*$ so that one checks axiom $(7')$,

$$[a, b^0] = 0 \qquad \forall\, a, b \in \mathcal{A}\ ,\ b^0 = J\, b^*\, J^{-1}\,.$$

As a first choice of D one can take $D = \partial\!\!\!/_M \otimes 1$, the tensor product of the Dirac operator $\partial\!\!\!/_M$ on M by the identity on $M_k(\mathbb{C})$. One can then easily check that one obtains a non commutative geometry in this way. This example is relevant to illustrate two facts which hold in general in the non commutative case. The first is that the group $\mathrm{Aut}(\mathcal{A})$ of $*$ automorphisms of the algebra $\mathcal{A}$, which plays in general the role of the group $\mathrm{Diff}(M)$ of diffeomorphisms of the manifold M and acts on the space of representations π by composition, has a natural normal subgroup

$$\mathrm{Int}\, M \subset \mathrm{Aut}\, M$$

where $\mathrm{Int}\, M$ is the group of inner automorphisms, i.e. automorphisms of the form,

$$\alpha_u(x) = u\, x\, u^* \qquad \forall\, x \in \mathcal{A}\,,$$

where u is an arbitrary element of the unitary group

$$\mathcal{U} = \{u \in \mathcal{A}\ ;\ u\, u^* = u^*\, u = 1\}\,.$$

Moreover the action of this group of *internal diffeomorphisms* on our geometries can be expressed in a simple manner. Indeed the replacement of the representation π by $\pi \circ \alpha_u^{-1}$ is equivalent to the replacement of the operator D by

$$\widetilde{D} = D + A + J\, A\, J^{-1}$$

where $A = u[D, u^*]$.

The desired unitary equivalence is given by the operator,

$$U = u\, J\, u\, J^{-1}\,.$$

(One checks that $U\, D\, U^* = \widetilde{D}$ and that $U\, \alpha_u^{-1}(x)\, U^* = x$ for any $x \in \mathcal{A}$.)

The above perturbation of the metric is a special case of *internal perturbation* given by,

$$\widetilde{D} = D + A + J\, A\, J^{-1}$$

where $A = A^*$ is now an arbitrary operator of the form

$$A = \Sigma\, a_i\, [D, b_i] \qquad a_i, b_i \in \mathcal{A}\,.$$

In the commutative case such perturbations of the metric all vanish because $A = A^*$ implies $A + J\,A\,J^{-1} = 0$. But in the non commutative case they do not. Thus in our example with $k > 1$ one gets that,

$$\mathrm{Int}\, M = C^\infty(M, SU(k))$$

is the group of local gauge transformations (internal symmetries) for an $SU(k)$ gauge theory on M. The internal perturbations of the metric are parametrized by $SU(k)$ gauge potentials, i.e. by the non tracial part of $A = \Sigma\, a_i\, [D, b_i]$, $a_i, b_i \in \mathcal{A}$. One can compute the effect of such internal perturbations to the distance between two pure states φ, ψ on $\mathcal{A}$. We take $x, y \in M$ and let φ and ψ correspond to two unit vectors (rays) in $\mathbb{C}^k$ by the equality

$$\varphi(a) = \langle a(x)\, \xi, \xi \rangle \ , \ \psi(a) = \langle a(y)\, \eta, \eta \rangle \qquad \forall\, a \in \mathcal{A}\,.$$

The distance $d(\varphi, \psi)$ for the metric $\widetilde{D}$ is defined as usual by

$$d(\varphi, \psi) = \mathrm{Sup}\, \{ |\varphi(a) - \psi(a)| \ ; \ a \in \mathcal{A} \ , \ \| [\widetilde{D}, a] \| \leq 1 \}\,.$$

This distance depends heavily on the gauge connection A. The latter defines an horizontal distribution H on the fibre bundle P over M whose fiber over each $x \in M$ is the pure state space $P_{k-1}(\mathbb{C})$ of $M_k(\mathbb{C})$. The metric d turns out to be equal to the Carnot metric ([G]) on P defined by the horizontal distribution H and the Euclidean structure on H given by the Riemannian metric of M,

$$d(\varphi, \psi) = \mathrm{Inf} \int_0^1 \| \pi_*(\dot\gamma(t)) \|\, dt \ , \ \gamma(0) = \varphi \ , \ \gamma(1) = \psi$$

where $\pi : P \to M$ is the projection and γ varies through all *horizontal* paths ($\dot\gamma(t) \in H \quad \forall\, t$) which join φ to ψ. In particular the finiteness of $d(\varphi, \psi)$ for $x = y$ is governed by the holonomy of the connection at the point x. The flat situation $A = 0$ corresponds to the product geometry of M by the finite geometry where the algebra is $M_N(\mathbb{C})$ and $D = 0$. For the latter since $D = 0$ the distance between any two $\varphi \neq \psi$ is $+\infty$. Thus for the product one gets that the connected components of the metric topology are the flat sections of the bundle P. Alternatively, if the holonomy at x is $SU(k)$, the metric on the fiber P_x is finite.

Our next example will be highly non commutative. It is a special case of a general construction that works for any isometry of a Riemannian Spin manifold, but we shall work it out in the very specific case where this isometry is the irrational rotation R_θ of the circle S^1. This will have the advantage of definitness but the whole discussion is general. The algebra is the irrational rotation smooth algebra,

$$\mathcal{A}_\theta = \{\Sigma\, a_{nm}\, U^n\, V^m \ ;\ a = (a_{n,m}) \in \mathcal{S}(\mathbb{Z}^2)\}$$

where $\mathcal{S}(\mathbb{Z}^2)$, the Schwartz space of $\mathbb{Z}^2$, is the space of sequences of rapid decay (i.e. $(1+|n|+|m|)^k\, a_{n,m}$ is bounded for any $k \geq 0$). The $*$ algebra structure is governed by the presentation,

$$U^* = U^{-1}\ ,\ V^* = V^{-1}\ ,\ V\,U = \lambda\,U\,V \text{ with } \lambda = \exp(2\pi i\theta)$$

with $\theta \in]0,1]$.

In order to specify the metric structure of our non commutative geometry we shall need, as for usual elliptic curves, a complex number τ, $\Im\tau > 0$. We can then describe the geometry as follows: the Hilbert space $\mathcal{H}$ is given by the sum of 2-copies of $L^2(\mathcal{A}_\theta, \tau_0)$ where τ_0 is the canonical normalized trace,

$$\tau_0(a) = a_{00} \qquad \forall\, a \in \mathcal{A}_\theta\,.$$

The Hilbert space $L^2(\mathcal{A}_\theta, \tau_0)$ is just the completion of $\mathcal{A}_\theta$ for the inner product $\langle a,b\rangle = \tau_0(b^*\,a) \quad \forall\, a,b \in \mathcal{A}_\theta$.

The representation of $\mathcal{A}_\theta$ in $\mathcal{H}$ is given by left multiplication, i.e. as 2 copies of the left regular representation,

$$a \to \begin{bmatrix} \lambda(a) & 0 \\ 0 & \lambda(a) \end{bmatrix}\ ,\ \lambda(a)\,b = a\,b \qquad \forall\, a,b \in \mathcal{A}_\theta\,.$$

The operator D depends explicitly on the choice of τ and is

$$D = \begin{bmatrix} 0 & \delta_1 + \tau\,\delta_2 \\ -\delta_1 - \overline{\tau}\,\delta_2 & 0 \end{bmatrix}$$

where the δ_j are the following derivations of $\mathcal{A}_\theta$,

$$\delta_1(U) = 2\pi i\,U\ ,\ \delta_1(V) = 0\ ;\ \delta_2(U) = 0\ ,\ \delta_2(V) = 2\pi i\,V\,.$$

The $\mathbb{Z}/2$ grading γ is just $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

To specify our geometry it remains to give the antilinear isometry. We let $J = \begin{bmatrix} 0 & J_0 \\ -J_0 & 0 \end{bmatrix}$ where J_0 is Tomita's involution ([Ta]),

$$J_0\, a = a^* \qquad \forall\, a \in L^2(\mathcal{A}_\theta, \tau_0)\,.$$

Note that $J_0\, \lambda(a^*)\, J_0^{-1} = \rho(a)$ is the **right** multiplication by $a \in \mathcal{A}_\theta$ so that our axiom $(7')$ is easy to check. The dimension n is equal to 2 so that $J\,D = D\,J$, $J^2 = -1$ and $J\,\gamma = -\gamma\, J$ as expected.

The other axioms are easy to check but axiom $(4')$ is not so trivial because it requires to exhibit a Hochschild 2-cycle, $c \in Z_2(\mathcal{A}_\theta, \mathcal{A}_\theta)$ such that $\pi(c) = \gamma$. It turns out to be,

$$c = (2i\pi)^{-2}\, (\tau - \overline{\tau})^{-1}\, (V^{-1}\, U^{-1} \otimes U \otimes V - U^{-1}\, V^{-1} \otimes V \otimes U)\,.$$

(Note that this is not the same as the antisymmetrization of $V^{-1}\, U^{-1} \otimes U \otimes V$ because of the phase factor in $V^{-1}\, U^{-1} \neq U^{-1}\, V^{-1}$.) One checks that $b\,c = 0$ and $\pi(c) = \gamma$.

Note also that the area, $⨍\, ds^2 = ⨍\, D^{-2}$ of the non commutative torus depends on τ in the same way as c, the reason is that it has homological significance,

$$⨍\, ds^2 = \langle \varphi, c \rangle$$

where the cyclic cocycle φ is rigidly fixed by its integrality as the chern character of our K-cycle.

It is also worthwile to check in some detail the Poincaré duality. The point is that while at a superficial level our geometry $\mathbb{T}_\theta^2$ looks like an ordinary torus, this impression quickly fades away if we realize that for $\theta \notin \mathbb{Q}$ the algebra $\mathcal{A}_\theta$ contains many non trivial idempotents ([Ri] [Po]) and smooth real elements $a = a^*$ of $\mathcal{A}_\theta$ can have Cantor spectrum (thus from the point of view of a the space $\mathbb{T}_\theta^2$ looks as totally disconnected). In fact at first it might seem that Poincaré duality might fail since there is no element $x \in K_0(\mathcal{A}_\theta)$, $x \neq 0$ of virtual dimension 0, $\tau_0(x) = 0$, the characteristic property of the Bott element. Indeed, when $\theta \notin \mathbb{Q}$, the K_0 group is $\mathbb{Z}^2$ ([Pi-V]) and the trace $\tau_0(n\, e_0 + m\, e_1) = n + m\,\theta$ never vanishes unless $n = m = 0$. The above specifies uniquely the basis $e_0 = [1]$ and e_1, $\tau_0(e_1) = \theta$ of K_0 and we take the basis U, V for K_1. In this basis the intersection form of axiom (6) is given by the symplectic matrix,

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

It follows that Poincaré duality is satisfied and that the Bott element β makes sense, as an element of $K_0(\mathcal{A}_\theta^0 \otimes \mathcal{A}_\theta)$. It is given by

$$\beta = e_0^0 \otimes e_1 - e_1^0 \otimes e_0 + u^0 \otimes v - v^0 \otimes u$$

where we denote by $\otimes$ the external cup product in K theory. One can check that with μ the K-homology class on $\mathcal{A}_\theta \otimes \mathcal{A}_\theta^0$ given by our K-cycle one has the Poincaré duality equation ([CS])

$$\mu \otimes_{\mathcal{A}_\theta^0} \beta = \mathrm{id}_{\mathcal{A}_\theta} \qquad \beta \otimes_{\mathcal{A}_\theta} \mu = \mathrm{id}_{\mathcal{A}_\theta^0} .$$

Finally the unique trace on $\mathcal{A}_\theta^0 \otimes \mathcal{A}_\theta$ is $\tau_0^0 \otimes \tau_0$ and it does vanish on β, $\langle \tau_0^0 \otimes \tau_0, \beta \rangle = 0$ which is the expected property of the Bott element.

In the general theory both $\beta \in K_*(\mathcal{A}_\theta^0 \otimes \mathcal{A}_\theta)$ and $\mu \in K^*(\mathcal{A}_\theta \otimes \mathcal{A}_\theta^0)$ make sense and satisfy the above equation. Moreover the local index theorem of [CM] allows to compute the pairing $\langle \mu, \beta \rangle$ by a local formula, i.e. a formula invoking $⨍$ of simple algebraic expressions. This yields the Gauss Bonnet theorem in our context, due to the equality,

$$\langle \mu, \beta \rangle = \mathrm{Rank}\, K_0 - \mathrm{Rank}\, K_1$$

(where the rank is the rank of the abelian group, i.e. $\dim_{\mathbb{Q}}(K_j \otimes \mathbb{Q})$). Using the automorphism σ of $\mathcal{A}_\theta$ associated to a unimodular integral matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z})$ by,

$$\sigma(U) = U^a \, V^b \ , \ \sigma(V) = U^c \, V^d$$

it is not difficult to check that, up to an irrelevant scale factor, the non commutative geometry $\mathbb{T}^2_{\theta,\tau}$ only depends upon the value of τ in $\mathbb{C}^+/PSL(2, \mathbb{Z})$. The new phenomenon which occurs in the non commutative case is the *Morita equivalence* of geometries which will correspond here to the action of the modular group $PSL(2, \mathbb{Z})$ on the module θ rather than τ. To change θ in this way one uses a finite projective right module $\mathcal{E}$ over the algebra $\mathcal{A}$ and then one replaces $\mathcal{A}$ by the algebra $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E})$. To be more specific let us change θ in $-\frac{1}{\theta}$ as follows. We take for $\mathcal{E}$ the following right $\mathcal{A}$-module ([Co]). As a vector space $\mathcal{E}$ is the Schwartz space $\mathcal{S}(\mathbb{R})$ of functions on $\mathbb{R}$. The right action of $\mathcal{A}_\theta$ on $\mathcal{S}$ is specified by the rules,

$$(\xi \, U)(s) = \xi(s + \theta) \qquad \forall \, \xi \in \mathcal{S}(\mathbb{R}) \ , \ \forall \, s \in \mathbb{R}$$

$$(\xi \, V)(s) = e^{2\pi i s} \, \xi(s) \qquad \forall \, \xi \in \mathcal{S}(\mathbb{R}) \ , \ \forall \, s \in \mathbb{R} .$$

One checks directly that the algebra $\mathcal{B} = \mathrm{End}_{\mathcal{A}_\theta}(\mathcal{S})$ is isomorphic to $\mathcal{A}_{-1/\theta}$ with generators given by the translation by 1 and the multiplication by the function $s \to \exp\left(\frac{2\pi i s}{\theta}\right)$. This $\mathcal{A}_\theta$- module $\mathcal{S}$ is naturally a hermitian module ([Co]) and we can now apply the following general operation of Morita equivalence in non commutative geometry.

Let $(\mathcal{A}, \mathcal{H}, D)$ be a given geometry and $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ a Morita equivalent algebra. Then a non commutative geometry, $(\mathcal{B}, \widetilde{\mathcal{H}}, \widetilde{D})$ is uniquely specified by a *hermitian connection* ([Co]) ∇ on $\mathcal{E}$. Such a connection is a linear map,

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$$

where Ω^1_D is the $\mathcal{A}$-bimodule,

$$\Omega^1_D = \{\Sigma\, a_i\, [D, b_i]\ ;\ a_i, b_i \in \mathcal{A}\} \subset \mathcal{L}(\mathcal{H})$$

and ∇ should satisfy the Leibniz rule and the compatibility with the hermitian structure,

1) $\nabla(\xi\, a) = (\nabla\, \xi)\, a + \xi \otimes [D, a] \qquad \forall\, a \in \mathcal{A}\ ,\ \xi \in \mathcal{E}$

2) $(\xi, \nabla\, \eta) - (\nabla\, \xi, \eta) = d(\xi, \eta) \qquad \forall\, \xi, \eta \in \mathcal{E}$

where $d\, a = [D, a]$ by definition.

To construct $\widetilde{\mathcal{H}}$ and $\widetilde{D}$ one proceeds as follows.

One lets $\widetilde{\mathcal{H}} = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \overline{\mathcal{E}}$ where $\overline{\mathcal{E}}$ is the conjugate module, with elements $\overline{\xi}$, $\xi \in \mathcal{E}$ and the module structure,

$$a\, \overline{\xi} = (\xi\, a^*)^- \qquad \forall\, \xi \in \mathcal{E}\ ,\ a \in \mathcal{A}\, .$$

Then $\widetilde{\mathcal{H}}$ has a natural Hilbert space structure (cf. [Co]) obtained using the hermitian structure of $\mathcal{E}$.

The operator $\widetilde{D}$ is given by the formula,

$$\widetilde{D}(\xi \otimes \eta \otimes \overline{\zeta}) = (\nabla\, \xi)\, \eta \otimes \overline{\zeta} + \xi \otimes D\, \eta \otimes \overline{\zeta} + \xi \otimes \eta(\overline{\nabla\, \zeta})$$

where we take advantage of the action of Ω^1_D in $\mathcal{H}$ in order to make sense of $(\nabla\, \xi)\, \eta$ for instance (cf. [Co]). The above formula is compatible with tensor products over $\mathcal{A}$. Similarly the real structure $\widetilde{J}$ is given by (with obvious notations),

$$\widetilde{J}\, (\xi \otimes \eta \otimes \overline{\zeta}) = \zeta \otimes J\, \eta \otimes \overline{\xi}\, .$$

One then checks that all our axioms are fulfilled by the triple $(\mathcal{B}, \widetilde{\mathcal{H}}, \widetilde{D})$ and $\widetilde{J}$ where the action of $\mathcal{B}$ is given by

$$b\,(\xi \otimes \eta \otimes \overline{\zeta}) = (b\,\xi) \otimes \eta \otimes \overline{\zeta}\,.$$

In our specific example of the right module $\mathcal{S}$ on $\mathcal{A}_\theta$ the $\mathcal{A}_\theta$ bimodule Ω^1_D is easily computed and (cf. [Co]) as a bimodule it is the sum of two copies of $\mathcal{A}_\theta$ which we write by identifying $\omega \in \Omega^1_D$ with a 2×2 matrix of the form,

$$\omega = \begin{bmatrix} 0 & \lambda(a) \\ \lambda(b) & 0 \end{bmatrix}$$

and writing $a = (\delta_1 + \tau\,\delta_2)(x)$, $b = (-\delta_1, -\overline{\tau}\,\delta_2)(x)$ for any $x \in \mathcal{A}$ and $\omega = dx$. A connection ∇ is uniquely specified by the two covariant derivatives ∇_j which correspond to δ_j. Thus a specific choice of connection on $\mathcal{S}$ is given by

$$(\nabla_1\,\xi)(s) = \frac{2\pi i}{\theta}\,s\,\xi(s)$$
$$(\nabla_2\,\xi)(s) = \xi'(s)\,.$$

We leave as an exercise to compute the corresponding value of the module τ' for the Morita equivalent geometry on $\mathbb{T}^2_{-1/\theta}$. The connection ∇ used in this example has constant curvature (cf. [C-R]) but there is a lot of freedom in the choice of the connection. Indeed the space of connections is naturally an affine space over the vector space $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D)$ (with the selfadjointness condition).

All this discussion applies in particular when $\mathcal{E} = \mathcal{A}$ and yields a whole new class of geometries, the *internal perturbation* which replace D by $D + A + J\,A\,J^{-1}$ where A is an arbitrary selfadjoint element of Ω^1_D.

These internal perturbations are trivial: $A + J\,A\,J^{-1} = 0$ when the geometry is commutative (theorem 1). They are highly non trivial for $\mathbb{T}^2_\theta$, $\theta \notin \mathbb{Q}$.

When $\theta \notin \mathbb{Q}$ the von Neumann algebra weak closure of $\mathcal{A}_\theta$ in $\mathcal{H}$ is the hyperfinite factor of type $\mathbb{T}_1$ and the antilinear isometry J_0 is Tomita's involution. In the general theory it is always true that the von Neumann algebra $\mathcal{A}''$ weak closure of $\mathcal{A}$ in $\mathcal{H}$ is finite and hyperfinite. This follows from axiom 5, the general properties of the Dixmier trace and the results of [Co]. In particular the von Neumann algebras involved in our geometries are completely classified up to isomorphism ([Co]).

The non commutative geometry of the standard model.

We shall now describe a simple *finite* geometry of dimension equal to 0, $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ (F for finite), whose product by ordinary Euclidean 4-dimensional geometry (or more generally by a 4-dimensional Spin manifold) will give the standard model (SM) in the following way:

1) The Hilbert space $\mathcal{H}$ will describe the (one particle) Fermionic sector of the SM.
2) The inner fluctuations of the metric $D \to D + A + J\,A\,J^{-1}$ give exactly the bosonic sector of SM with the correct quantum numbers and hypercharges for the coupling with the fermions: $\langle D\psi, \psi\rangle$.
3) The *spectral action* Trace $(\varphi(D^{-1})) + \langle D\psi, \psi\rangle$ restricted to the inner fluctuations of the metric, gives the SM Lagrangian.

We postpone the proof of (3) and the analysis of its relation to gravity to our collaboration with A.C. [CC].

In this paper we shall describe the geometry $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ and check (2) in some detail.

We let $\mathcal{H}_F$ be the Hilbert space with basis the list of elementary Fermions. Thus each generation of Fermions contributes by a space of dimension $15 + 15$ where $15 = 12 + 3$. The 12 corresponds to the quarks and $12 = 4 \times 3$ where the 4 is given by the table

$$\begin{matrix} u_R & u_L \\ d_R & d_L \end{matrix}$$

of up and down particles of left or right chirality, while the 3 is given by the color index. The 3 in $15 = 12 + 3$ corresponds to the leptons and is given by the table

$$\begin{matrix} & \nu_L \\ e_R & e_L \end{matrix}\,.$$

The second 15 in $15 + 15$ corresponds to antiparticles and is obtained by putting an $\overline{f}$ instead of an f for any f in the above basis. This gives us the antilinear isometry $J = J_F$ in $\mathcal{H}_F$, by

$$J_F(\Sigma\,\lambda_i\,f_i + \Sigma\,\mu_j\,\overline{f}_j) = \Sigma\,\overline{\mu}_j\,f_j + \Sigma\,\overline{\lambda}_i\,\overline{f}_i$$

for any $\lambda_i, \mu_j \in \mathbb{C}$.

We let $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ be the direct sum the real involutive algebras $\mathbb{C}$ of complex numbers, $\mathbb{H}$ of quaternions, and $M_3(\mathbb{C})$ of 3×3 matrices. (Recall that quaternions q can be represented as 2×2 matrices of the form $\begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}$ where $\alpha, \beta \in \mathbb{C}$.)

Let us give the action of $\mathcal{A}$ in $\mathcal{H}_F$. We let, for $a=(\lambda,q,m)\in\mathcal{A}$,

$$a\,u_R=\lambda\,u_R \qquad a\,u_L=\alpha\,u_L-\overline{\beta}\,d_L$$

$$a\,d_R=\overline{\lambda}\,d_R \qquad a\,d_L=\beta\,u_L+\overline{\alpha}\,d_L\,.$$

(Independently of the color index), while for leptons the formula is the same but there is no u_R.

This fixes completely the action of $\mathcal{A}$ on particles. The action on antiparticles $\overline{f}$ is given by: (for $a=(\lambda,q,m)$ as above)

$$a\,\overline{f}=\lambda\,\overline{f} \quad \text{if } f \text{ is a lepton}$$

$$a\,\overline{f}=m\,\overline{f} \quad \text{if } f \text{ is a quark}\,.$$

Here the 3×3 matrix is acting in the obvious way on the color index.

For the operator D_F we take $D_F=\begin{bmatrix} Y & 0\\ 0 & \overline{Y}\end{bmatrix}$ where Y is the Yukawa coupling matrix, which has the dimension of a mass, and is of the form:

$$Y=Y_q\otimes 1_3\oplus Y_f$$

with

$$Y_q=\begin{bmatrix} 0 & 0 & M_u & 0\\ 0 & 0 & 0 & M_d\\ M_u^* & 0 & 0 & 0\\ 0 & M_d^* & 0 & 0\end{bmatrix}\ ,\ Y_f=\begin{bmatrix} 0 & 0 & M_e\\ 0 & 0 & 0\\ M_e^* & 0 & 0\end{bmatrix}\,.$$

For one generation M_u, M_d and M_e would just be scalars but for 3 generations they are matrices which encode both the masses of the Fermions and their mixing properties.

Let us now check our axioms for this 0-dimensional geometry $(\mathcal{A}_F,\mathcal{H}_F,D_F)$. We begin by checking $7')$. One has $J_F^2=1$, $J_F\,D_F=D_F\,J_F$ and it is clear that J_F commutes with the natural $\mathbb{Z}/2$ grading given by chirality:

$$\gamma(f_R)=f_R\ ,\ \gamma(f_L)=-f_L \qquad (f \text{ particle or antiparticle})\,.$$

But the important property is that $[a,b^0]=0$ for any $a,b\in\mathcal{A}$ (cf. $(7')$), where $b^0=J\,b^*\,J^{-1}\quad\forall\,b\in\mathcal{A}$. To check it, it is enough to see what happens on one generation of particles. For quarks the right action b^0 of $b\in\mathcal{A}\quad b=(\lambda,q,m)$ is given by m^t acting on the color index which obviously commutes with the left action of $\mathcal{A}$. For leptons the right action is by the scalar λ which also commutes with the left action of $\mathcal{A}$.

We have thus checked $(7')$. Since $n = 0$ and $\mathcal{H}_F$ is finite dimensional the axiom (1) is obvious. To check $(2')$ we need to show that $[[D, a], b^0] = 0 \quad \forall\, a, b \in \mathcal{A}$.

Once again it is enough to check it for one generation of particles, it is clear for leptons and it is true for quarks exactly because the color is unbroken so that both a and D exactly commute with the right action of $\mathcal{A}$.

Being in finite dimension the axiom 3 is obvious.

To check $(4')$ one verifies that $\gamma = \varepsilon\, J\, \varepsilon\, J$ where ε is the following element of $\mathcal{A}$, $\varepsilon = (1, -1, 1)$. Thus one has $c = \varepsilon \otimes \varepsilon^0 \in \mathcal{A} \otimes \mathcal{A}^0$.

Note that our algebra $\mathcal{A}$ is real so that (5) has to be stated for the complex algebra generated by $\mathcal{A}$ in $\mathcal{H}_F$. It is then clear. Finally when we compute the intersection form on $K_0(\mathcal{A}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ we find, with N the number of generators, the 3×3 matrix,

$$Q = 2N \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} .$$

The above matrix is invertible with inverse given by

$$Q^{-1} = (2N)^{-1} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

so that (6) only holds rationally.

We now consider a 4-dimensional smooth compact Riemannian manifold M with a fixed spin structure and consider its product with the above finite geometry. One can prove that our notion of geometry is stable by products. When one of the two geometries is even (i.e. it possesses a $\mathbb{Z}/2$ grading γ_1), the product geometry is given by the rules,

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \ , \ \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \ , \ D = D_1 \otimes 1 + \gamma_1 \otimes D_2 \, .$$

To check axiom $(4')$ for instance one uses the shuffle product in Hochschild homology (cf. Loday).

For the product of the manifold M by the finite geometry F we thus have $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F = C^\infty(M, \mathcal{A}_F)$, $\mathcal{H} = L^2(M, S) \otimes \mathcal{H}_F = L^2(M, S \otimes \mathcal{H}_F)$ and $D = \partial\!\!\!/_M \otimes 1 + \gamma_5 \otimes D_F$ where $\partial\!\!\!/_M$ is the Dirac operator on M.

Let us check that the inner fluctuations of the metric give us the gauge bosons of the standard model with their correct quantum numbers. We first have to compute $A = \Sigma\, a_i [D, a_i'] \quad a_i, a_i' \in \mathcal{A}$. Since $D = \partial\!\!\!/_M \otimes 1 + \gamma_5 \otimes D_F$ decomposes as a sum of two terms, so does A and we first consider the discrete part $A^{(0,1)}$ coming from commutators with $\gamma_5 \otimes D_F$. Let $x \in M$ and let $a_i(x) = (\lambda_i, q_i, m_i)$, $a_i'(x) = (\lambda_i', q_i', m_i')$, the computation of $\Sigma\, a_i [\gamma_5 \otimes D_F, a_i']$ at x gives γ_5 tensored by the following matrices,

$$\begin{bmatrix} 0 & X \\ X' & 0 \end{bmatrix}, \ X = \begin{bmatrix} M_u\, \varphi_1 & M_u\, \varphi_2 \\ -M_d\, \overline{\varphi}_2 & M_d\, \overline{\varphi}_1 \end{bmatrix}, \ X' = \begin{bmatrix} M_u^*\, \varphi_1' & M_d^*\, \varphi_2' \\ -M_u^*\, \overline{\varphi}_2' & M_d^*\, \overline{\varphi}_1' \end{bmatrix}$$

for the quark part, with $\varphi_1 = \Sigma\, \lambda_i (\alpha_i' - \lambda_i')$, $\varphi_2 = \Sigma\, \lambda_i\, \beta_i'$

$$\varphi_1' = \Sigma\, \alpha_i (\lambda_i' - \alpha_i') + \beta_i\, \overline{\beta}_i' \ , \ \varphi_2' = \Sigma\, (-\alpha_i\, \beta_i' + \beta_i (\overline{\lambda}_i' - \overline{\alpha}_i')) \, .$$

For the lepton part one gets the 3×3 matrix,

$$\begin{bmatrix} 0 & -M_d\, \overline{\varphi}_2 & M_d\, \overline{\varphi}_1 \\ M_d^*\, \varphi_2' & 0 & 0 \\ M_d^*\, \overline{\varphi}_1' & 0 & 0 \end{bmatrix}$$

where φ_1, φ_2, φ_1' and φ_2' are as above.

Let $q = \varphi_1 + \varphi_2\, j$, $q' = \varphi_1' + \varphi_2'\, j$ where j is the quaternion $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The selfadjointness condition $A = A^*$ is equivalent to $q' = q^*$ and we see that the discrete part $A^{(0,1)}$ is exactly given by a quaternion valued function, $q(x) \in \mathbb{H}$ on M. This pair of complex fields is the Higgs doublet and one checks that it has the right quantum numbers. The antiparticle sector does not contribute to $A^{(0,1)}$ because the left action of $\mathcal{A}_F$ on this sector exactly commutes with D_F.

Let us now determine the other part $A^{(1,0)}$ of A, i.e.

$$A^{(1,0)} = \Sigma\, a_i [(\partial\!\!\!/_M \otimes 1), a_i'] \, .$$

With obvious notations, $a_i = (\lambda_i, q_i, m_i)$, $a_i' = (\lambda_i', q_i', m_i')$ we obtain,

A $U(1)$ gauge field $\Lambda = \Sigma\, \lambda_i\, d\, \lambda_i'$

A $SU(2)$ gauge field $Q = \Sigma\, q_i\, d\, q_i'$

A $U(3)$ gauge field $V = \Sigma\, m_i\, d\, m_i'$.

The computation of $A+J\,A\,J^{-1}$ gives the following matrices on quarks and leptons, where we omit the symbol of Clifford multiplication,

$$\begin{bmatrix} \Lambda+V & 0 & 0 & 0 \\ 0 & -\Lambda+V & 0 & 0 \\ 0 & 0 & Q_{11}+V & Q_{12} \\ 0 & 0 & Q_{21} & Q_{22}+V \end{bmatrix}$$

$$\begin{bmatrix} -2\Lambda & 0 & 0 \\ 0 & Q_{11}-\Lambda & Q_{12} \\ 0 & Q_{21} & Q_{22}-\Lambda \end{bmatrix}$$

where the matrix for quarks is a 4×4 matrix of 3×3 matrices because V is a 3×3 matrix, (ignoring the flavor index). Since we are only interested in the fluctuation of the metric we shall write the total 15×15 matrix as the sum of a traceless matrix plus a scalar multiple of the identity matrix. Since the latter does not affect the metric we shall ignore it. This amounts to remove a $U(1)$ by imposing the condition that the full matrix is traceless, i.e. that 4 trace $V - 4\Lambda = 0$, i.e. trace $V = \Lambda$. Thus we get $V = V' + \frac{1}{3}\,\Lambda$ with trace $V' = 0$ so that V' is an $SU(3)$ gauge potential. The $U(1)$ field Λ is the generator of hypercharge and we obtain the following matrices for the inner fluctuation $A + J\,A\,J^{-1}$ of the metric (vector part),

$$\begin{matrix} u_R \\ d_R \\ u_L \\ d_L \end{matrix} \begin{bmatrix} \frac{4}{3}\,\Lambda+V' & 0 & 0 & 0 \\ 0 & -\frac{2}{3}\,\Lambda+V' & 0 & 0 \\ 0 & 0 & Q_{11}+\frac{1}{3}\,\Lambda+V' & Q_{12} \\ 0 & 0 & Q_{21} & Q_{22}+\frac{1}{3}\,\Lambda+V' \end{bmatrix}$$

$$\begin{matrix} e_R \\ \nu_L \\ e_L \end{matrix} \begin{bmatrix} -2\Lambda & 0 & 0 \\ 0 & Q_{11}-\Lambda & Q_{12} \\ 0 & Q_{21} & Q_{22}-\Lambda \end{bmatrix}.$$

Together with the above Higgs doublet which gives the scalar part of $A+J\,A\,J^{-1}$ we thus obtain exactly the gauge bosons of the standard model coupled with the correct hypercharges Y_L, Y_R. They are such that the electromagnetic charge Q_{em}, is determined by $2\,Q_{em} = Y_R$ for right handed particles and $2\,Q_{em} = Y_L + 2\,I_3$ where I_3 is the $3^{\rm rd}$ generator of the weak isospin group $SU(2)$. For Q_{em} one gets the same answer for the left and right components of each particle and $\frac{2}{3}$, $-\frac{1}{3}$ for u, d respectively and $0, -1$ for ν and e respectively.

We showed in [C] that one obtains the full Lagrangian of the standard model from the sum $⨍\, \theta^2\, ds^4 + \langle (D+A+J\,A\,J^{-1})\psi, \psi\rangle$ where θ is the curvature of the connection A. However this requires the definition of the curvature and is still in the spirit of gauge theories. What the present paper shows is that one should consider the internal gauge symmetries as part of the diffeomorphism group of the non commutative geometry, and the gauge bosons as the internal fluctuations of the metric. It follows then that the action functional should be of purely gravitational nature. We state the principle of spectral invariance, stronger than the invariance under diffeomorphisms, which requires that the action functional only depends on the spectral properties of $D = ds^{-1}$ in $\mathcal{H}$. This is verified by the action,

$$I = \mathrm{Trace}\,(\varphi(ds/\ell_p)) + \langle D\,\psi, \psi\rangle$$

for any nice function φ from $\mathbb{R}_+^*$ to $\mathbb{R}$.

We shall show in [CC] that this action gives the SM Lagrangian coupled with gravity. It would seem at first sight that the algebra $\mathcal{A}$ has disappeared from the scene when one writes down the above action, the point is that it is still there because it imposes the constraints $[[D,a],b^0] = 0 \quad \forall\, a,b \in \mathcal{A}$ and $\Sigma\, a_i^0 [D,a_i^1] \ldots [D,a_i^4] = \gamma$ coming from axioms 2 and 4.

It is important at this point to note that the *integrality*, $n \in \mathbb{N}$ of the dimension of a non commutative geometry appears to be essential to define the Hochschild cycle $c \in Z_n$ and in turns the chirality γ. This is very similar to the obstruction which appears when one tries to apply dimensional regularization to chiral gauge theories.

The relations (axioms 2 and 4) which relate the algebra $\mathcal{A}$ with the infinitesimal length element ds are very simple but it is clear that more involved relations of a quartic type are necessary to cover the hypoelliptic situation encountered in [CM]. Also the algebra $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$ used in the description of the standard model is only slightly non commutative. This commutativity should fade away when one gets to energies $\Lambda \sim m_p$ so that the inner automorphisms $\mathrm{Int}\,\mathcal{A}$ exhaust more and more automorphisms of $\mathcal{A}$. It is tempting to follow the approximation scheme given in [GKP] but the main difficulty is that the above non linear constraints between the algebra $\mathcal{A}$ and the operator D do not have finite dimensional realizations since the Hochschild homology $H_4(\mathcal{A},\mathcal{A})$ vanishes for any finite dimensional algebra $\mathcal{A}$.

As a final remark we look for an explanation of the remaining non commutativity of the algebra $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$ from the theory of quantum groups at roots of unity. The simple fact is that the Spin covering of $SO(4)$, i.e. $\mathrm{Spin}(4)$ is not the maximal covering in the world of quantum groups. Indeed, $\mathrm{Spin}(4) = SU(2) \times SU(2)$ and even the single group $SU(2)$ admits, in the sense of non commutative geometry (cf. [M]) non trivial extensions of finite order, (Frobenius at ∞)

$$1 \to H \to SU(2)_q \to SU(2) \to 1$$

where q is any root of unity, $q^m = 1$, of odd order. The simplest instance is when $m = 3$ so that $q = \exp\left(\frac{2\pi i}{3}\right)$. The quantum group H has a finite dimensional Hopf algebra which is closely related to the algebra $\mathcal{A}_F$. The Spin representation of H defines, like any other representation of H, a bimodule over the Hopf algebra of H. The structure of this bimodule turns out to be very similar to the structure of the bimodule $\mathcal{H}_F$ over $\mathcal{A}_F$ that we described above. The details of the adaptation of these ideas to $\mathrm{Spin}(4) = SU(2) \times SU(2)$ still remain to be elucidated.

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