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Insertion and Elimination: the doubly infinite Lie algebra of Feynman graphs

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Abstract

The Lie algebra of Feynman graphs gives rise to two natural representations, acting as derivations on the commutative Hopf algebra of Feynman graphs, by creating or eliminating subgraphs. Insertions and eliminations do not commute, but rather establish a larger Lie algebra of derivations which we here determine.

Introduction

The algebraic structure of perturbative QFT [1, 2, 3, 4] gives rise to commutative Hopf algebras $\mathcal{H}$ and corresponding Lie-algebras $\mathcal{L}$, with $\mathcal{H}$ being the dual of the universal enveloping algebra of $\mathcal{L}$. $\mathcal{L}$ can be represented by derivations of $\mathcal{H}$, and two representations are most natural in this respect: elimination or insertion of subgraphs.

Perturbation theory is indeed governed by a series over one-particle irreducible graphs. It is then a straightforward question how the basic operations of inserting or eliminating subgraphs act. These are the basic operations which are needed to construct the formal series over graphs which solve the Dyson–Schwinger equations. We give an account of these actions here as a further tool in the mathematician's toolkit for a comprehensible description of QFT. We introduce these structures by first considering the case of undecorated rooted trees. In that case one is lead naturally to the two basic operations of grafting and trimming using the relation between the Hopf algebras $\mathcal{H}_{cm}$ and $\mathcal{H}_{rt}$ ([2]). The Hopf algebra $\mathcal{H}_{cm}$ is neither commutative nor cocommutative but admits a finite set of generators with simple relations. The basic relation ([2]) between a commutative subalgebra $\mathcal{H}^1_{cm}$ of $\mathcal{H}_{cm}$ and the Hopf algebra $\mathcal{H}_{rt}$ was obtained using the "natural growth operation" on trees. By extending this "natural growth operation" to the grafting of arbitrary trees we show how to enlarge $\mathcal{H}_{rt}$ to a

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Hopf algebra $\mathcal{H}_{rtt}$ whose relation to $\mathcal{H}_{rt}$ is the same as the relation of $\mathcal{H}_{cm}$ with $\mathcal{H}^1_{cm}$. In particular it is neither commutative nor cocommutative. We show that it is obtained as a "bicrossed product" construction from a doubly infinite Lie algebra of rooted trees, similar to the Lie algebra of formal vector fields. Since most of the information is then contained in that Lie algebra, which can be concretely described from grafting and trimming operations, we then turn to Feynman graphs, and only discuss the Lie algebra aspect in that case.

1 Undecorated rooted trees

The Hopf algebras $\mathcal{H}_{cm}$ and $\mathcal{H}_{rt}$

Let us first recall the constructions of the basic Hopf algebras involved in [5] and [2], and compare their properties.
As an algebra $\mathcal{H}_{cm}$ is noncommutative but finitely generated.
It is generated by three elements Y, X, δ_1. To describe the relations between these three generators, one lets δ_n, $n \geq 1$ be defined by induction by,

$$[X, \delta_n] = \delta_{n+1} \ \forall\, n \geq 1\,, \tag{1}$$

then the presentation of the relations in $\mathcal{H}_{cm}$ is the following,

$$[Y, X] = X, [Y, \delta_n] = n\, \delta_n, [\delta_n, \delta_m] = 0 \ \forall\, n, m \geq 1, \tag{2}$$

The coproduct Δ in $\mathcal{H}_{cm}$ is defined by

$$\Delta\, Y = Y \otimes 1 + 1 \otimes Y \ , \ \Delta\, X = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y \ , \ \Delta\, \delta_1 = \delta_1 \otimes 1 + 1 \otimes \delta_1 \tag{3}$$

and the equality,

$$\Delta(h_1\, h_2) = \Delta h_1\, \Delta h_2 \qquad \forall\, h_j \in \mathcal{H}_T\,. \tag{4}$$

The Hopf algebra $\mathcal{H}_{cm}$ is neither commutative nor cocommutative but is obtained in a simple manner from the commutative subalgebra $\mathcal{H}^1_{cm}$ generated by the δ_n.

Theorem. ([5]) *Let G_2 be the group of formal diffeomorphisms of the real line of the form $\psi(x) = x + o(x)$. For each n, let γ_n be the functional on G_2 defined by,*

$$\gamma_n(\psi^{-1}) = (\partial_x^n \log \psi'(x))_{x=0}\,.$$

The equality $\Theta(\delta_n) = \gamma_n$ determines a canonical isomorphism Θ of the Hopf algebra $\mathcal{H}^1_{cm}$ with the Hopf algebra of coordinates on the group G_2. The Hopf algebra $\mathcal{H}_{cm}$ is the bicrossed product associated to the formal decomposition $G = G_1\, G_2$ associated to the decomposition Lie G = Lie G_1+Lie G_2 of formal vector fields in their affine part (Lie G_1) and nilpotent part (Lie G_2).

The Hopf algebra $\mathcal{H}_{rt}$ of rooted trees is commutative but not finitely generated.

Recall that a rooted tree T is, by definition, a finite, connected, simply connected, one dimensional simplicial complex with a base point $* \in \Delta^0(T) = \{\text{set of vertices of } T\}$. This base point is called the root. By the degreee of the tree we mean

$$|T| = \text{Card}\Delta^0(T) = \# \text{ of vertices of } T\,. \tag{5}$$

By a *simple* cut of a rooted tree T we mean a subset $c \subset \Delta^1(T)$ of the set of edges of T such that,

$$\text{for any } x \in \Delta^0(T) \text{ the path } (*, x) \text{ only contains at most one element of } c\,. \tag{6}$$

Thus what is excluded is to have two cuts of the same path or branch. Given a cut c the new simplicial complex T_c with $\Delta^0(T_c) = \Delta^0(T)$ and

$$\Delta^1(T_c) = \Delta^1(T) \backslash c\,, \tag{7}$$

is no longer connected, unless $c = \emptyset$. We let $R_c(T)$ be the connected component of $*$ with the same base point and call it the trunk. We endow each other connected component, called a cut branch, with the base point coming from the cut. We obtain in this way a set (with multiplicity) of rooted trees, which we denote by $P_c(T)$. We let Σ be the set of rooted trees up to isomorphism, and let $\mathcal{H}_{rt}$ be the polynomial commutative algebra generated by the symbols,

$$\delta_T\ ,\ T \in \Sigma\,. \tag{8}$$

One defines a coproduct on $\mathcal{H}_{rt}$ by,

$$\Delta\,\delta_T = \delta_T \otimes 1 + 1 \otimes \delta_T + \sum_c \left(\prod_{P_c(T)} \delta_{T_i} \right) \otimes \delta_{R_c(T)}\,, \tag{9}$$

where the last sum is over all non trivial simple cuts ($c \neq \emptyset$) of T, while the product $\prod\limits_{P_c(T)}$ is over the cut branches.

Equivalently, one can write (9) as,

$$\Delta\,\delta_T = \delta_T \otimes 1 + \sum_c \left(\prod_{P_c(T)} \delta_{T_i} \right) \otimes \delta_{R_c(T)}\,, \tag{10}$$

where the last sum is over all simple cuts.

This defines Δ on generators and it extends uniquely as an algebra homomorphism,

$$\Delta : \mathcal{H}_{rt} \to \mathcal{H}_{rt} \otimes \mathcal{H}_{rt}\,. \tag{11}$$

The first basic relation between the Hopf algebras $\mathcal{H}_{cm}$ and $\mathcal{H}_{rt}$ is the Hopf algebra homomorphism ([2]) obtained using the "natural growth" operator N defined as the unique derivation of the commutative algebra $\mathcal{H}_{rt}$ such that,

$$N\,\delta_T = \sum \delta_{T'} \tag{12}$$

where the trees T' are obtained by adding one vertex and one edge to T in all possible ways without changing the base point. It is clear that the sum (12) contains $|T|$ terms.

Theorem. ([2]) *The equality* $\Lambda(\delta_n) = N^n(\delta_*)$ *determines a canonical homomorphism* Λ *of the Hopf algebra* $\mathcal{H}^1_{cm}$ *into the Hopf algebra* $\mathcal{H}_{rt}$ *.*

This theorem suggests, as we did in ([2]) to enlarge the Hopf algebra $\mathcal{H}_{rt}$ in the same way as $\mathcal{H}^1_{cm}$ is naturally enlarged to $\mathcal{H}_{cm}$, by adjoining the elements Y, X implementing both the grading and the natural growth operators. We shall now show that it is indeed possible to do much more by extending the natural growth operator N to the grafting of arbitrary trees.

The derivations N_T of $\mathcal{H}_{rt}$

Let us first extend the construction of the natural growth operator N to get operators N_T labelled by arbitrary trees.
For a given rooted tree T we consider the unique derivation N_T of $\mathcal{H}_{rt}$ such that, for any $t \in \Sigma$,

$$N_T\,(\delta_t) = \sum_v \delta_{(t\,\cup_v\,T)} \tag{13}$$

where in the summation, v runs through the vertices $v \in \Delta^0(t)$ and where the rooted tree $t' = t \cup_v T$ is obtained as the union of t and T, with the root $*$ of T identified with v. One has

$$\Delta^1(t \cup_v T) = \Delta^1(t) \cup \Delta^1(T), \tag{14}$$

$$\mathrm{root}(t \cup_v T) = \mathrm{root}(t), \tag{15}$$

and the number of vertices of $(t \cup_v T)$ is,

$$|t'| = |t| + |T| - 1\,. \tag{16}$$

When $T = *$ has one element we see that,

$$N_*(\delta_t) = |t|\,\delta_t \tag{17}$$

thus we get the derivation Y.
When $T = \begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}$ is the rooted tree with one edge, we just get the natural growth operation: $N_{\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}} = N$.
Since N_T is extended as a derivation one has,

$$N_T\left(\prod \delta_{t_i}\right) = \sum_1^n \delta_{t_1} \ldots N(\delta_{t_k}) \ldots \delta_{t_n}\,. \tag{18}$$

Let us now prove,

Lemma.

$$\Delta(N_T(a)) = \left(N_T \otimes \mathrm{id} + \mathrm{id} \otimes N_T + \sum_c \prod_{P_c(T)} \delta_{t_j} \otimes N_{R_c(T)} \right) \Delta(a) \, .$$

Proof. Both sides of the equation are linear maps from $\mathcal{H}_{rt}$ to $\mathcal{H}_{rt} \otimes \mathcal{H}_{rt}$ which satisfy the derivation rule, $\rho(ab) = \rho(a)\,\Delta(b) + \Delta(a)\,\rho(b)$. Indeed $\mathcal{H}_{rt}$ is a commutative algebra so that the multiplication by a product of $\delta_{t_j} \otimes 1$ does not alter the derivation rule. Thus it is enough to check the lemma for $a = f\delta_t, t \in \Sigma$.
Now, by definition of the coproduct,

$$\Delta\, N_T(\delta_t) = N_T(\delta_t) \otimes 1 + 1 \otimes N_T(\delta_t) + \sum_{v_0, c} \prod \delta_{t'_j} \otimes \delta_{R_c(t')}$$

where v_0 varies in $\Delta^0(t)$ and the c varies through simple cuts of $t' = t \cup_{v_0} T$.
Let us first consider the partial sum over pairs (v_0, c) with $v_0 \notin R_c(t')$ i.e. $v_0 \in \cup t'_i$.
This means that the segment $[*, v_0]$ is cut somewhere and hence that $c \cap T = \emptyset$ since otherwise the cut would not be simple.
We thus have $c \subset t$ so that we can view c as a cut of t. Thus $R_c(t') = R_c(t)$. Also $v_0 \in \cup t'_i$ and the sum over v_0 decomposes as a sum over i and yields for each i the value

$$\prod \delta_{t'_j} = N_T(\delta_{t_i}) \prod_{j \neq i} \delta_{t_i} \tag{19}$$

Thus, since N_T is a derivation, the partial sum gives

$$\sum_{c\,(\mathrm{cut\,of}\,t)} N_T \left(\prod_{P_c(t)} \delta_{t_i} \right) \otimes \delta_{R_c(t)} \, . \tag{20}$$

Now this equals $N_T \left(\sum_c \prod_{P_c(t)} \delta_{t_i} \right) \otimes \delta_{R_c(t)}$ and we can group this sum with $N_T(\delta_t) \otimes 1$, using $N_T(1) = 0$ to get,

$$(N_T \otimes \mathrm{id})\, \Delta(\delta_t) \, . \tag{21}$$

which is the first term in the right hand side of the equation of the lemma.
We then consider the partial sum over pairs (v_0, c) with $v_0 \in R_c(t')$ and $c \cap T = \emptyset$. Then c is a cut of t as above, while v_0 now varies among the vertices of $R_c(t)$. One has $t'_i = t_i$ and $R_c(t') = R_c(t) \cup_{v_0} T$. Thus the sum over v_0 replaces $\delta_{R_c(t)}$ by $N_T(\delta_{R_c(t)})$ without touching the δ_{t_i}. We can group this with $1 \otimes N_T(\delta_t)$ and get,

$$(\mathrm{id} \otimes N_T)\, \Delta(\delta_t) \, . \tag{22}$$

which is the second term in the right hand side of the equation of the lemma.

We are now left only with the partial sum over pairs (v_0, c) such that $c \cap T \neq \emptyset$ (in which case $v_0 \in R_c(t')$). Let us then fix the nonempty simple cut of T,

$$c' = c \cap \Delta^1(T)\,, \tag{23}$$

and show that the corresponding partial sum is equal to,

$$\prod_{t_j \in P_{c'}(T)} \delta_{t_j} \otimes N_{R_{c'}(T)}(\Delta\,\delta_t)\,. \tag{24}$$

Since $\Delta^1(t') = \Delta^1(t) \cup \Delta^1(T)$, one has $c = c_1 \cup c'$ where c_1 now varies among (possibly empty) simple cuts of t. Moreover v_0 now varies in $R_{c_1}(t)$.
To each $\varepsilon \in c = c_1 \cup c'$ there is a corresponding fallen branch t_ε. For $\varepsilon \in c_1$ it is a fallen branch of t for c_1 while for $\varepsilon \in c'$ it is a fallen branch of T for c'. Thus the product of fallen branches is

$$\prod_{t_i \in P_{c_1}(t)} \delta_{t_i} \prod_{t_j \in P_{c'}(T)} \delta_{t_j}\,. \tag{25}$$

One has

$$\Delta\,\delta_t = \delta_t \otimes 1 + \sum_{c_1} \left(\prod_{P_{c_1}(t)} \delta_{t_i} \right) \otimes \delta_{R_{c_1}(t)}\,, \tag{26}$$

where c_1 varies among (possibly empty) simple cuts of t. Let $P = \prod_{t_j \in P_{c'}(T)} \delta_{t_j}$ and let us look at the terms in,

$$(P \otimes N_{R_{c'}(T)})(\Delta\,\delta_t)\,. \tag{27}$$

The term $\delta_t \otimes 1$ does not contribute since $N(1) = 0$. When we apply $P \otimes N_{R_{c'}(T)}$ to the term $\prod_{P_{c_1}(t)} \delta_{t_j} \otimes \delta_{R_{c_1}(t)}$ in $\Delta\,\delta_t$, we get

$$\sum_{v_0} P \prod_{P_{c_1}(t)} \delta_{t_j} \otimes \delta_{R_{c_1}(t) \cup_{v_0} R_{c'}(t')}\,. \tag{28}$$

where v_0 varies in $R_{c_1}(t)$.
With $t' = t \cup_{v_0} T$, one has $R_{c_1}(t) \cup_{v_0} R_{c'}(T) = R_c(t')$, for $c = c_1 \cup c'$. Thus we get the corresponding term of $\Delta(N_T(\delta_t))$, namely,

$$P \prod_{t_j \in P_{c_1}(t)} \delta_{t_j} \otimes \delta_{R_c(t')}\,. \tag{29}$$

Taking the sum over pairs (v_0, c_1) such that $v_0 \in R_{c_1}(t)$ yields the required equality and completes the proof of the lemma. $\|$

It is then natural to enlarge the Hopf algebra $\mathcal{H}_{rt}$ by introducing new generators $X_T, T \in \Sigma$ such that,

$$[X_T, \delta_t] = N_T(\delta_t) \tag{30}$$

and with coproduct rule given by,

$$\Delta X_T = X_T \otimes 1 + 1 \otimes X_T + \sum_c \prod_{P_c(T)} \delta_{t_j} \otimes X_{R_c(T)} \,. \tag{31}$$

This coproduct is superficially similar to (26), but the right hand side now involves both the δ's and the X's. In order to complete the presentation of the extended Hopf algebra $\mathcal{H}_{rtt}$, we need to compute the Lie bracket of the derivations N_T. This is straightforward and given by

Lemma.

$$[N_{T_1}, N_{T_2}] = \sum_{v_2 \in \Delta^0(T_2)} N_{T_2 \cup_{v_2} T_1} - \sum_{v_1 \in \Delta^0(T_1)} N_{T_1 \cup_{v_1} T_2} \,. \tag{32}$$

We are dealing with derivations of $\mathcal{H}_{rt}$ and it is thus enough to consider the action of both sides on δ_t. One has,

$$N_{T_1}(N_{T_2}(\delta_t)) = \sum_{v_0 \in \Delta^0(t)} N_{T_1}(\delta_{t \cup_{v_0} T_2}) = \sum_{v_0 \in \Delta^0(t)} \sum_{v_1 \in \Delta^0(t \cup_{v_0} T_2)} \delta_{(t \cup_{v_0} T_2) \cup_{v_1} T_1}$$

$$= \sum_{v_0 \in \Delta^0(t)} \sum_{v_1 \in \Delta^0(T_2)} \delta_{t \cup_{v_0} (T_2 \cup_{v_1} T_1)} + \sum_{\substack{v_0, v_1 \in \Delta^0(t) \\ v_0 \neq v_1}} \delta_{t \cup_{v_0} T_1 \cup_{v_1} T_2} \,.$$

The last term is symmetric in T_1, T_2 and thus does not contribute to the commutator which is thus given by the formula of the lemma.

We can thus complete the presentation of the Hopf algebra $\mathcal{H}_{rtt}$ by the rule,

$$[X_{T_1}, X_{T_2}] = \sum_{v_2 \in \Delta^0(T_2)} X_{T_2 \cup_{v_2} T_1} - \sum_{v_1 \in \Delta^0(T_1)} X_{T_1 \cup_{v_1} T_2} \,. \tag{33}$$

and define $\mathcal{H}_{rtt}$ as the envelopping algebra of the Lie algebra which is the linear span of the $X_T, \delta_t, T, t \in \Sigma$, with bracket given by (33), (30) and the commutativity of the δ's. We define a coproduct on $\mathcal{H}_{rtt}$ by (26) and (31). We thus get,

Theorem. *Endowed with the above structure $\mathcal{H}_{rtt}$ is a Hopf algebra. The equalities $\Lambda(\delta_n) = N^n(\delta_*)$, $\Lambda(Y) = X_*$, $\Lambda(X) = (X_{\substack{\bullet \\ \bullet}})$ determine a canonical homomorphism Λ of the Hopf algebra $\mathcal{H}_{cm}$ in the Hopf algebra $\mathcal{H}_{rtt}$.*

The best way to comprehend the Hopf algebra structure of $\mathcal{H}_{rtt}$ is to consider the natural action of $\mathcal{H}_{rtt}$ as an algebra on the dual of $\mathcal{H}_{rt}$, obtained by transposition. The compatibility of the algebra structures dictates the Hopf algebra structure, by transposing multiplication to comultiplication. Combining the basic Hopf algebra identity, $m(S \otimes Id)\Delta = \epsilon$ with equation (31) yields the following explicit formula for the antipode $S(X_T)$, $T \in \Sigma$,

$$S(X_T) = -X_T - \sum_c \prod_{P_c(T)} S(\delta_{t_j}) X_{R_c(T)} \,, \tag{34}$$

using the known formula for $S(\delta_{t_j})$ in the subalgebra $\mathcal{H}_{rt}$.
The reader should note that $S^2 \neq 1$ for the antipode S in $\mathcal{H}_{rtt}$, as this algebra is neither commutative nor cocommutative, comparable to the situation in $\mathcal{H}_{cm}$. Indeed, we now have a large supply of natural growth operators in generalization of that situation.

Let $\overline{\Delta}(X_T) = \Delta(X_T) - X_T \otimes 1 - 1 \otimes X_T$. For the multiple application of that subtracted coproduct, we can still uniquely write

$$\overline{\Delta}^n(X_T) = X_{T'} \otimes \cdots \otimes X_{T'\cdots'}, \ {}_{n+1\ '\mathrm{s}}.$$

It is obvious that the Hopf algebra endomorphism S^2 fulfills $S^2(\delta_t) = \delta_t$, while for the generators X_T we have

Proposition.

$$S^2(X_T) = X_T + N_{T''}(\delta_{T'}) + S(\delta_{T''})N_{T'''}(\delta_{T'}).$$

Proof: In the above notation, $S(X_T) = -X_T - S(\delta_{T'})X_{T''}$, and also $S(\delta_{T'})X_{T''} = \delta_{T'}S(X_{T''})$. Thus

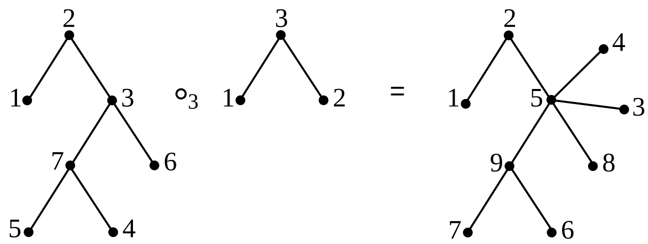
$$\begin{aligned} S^2(X_T) &= S[-X_T - S(\delta_{T'})X_{T''}] \\ &= X_T + S(\delta_{T'})X_{T''} - S(X_{T''})\delta_{T'} \\ &= X_T + S(\delta_{T'})X_{T''} - \delta_{T'}S(X_{T''}) - [S(X_{T''}), \delta_{T'}] \\ &= X_T - [S(X_{T''}), \delta_{T'}] \\ &= X_T + N_{T''}(\delta_{T'}) + S(\delta_{T''})N_{T'''}(\delta_{T'}), \end{aligned}$$

using $\overline{\Delta}^2$. ‖

It is of course desirable to extend to the Hopf algebra $\mathcal{H}_{rtt}$ the description of $\mathcal{H}_{cm}$ as a bicrossed product associated to the decomposition Lie G = Lie G_1+Lie G_2 of the Lie algebra of formal vector fields. Our next task will be to describe the Lie algebra $\mathcal{L}$ that will play the role of the Lie algebra of formal vector fields. As a preliminary remark, let us relate the Lie algebra structure $\mathcal{L}_1$ on the X_T given by (33) to an operad $\mathcal{P}$. This insertion operad [6] underlies the pre-Lie structure, whose antisymmetrization is the Lie bracket (33). The operad is obtained by considering as elements of $\mathcal{P}(n)$ a pair of rooted tree t and a bijection,

$$\sigma : \{1, \ldots, n\} \to \Delta^0(t)\,. \tag{35}$$

We then define $t \circ_i t'$ as $t \cup_{\sigma(i)} t'$, for $i \in \{1, \ldots, n\}$ and where the new bijection is obtained by shifting the labels of the vertices $\sigma(i+1) \ldots \sigma(n)$ to $i+n', \ldots, n+n'-1$ as well as the labels of the vertices $\sigma'(1) \ldots \sigma'(n')$ to $i, i+1, \ldots, i+n'-1$.



One has a natural action of S_n the group of permutations of $\{1, \ldots, n\}$ which replaces σ by $\sigma \circ \pi^{-1}$, i.e. replaces the labelling σ^{-1} of the vertices by $\pi \circ \sigma^{-1}$. One checks that,

$$t^{\pi} \circ_{\pi(i)} t'^{\rho} = (t \circ_i t')^{\alpha} \tag{36}$$

where α is obtained from the permutations π of $\{1, \ldots, n\}$, ρ of $\{1, \ldots, n'\}$ and $i \in \{1, \ldots, m\}$ by acting by ρ in $\{i, i+1, \ldots, i+n'-1\}$ and by π after collapsing the above interval to $\{i\}$.
One also checks the following two equalities for $\lambda \in \mathcal{P}(\ell)$, $\mu \in \mathcal{P}(m)$, $\nu \in \mathcal{P}(n)$

$$(\lambda \circ_i \mu) \circ_{j+m-1} \nu = (\lambda \circ_j \nu) \circ_i \mu \qquad 1 \leq i < j \leq \ell \tag{37}$$

$$(\lambda \circ_i \mu) \circ_{i-1+j} \nu = \lambda \circ_i (\mu \circ_j \nu) \qquad 1 \leq i \leq \ell\,,\ 1 \leq j \leq m\,. \tag{38}$$

The first is the independence of two graftings at two distinct vertices, and the second is a kind of associativity of grafting.

The Lie algebra $\mathcal{L}$

We shall now describe the Lie algebra $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ playing the role of the Lie algebra of formal vector fields, in the case of rooted trees, i.e. baring the same relation with $\mathcal{H}_{rtt}$ as the Lie algebra of formal vector fields does with $\mathcal{H}_{cm}$. We already know the Lie subalgebra $\mathcal{L}_1$ of the X_T's. The Lie algebra $\mathcal{L}_2$ is the Lie algebra of primitive elements in the dual of $\mathcal{H}_{rt}$. In order to obtain $\mathcal{L}$ we consider the natural actions of both $\mathcal{L}_1$ and $\mathcal{L}_2$ as derivations of the commutative algebra $\mathcal{H}_{rt}$. We already saw the action N of $\mathcal{L}_1$. The action of $\mathcal{L}_2$ is the canonical action of the Lie algebra of primitive elements of the dual of $\mathcal{H}_{rt}$ on the commutative algebra $\mathcal{H}_{rt}$. It is given by the following derivations M_T of $\mathcal{H}_{rt}$,

$$M_T(a) = \langle Z_T \otimes \mathrm{id}, \Delta(a) \rangle \qquad \forall\, a \in \mathcal{H}\,, \tag{39}$$

where, for $T \in \Sigma$, Z_T is the primitive element of the dual $\mathcal{H}_{rt}^*$ given by the linear form on $\mathcal{H}_{rt}$ which vanishes on any monomial $\delta_1\, \delta_2 \ldots\ldots \delta_n$ except for δ_T, with

$$\langle Z_T, \delta_T \rangle = 1. \tag{40}$$

One has $Z_T(ab) = Z_T(a)\, \varepsilon(b) + \varepsilon(b)\, Z_T(b)$ so that by construction M_T is a derivation of $\mathcal{H}$.

The Lie bracket of the Z_T's is given by the Lie algebra of rooted trees, i.e.

$$[Z_{T_1}, Z_{T_2}] = \sum (n(T_1, T_2; T) - n(T_2, T_1; T))\, Z_T\,. \tag{41}$$

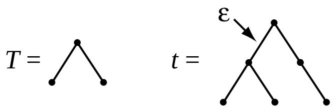
where $n(T_1, T_2; T)$ is the number of cuts c of T of cardinality one ($|c| = 1$) such that $P_c(T) = T_1$, $R_c(T) = T_2$.
For $a = \delta_t$ we get,

$$M_T(\delta_t) = \sum_{\substack{|c|=1 \\ P_c(t)=T}} \delta_{R_c(t)} \qquad \text{if } t \neq T \tag{42}$$

and,

$$M_T(\delta_T) = 1\,. \tag{43}$$

Thus $M_T(\delta_t) = 0$ unless $t = T$ or t admits an edge $\varepsilon \in \Delta^1(t)$ such that $P_\varepsilon(t) = T$.



By construction M is a representation of the Lie algebra $\mathcal{L}_2$ in the Lie algebra of derivations D of $\mathcal{H}_{rt}$ which preserve the linear span $\mathcal{D} = \{\Sigma\, \lambda_T\, \delta_T + \lambda_1 1\}$,

$$D\,(\mathcal{D}) \subset \mathcal{D}. \tag{44}$$

Similarly the representation N of $\mathcal{L}_1$ is given by derivations fulfilling (44). In order to show that $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ is a Lie algebra, let us now compute the commutator

$$M_{T_1}\, N_{T_2} - N_{T_2}\, M_{T_1}\,. \tag{45}$$

Let us first consider the case where T_1 and T_2 are not comparable, i. e. we assume that $T_1 \neq P_c(T_2)$ for all cuts c, $|c| = 1$ of T_2 and that $T_1 \neq t_1 \cup_v T_2$ for any tree t_1 and vertex $v \in \Delta^0(t_1)$. Let us show that in that case M_{T_1} and N_{T_2} actually commute. The nonzero terms in $M_{T_1}\, N_{T_2}(t)$ are given by $\delta_{P_\varepsilon(t \cup_{v_0} T_2)}$ for a vertex $v_0 \in \Delta^0(t)$ and an edge $\varepsilon \in \Delta^1(t \cup_{v_0} T_2)$ such that $P_\varepsilon(t \cup_{v_0} T_2) = T_1$. Now $\Delta^1(t \cup_{v_0} T_2) = \Delta^1(t) \cup \Delta^1(T_2)$, and if $\varepsilon \in \Delta^1(T_2)$ would yield a nonzero term, then T_1 would appear as $P_\varepsilon(T_2)$. Thus $\varepsilon \in \Delta^1(t)$.
Next if $v_0 \notin R_\varepsilon(t)$ then $v_0 \in P_\varepsilon(t)$ and $P_\varepsilon(t \cup_{v_0} T_2) = P_\varepsilon(t) \cup_{v_0} T_2$. But by hypothesis this cannot be T_1 so we get 0.
The only remaining case is $v_0 \in R_\varepsilon(t)$ so that $P_\varepsilon(t \cup_{v_0} T_2) = P_\varepsilon(t)$ while $R_\varepsilon(t \cup_{v_0} T_2) = R_\varepsilon(t) \cup_{v_0} T_2$, thus we get,

$$M_{T_1}\, N_{T_2}(t) = \sum_{\substack{v_0 \in \Delta^0(t), \varepsilon \in \Delta^1(t) \\ v_0 \in R_\varepsilon(t), P_\varepsilon(t) = T_1}} \delta_{R_\varepsilon(t) \cup_{v_0} T_2}\,. \tag{46}$$

But we have

$$M_{T_1}(t) = \sum_{\varepsilon \in \Delta^1(t), P_\varepsilon(t) = T_1} \delta_{R_\varepsilon(t)} \tag{47}$$

and

$$N_{T_2}(M_{T_1}(t)) = \sum_{\substack{\varepsilon\in\Delta^1(t), v_0\in R_\varepsilon(t) \\ P_\varepsilon(t)=T_1}} \delta_{R_\varepsilon(t)\cup_{v_0} T_2}\,. \tag{48}$$

Thus we see that if T_1 and T_2 are not comparable we get

$$M_{T_1}\, N_{T_2} = N_{T_2}\, M_{T_1}\,. \tag{49}$$

In general, given $t, T_1, T_2 \in \Sigma$ we define the integers $N(t, T_2; T_1)$ and $M(T_1, T_2; t)$ by,

$$N(t, T_2; T_1) = \langle N_{T_2}(\delta_t), Z_{T_1}\rangle \tag{50}$$

and

$$M(T_1, T_2; t) = \langle M_{T_1}(\delta_{T_2}), Z_t\rangle. \tag{51}$$

By construction $N(t, T_2; T_1)$ is the number of times T_1 occurs as $t \cup_v T_2$ while $M(T_1, T_2; t)$ is the number of times T_1 occurs as $P_c(T_2)$ with $|c| = 1$ and $R_c(T_2) = t$. We then get,

Lemma.

$$[M_{T_1}, N_{T_2}] = \sum_t N(t, T_2; T_1)\, M_t + \sum_t M(T_1, T_2; t)\, N_t\,. \tag{52}$$

First assume $|T_1| \geq |T_2|$ so that T_1 cannot be a $P_c(T_2)$, for $|c| = 1$ and $M(T_1, T_2; t) = 0$. Then the same computation of $[M_{T_1}, N_{T_2}](\delta_t)$ as above gives the sum of the $\delta_{R_\varepsilon(t)}$ such that T_1 occurs as a $P_\varepsilon(t)\cup_{v_0} T_2$. Fixing then $t_1 = P_\varepsilon(t)$ we see that we obtain the sum of the M_{t_1} with multiplicity given by the number of solutions of

$$t_1 \cup_v T_2 = T_1\,. \tag{53}$$

Next assume that $|T_1| < |T_2|$ so that T_1 can occur as $P_c(T_2)$, $|c| = 1$, but cannot occur as $t_1 \cup_v T_2$, so that $N(t, T_2; T_1) = 0$. Then in the above computation of $[M_{T_1}, N_{T_2}](\delta_t)$ the case $v_0 \in P_\varepsilon(t)$ above only gives 0 and the only nonzero contribution comes when $\varepsilon \in \Delta^1(T_2)$. One then has $R_\varepsilon(t\cup_{v_0} T_2) = t\cup_{v_0} R_\varepsilon(T_2)$ and $P_\varepsilon(t\cup_{v_0} T_2) = P_\varepsilon(T_2)$ which must be T_1 to yield a non zero result. Thus we obtain the sum of the $\delta_{R_\varepsilon(t\cup_{v_0} T_2)}$ where $P_\varepsilon(T_2) = T_1$. This equals the sum of the $\delta_{t\cup_{v_0} R_\varepsilon(T_2)}$ and hence, letting $t_2 = R_\varepsilon(T_2)$ the sum of the $M(T_1, T_2; t_2)\, N_{t_2}(\delta_t)$. We need to take care of (43), i.e. to consider the case where M_{T_1} is applied to some $t \cup_{v_0} T_2 = T_1$ which only occurs when $|T_1| \geq |T_2|$. For each such term one takes $c = \emptyset$ so the above discussion does not apply, but one can check that the additional contribution to both sides of (52) do agree when evaluated on t fulfilling (53) for some $v \in \Delta^0(t)$.

We can now define the full Lie algebra $\mathcal{L}$ of rooted trees by introducing new generators of the form, Z_{-t} where t is a rooted tree, and extending the Lie bracket (41) based on the above lemma. We associate Z_{-T} with $-N_T$ and Z_T with M_T and work out the Lie brackets so that we get a representation. In particular the elements Z_0, Z_{-1} now become,

$$Z_0 = Z_{-*}, Z_{-1} = Z_{-T}\,. \tag{54}$$

We use the $-$ sign, $-N_T$ to get that the commutator with Z_{-*} does give the grading of the Lie algebra. Indeed if we apply (52) for $T_2 = *$ we get

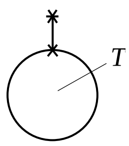
$$[-N_*, M_T] = |T|\, M_T\,, \tag{55}$$

while one has,

$$[-N_*, N_T] = (1 - |T|)\, N_T\,. \tag{56}$$

Theorem. $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ *is a Lie algebra.*
The Hopf algebra $\mathcal{H}_{rtt}$ *is the bicrossed product associated to the decomposition* $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$.

As a final remark, note that the Lie subalgebra $\mathcal{L}_2$ generated by the Z_T is naturally isomorphic to a subalgebra of $\mathcal{L}_1$ generated by the Z_{-T}. Indeed one lets $*T$ be the new rooted tree given by,

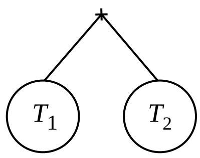


(57)

Then the following map is an inclusion $\mathcal{L}_2 \subset \mathcal{L}_1$,

$$Z_T \to \frac{1}{S_T}\, Z_{-(*T)}\,. \tag{58}$$

By (41) we see that this is a Lie algebra homomorphism since the grafting at $*$ gives a symmetric result, which drops out of the bracket:



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2 Graphs

Formal Definitions

We only consider graphs without self-loops: no edge starts and ends in the same vertex. We allow for multiple edges though: two vertices might be connected by more than one edge.
First, we define n-particle irreducible (n-PI) graphs.

Definition. *A* n*-particle irreducible graph* Γ *is graph such that upon removal of any set of* n *of its edges it is still connected. Its set of edges is denoted by* $\Gamma^{[1]}$ *and its set of vertices is denoted by* $\Gamma^{[0]}$*, edges and vertices can be of various* *different type.*

The type of an edge is often indicated by the way we draw it: (un-)oriented straight lines, curly lines, dashed lines and so on. These types of edges, often called propagators in physicists parlance, are chosen in accordance to Lorentz covariant wave equations: the propagator as the analytic expression assigned to an edge is an inverse wave operator with boundary conditions typically chosen in accordance with causality.
The types of vertices are determined by the types of edges to which they are attached:

Definition. *For any vertex* $v \in \Gamma^{[0]}$ *we call the set* $f_v := \{f \in \Gamma^{[1]} \mid v \cap f \neq \emptyset\}$ *its type.*

Note that f_v is a set of edges.
Of particular importance are the 1PI graphs. They decompose into disjoint graphs upon removal of an edge. Note that any n-PI graphs is also $(n-1)$-PI, $\forall n \geq 1$. A graph which is not 1-PI is called reducible. Also, any connected graph is considered as 0-PI.
A further notion needed is the one of external and internal edges.

Definition. *An edge* $f \in \Gamma^{[1]}$ *is internal, if* $\{v_f\} := f \cap \Gamma^{[0]}$ *is a set of two elements.*

So, internal edges connect two vertices of the graph Γ.

Definition. *An edge* $f \in \Gamma^{[1]}$ *is external, if* $f \cap \Gamma^{[0]}$ *is a set of one element.*

As we exclude self-loops, this means that an external edge has an open end. Thus external edges are associated with a single vertex of the graph. These edges correspond to external particles interacting in the way prescribed by the graph. There are obvious gluing operations combining 1PI graphs into reducible graphs, by identifying two open ends of edges of the same type originating from different 1PI graphs. We will make no use of reducible graphs here but note that the Hopf and Lie algebra structures could be set up in this context as well.
$\Gamma^{[1]}$ obviously decomposes into the set of internal edges and the set of external edges of a graph Γ,

$$\Gamma^{[1]} = \Gamma^{[1]}_{\text{ext}} \cup \Gamma^{[1]}_{\text{int}}.$$

We now turn to the possibilities of inserting graphs into each other. Our first requirement is to establish bijections between sets of edges so that we can define gluing operations.

Definition. *We call two sets of edges* I_1, I_2 *compatible,* $I_1 \sim I_2$*, iff they contain the same number of edges, of the same type.*

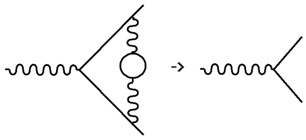
Compatibility is an equivalence relation. We will utilize it to glue graphs into each other. To compare vertices, we look at the adjacent edges:

Definition. *Two vertices* v_1, v_2 *are of the same type, if* f_{v_1} *is compatible with* f_{v_2}*.*

Quite often, we will shrink a graph to a point. The only useful information still available after that process is about its set of external edges:

Definition. *We define* $\mathbf{res}(\Gamma)$ *to be the result of identifying* $\Gamma^{[0]} \cup \Gamma^{[1]}_{\rm int}$ *with a point in* Γ.

An example is



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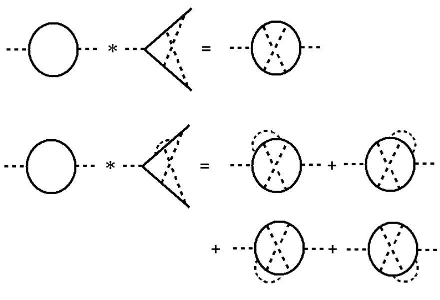
Note that $\mathbf{res}(\Gamma)^{[1]} \equiv \mathbf{res}(\Gamma)^{[1]}_{\rm ext} \sim \Gamma^{[1]}_{\rm ext}$. By construction all graphs which have compatible sets of external edges have the same residue.
If the set $\Gamma^{[1]}_{\rm ext}$ is empty, we call Γ a vacuumgraph, if it contains a single element we call the graph a tadpole graph. Vacuum graphs and tadpole graphs will be discarded in most of what follows. If this set contains two elements, we call Γ a self-energy graph, if it contains more than two elements, we call it an interaction graph. Further we restrict ourselves to graphs which have vertices such that the cardinality of their types is ≥ 2. If needed, for example in the presence of external fields, this can be relaxed.
A further important notion is the gluing of graphs into each other. It is the opposite of the shrinking of a graph to its residue. While in that process, a graph is reduced to a vertex of a specified type, we can replace any vertex $v \in \Gamma^{[0]}$ of type f_v by a graph γ, as long as $f_v \sim \gamma^{[1]}_{\rm ext}$ - a vertex will be replaced by a graph which has external edges compatible with its type.
To specify such a gluing of γ into Γ we first have to choose an internal vertex v where we wish to glue. If the type of v is incompatible with $\gamma^{[1]}_{\rm ext}$, we define the result to vanish. If the two sets of edges are compatible, we will have in general to choose a bijection between the two sets of edges. Summing over all places and bijections defines an operation $\Gamma \star \gamma$ which sums over all ways of inserting γ into Γ. We impose a normalization such that topologically different graphs are generated with unit multiplicity. The following picture illustrates this process.



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Proposition. *This gluing operation is pre-Lie.*

Proof: It suffices to show that for 1PI graphs Γ_i, $i = 1, 2, 3$, we have

$$\Gamma_1 \star (\Gamma_2 \star \Gamma_3) - (\Gamma_1 \star \Gamma_2) \star \Gamma_3 = \Gamma_1 \star (\Gamma_3 \star \Gamma_2) - (\Gamma_1 \star \Gamma_3) \star \Gamma_2.$$

This is elementary using that both sides reduce to the sum over all ways of gluing Γ_2 and Γ_3 simultaneously into Γ_1 at disjoint places. $\|$

Note that this pre-Lie operation can be extended to the insertion at internal edges (self-energies). Furthermore, external structures [3] can be incorporated easily, using coloured types of vertices.
Choices of types of lines and vertices are typically dictated by a chosen QFT, where, in particular, one often only consider superficially divergent graphs. External structures reflect their powercounting degree of divergence.
We let $\mathcal{L}_{FG}$ be any such chosen Lie-algebra generated from this pre-Lie product, and $\mathcal{H}_{FG}$ be the commutative Hopf algebra which we obtain as the dual of the universal enveloping algebra of $\mathcal{L}_{FG}$.

Derivations on the Hopf algebra

We have the decomposition of $\mathcal{H}_{FG}$ by the bidegree $\mathcal{H}_{FG} = \oplus_{k=0}^{\infty} \mathcal{H}_{FG}^{[k]}$, reduced to scalars $\in \mathcal{H}_{FG}^{[0]}$ by the counit. The linear basis of $\mathcal{H}_{FG}$ is denoted by $\mathcal{H}_{FG,L}$. It is spanned by generators δ_Γ, where Γ is a 1PI graph. Elements of $\mathcal{H}_{FG}$ are polynomials in these commutative variables.
We write Z_Γ for the dual basis of the universal enveloping algebra with pairing

$$\langle Z_\Gamma, \delta_{\Gamma'} \rangle = \delta^K_{\Gamma,\Gamma'},$$

where on the rhs we have the Kronecker δ^K, and extend the pairing by means of the coproduct

$$\langle Z_{\Gamma_1} Z_{\Gamma_2}, X \rangle = \langle Z_{\Gamma_1} \otimes Z_{\Gamma_2}, \Delta(X) \rangle.$$

For $X = \sum_i c_i \Gamma_i$, we extend by linearity so that $\delta_X = \sum_i c_i \delta_{\Gamma_i}$, and similarly for Z_X.
Quite often, we want to refer to the graph(s) which index an element in $\mathcal{H}_{FG}$ or $\mathcal{L}_{FG}$. For that purpose, for each element in $\mathcal{H}_{FG}$ and each element in $\mathcal{L}_{FG}$ we introduce a map to graphs:

$$\overline{Z_X} = X, \overline{\delta_X} = X.$$

Further, we write $\Delta(X) = \sum_i X'_{(i)} \otimes X''_{(i)}$ for the coproduct in the Hopf algebra $\mathcal{H}_{FG}$.
The Lie algebra $\mathcal{L}_{FG}$ gives rise to two representations acting as derivations on the Hopf algebra $\mathcal{H}_{FG}$:

$$Z^+_\Gamma \times \delta_X = \delta_{X \star \Gamma}$$

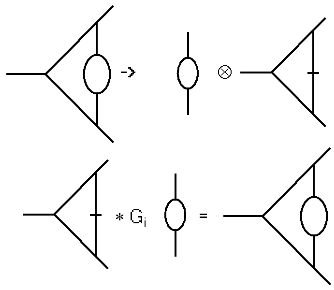
and

$$Z^-_\Gamma \times \delta_X = \sum_i \langle Z^+_\Gamma, X'_{(i)} \rangle X''_{(i)}.$$

Furthermore, any term in the coproduct of a 1PI graph Γ determines gluing data G_i such that

$$\Gamma = \Gamma''_{(i)} \star_{G_i} \Gamma'_{(i)}, \forall i.$$

Here, G_i specifies vertices in $\Gamma''_{(i)}$ and bijections of their types with the elements of $\Gamma'_{(i)}$ such that Γ is regained from its parts:



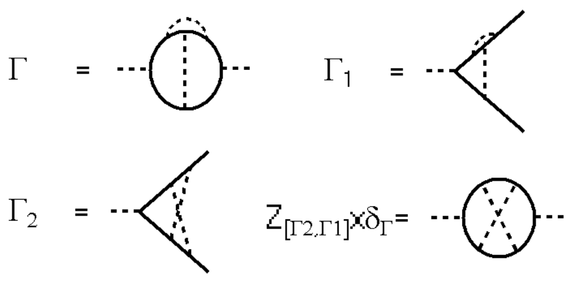
The first line gives a term (i) in the coproduct, decomposing this graph into its only divergent subgraph (assuming we have chosen ϕ^3 in six dimensions, say) and the corresponding cograph, the second line shows the gluing G_i for this term, in this example .
We want to understand the commutator

$$[Z^+_{\Gamma_1}, Z^-_{\Gamma_2}],$$

acting as a derivation on the Hopf algebra element δ_X. To this end introduce

$$Z_{[\Gamma_1,\Gamma_2]} \times \delta_X = \sum_i \langle Z^+_{\Gamma_2}, X'_{(i)}\rangle X''_{(i)} \star_{G_i} \Gamma_1.$$

Here, the gluing operation G_i still acts such that each topologically different graph is generated with unit multiplicity.



Note that if $\mathbf{res}(\Gamma_1) \not\sim \mathbf{res}(\Gamma_2)$, $Z_{[\Gamma_1,\Gamma_2]} \times \delta_X$ vanishes, as the existence of a bijection between edges adjacent to Γ_2 in X and $\Gamma^{[1]}_{1,\mathrm{ext}}$ demands the compatibility of the residues of Γ_1, Γ_2.

Let X, Y be related as $\overline{Z_{[\Gamma_1,\Gamma_2]} \times \delta_X} = Y$, for a 1PI graph X. Then, Y is a sum of say k 1PI graphs. We immediately have thanks to our gluing conventions

Proposition. $\overline{Z_{[\Gamma_2,\Gamma_1]} \times \delta_Y} = k\ X$.

Let us now consider

$$[Z_{[\Gamma_1,\Gamma_2]}, Z_{[\Gamma_3,\Gamma_4]}] \times \delta_X.$$

We first define

$$Y_{234} := \{Y \in \mathcal{H}_{FG,L} | \langle Z_{\Gamma_2}, Z_{[\Gamma_3,\Gamma_4]} \times \delta_Y \rangle = 1\}$$

and

$$Y_{412} := \{Y \in \mathcal{H}_{FG,L} | \langle Z_{\Gamma_4}, Z_{[\Gamma_1,\Gamma_2]} \times \delta_Y \rangle = 1\}.$$

Let $\Delta_\Gamma : \mathcal{H}_{FG} \to \mathcal{H}_{FG} \otimes \mathcal{H}_{FG}$ be the map

$$X \to \sum_i X'_{(i)} \otimes [X''_{(i)} \star_{G_i} \Gamma] \tag{59}$$

and let us write ∂_2 for the map $X \to \langle Z^+_{\Gamma_2}, X \rangle$. Then,

$$Z_{[\Gamma_1,\Gamma_2]} \times \delta_X = (\partial_2 \otimes \mathrm{id}) \circ \Delta_{\Gamma_1}$$

which justifies the shorthand notation $1^+\partial_2 X$ for the above.
Then, the desired commutator is

$$[1^+\partial_2 3^+\partial_4 - 3^+\partial_4 1^+\partial_2]X.$$

Let us consider $1^+\partial_2 3^+\partial_4 X$ first. We want to compare it with $1^+3^+\partial_{2,4}X$. These are the terms generated by shrinking Γ_2, Γ_4 at disjoint places, and gluing Γ_1 for the residue of Γ_2, and Γ_3 for the residue of Γ_4.
What we now need to know is the commutator $1^+[\partial_2, 3^+]\partial_4$. There are two cases:
i) Γ_2 is a proper subgraph of Γ_3, $\Gamma_2 \subset \Gamma_3$. Then,

$$1^+\partial_2 3^+\partial_4 X = (1^+\partial_2\Gamma_3)^+\partial_4 X + 1^+3^+\partial_{2,4}X.$$

ii) $\Gamma_2 \not\subset \Gamma_3$. Then, for any $X'_{(i)}$ such that $X'_{(i)} = Y$, $Y \in Y_{234}$, we have a contribution as $3^+\partial_4 Y = \Gamma_2$, and by the previous proposition, $\Gamma_2 = 4^+\partial_3 Y$. Hence

$$1^+\partial_2 3^+\partial_4 X = 1^+\partial_{4^+\partial_3 2}X + 1^+3^+\partial_{2,4}X.$$

Consider now $3^+\partial_4 1^+\partial_2 X$. Similarly, we find two cases:
i) Γ_4 is a proper subgraph of Γ_1, $\Gamma_4 \subset \Gamma_1$. Then,

$$3^+\partial_4 1^+\partial_2 X = (3^+\partial_4\Gamma_1)^+\partial_2 X + 3^+1^+\partial_{4,2}X.$$

ii) $\Gamma_4 \not\subset \Gamma_1$. Then, for any $X'_{(i)}$ such that $X'_{(i)} = Y$, $Y \in Y_{412}$, we have a contribution as $1^+\partial_2 Y = \Gamma_4$, and by the proposition again, $\Gamma_4 = 2^+\partial_1 Y$. Hence

$$3^+\partial_4 1^+\partial_2 X = 3^+\partial_{2^+\partial_1 4}X + 3^+1^+\partial_{4,2}X.$$

As

$$1^+3^+\partial_{2,4}X = 3^+1^+\partial_{4,2}X,$$

we get for the commutator, returning to the full fledged notation,

$$\begin{aligned}[Z_{[\Gamma_1,\Gamma_2]}, Z_{[\Gamma_3,\Gamma_4]}] &= +Z_{[\overline{Z_{[\Gamma_1,\Gamma_2]}\times\delta_{\Gamma_3}},\Gamma_4]} - Z_{[\Gamma_3,\overline{Z_{[\Gamma_2,\Gamma_1]}\times\delta_{\Gamma_4}}]} \\ &\quad -Z_{[\overline{Z_{[\Gamma_3,\Gamma_4]}\times\delta_{\Gamma_1}},\Gamma_2]} + Z_{[\Gamma_1,\overline{Z_{[\Gamma_4,\Gamma_3]}\times\delta_{\Gamma_2}}]} \\ &\quad -\delta^K_{\Gamma_2,\Gamma_3} Z_{[\Gamma_1,\Gamma_4]} + \delta^K_{\Gamma_1,\Gamma_4} Z_{[\Gamma_2,\Gamma_3]}.\end{aligned}$$

Let us check that this bracket fulfills a Jacobi identity. Equivalently, we can check that

$$Z_{[\Gamma_1,\Gamma_2]} \star Z_{[\Gamma_3,\Gamma_4]} := Z_{[\overline{Z_{[\Gamma_1,\Gamma_2]}\times\delta_{\Gamma_3}},4]} + Z_{[\Gamma_1,\overline{Z_{[\Gamma_4,\Gamma_3]}\times\delta_{\Gamma_2}}]}$$

defines a right or left pre-Lie product. Indeed, we find, returning to our shorthand notation:

$$\begin{aligned} & (1^+\partial_2 3^+\partial_4)5^+\partial_6 - 1^+\partial_2(3^+\partial_4 5^+\partial_6) \\ = & +(1^+(\partial_2 3))^+\partial_4 5^+\partial_6 + 1^+\partial_{4+\partial_3 2}5^+\partial_6 \\ & -1^+\partial_2(3^+\partial_4 5)^+\partial_6 - 1^+\partial_2 3^+\partial_{6+\partial_5 4} \\ = & +\underbrace{((1^+\partial_2 3)^+\partial_4 5)^+\partial_6}_{b_1} + \underbrace{(1^+\partial_2 3)^+\partial_{6+\partial_5 4}}_{a} \\ & +\underbrace{(1^+\partial_{4+\partial_3 2}5)^+\partial_6}_{b_2} + \underbrace{1^+\partial_{6+\partial_5(4+\partial_3 2)}}_{c_3} \\ & -\underbrace{(1^+\partial_2(3^+\partial_4 5))^+\partial_6}_{b_3} - \underbrace{1^+\partial_{6+\partial_{3+\partial_4 5}2}}_{c_1} \\ & -\underbrace{(1^+\partial_2 3)^+\partial_{6+\partial_5 4}}_{a} - \underbrace{1^+\partial_{(6+\partial_5 4)+\partial_3 2}}_{c_2}\end{aligned}$$

The two "a" terms cancel, while the terms b_1, b_2, b_3 add up to a contribution $(1^+3^+\partial_{2,4}5)^+\partial_6$ which is symmetric under exchange of the index pair $(1,2)$ with $(3,4)$. This term only contributes when Γ_2 appears as a subgraph of Γ_5. The terms c_1, c_2, c_3 add up to a contribution $1^+\partial_{6+4+\partial_{5,3}2}$ which only contributes when Γ_5 appears as a subgraph of Γ_2, and is symmetric under exchange of the index pair $(3,4)$ with $(5,6)$. The b_i-terms and the c_i terms are mutually exclusive. Furthermore, when the b_i terms contribute, we get a right pre-Lie product, while when the c_i terms contribute, we get a left pre-Lie product. In all cases, we then fulfill the Jacobi identity. ‖

Hence, we have established the following theorem:

Theorem. *For all 1PI graphs* Γ_i*, s.t.* $\mathbf{res}(\Gamma_1) = \mathbf{res}(\Gamma_2)$ *and* $\mathbf{res}(\Gamma_3) = \mathbf{res}(\Gamma_4)$*, the bracket*

$$[Z_{[\Gamma_1,\Gamma_2]}, Z_{[\Gamma_3,\Gamma_4]}] = +Z_{[\overline{Z_{[\Gamma_1,\Gamma_2]}\times\delta_{\Gamma_3}},\Gamma_4]} - Z_{[\Gamma_3,\overline{Z_{[\Gamma_2,\Gamma_1]}\times\delta_{\Gamma_4}}]}$$

$$
\begin{aligned}
&-Z_{[\overline{Z_{[\Gamma_3,\Gamma_4]}\times\delta_{\Gamma_1}},\Gamma_2]}+Z_{[\Gamma_1,\overline{Z_{[\Gamma_4,\Gamma_3]}\times\delta_{\Gamma_2}}]}\\
&-\delta^K_{\Gamma_2,\Gamma_3}Z_{[\Gamma_1,\Gamma_4]}+\delta^K_{\Gamma_1,\Gamma_4}Z_{[\Gamma_3,\Gamma_2]}.
\end{aligned}
$$

defines a Lie algebra of derivations acting on the Hopf algebra $\mathcal{H}_{FG}$ via

$$
Z_{[\Gamma_i,\Gamma_j]}\times\delta_X=\sum_I\langle Z^+_{\Gamma_2},\delta_{X'_{(i)}}\rangle\delta_{X''_{(i)}\star_{G_i}\Gamma_1},
$$

where the gluing data G_i are normalized as before.

The Kronecker δ^K terms just eliminate the overcounting when combining all cases in a single equation.
We note that $Z_{[\Gamma,\Gamma]}\times\delta_X=k_\Gamma\delta_X$, where k_Γ is the number of appearances of Γ in X and where we say that a graph Γ appears k times in X if k is the largest integer such that

$$
\langle\Gamma^k\otimes\mathrm{id},\Delta(\delta_X)\rangle
$$

is non-vanishing.
Furthermore, we note that $I:Z_{[\Gamma_1,\Gamma_2]}\to Z_{[\Gamma_2,\Gamma_1]}$ is an anti-involution such that

$$
I([Z_{[\Gamma_1,\Gamma_2]},Z_{[\Gamma_3,\Gamma_4]}])=-[I(Z_{[\Gamma_1,\Gamma_2]}),I(Z_{[\Gamma_3,\Gamma_4]})],
$$

by inspection. We have

$$
[Z_{[\Gamma_1,\Gamma_2]},Z_{[\Gamma_2,\Gamma_1]}]=Z_{[\Gamma_1,\Gamma_1]}-Z_{[\Gamma_2,\Gamma_2]}.
$$

Further structural analysis is left to future work.
By construction, we have

Proposition.

$$
\begin{aligned}
Z^+_\Gamma&\equiv Z_{[\Gamma,\mathbf{res}(\Gamma)]},\\
Z^-_\Gamma&\equiv Z_{[\mathbf{res}(\Gamma),\Gamma]}.
\end{aligned}
$$

Also, we immediately conclude

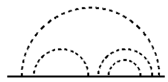
Corollary. $[Z^-_X,Z^-_Y]=-Z^-_{\overline{[Z^+_X,Z^+_Y]}}$.
Finally, we get the desired commutator

Corollary.

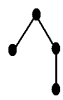
$$
\begin{aligned}
[Z_{[\Gamma_1,\mathbf{res}(\Gamma_1)]},Z_{[\mathbf{res}(\Gamma_2),\Gamma_2]}] &= +Z_{[\overline{Z_{[\Gamma_1,\mathbf{res}(\Gamma_1)]}\times\delta_{\mathbf{res}(\Gamma_2)}},\Gamma_2]}-Z_{[\mathbf{res}(\Gamma_2),\overline{Z_{[\mathbf{res}(\Gamma_1),\Gamma_1]}\times\delta_{\Gamma_2}}]}\\
&\quad -Z_{[\overline{Z_{[\mathbf{res}(\Gamma_2),\Gamma_2]}\times\delta_{\Gamma_1}},\mathbf{res}(\Gamma_1)]}+Z_{[\Gamma_1,\overline{Z_{[\Gamma_2,\mathbf{res}(\Gamma_2)]}\times\delta_{\mathbf{res}(\Gamma_1)}}]}\\
&\quad -\delta^K_{\mathbf{res}(\Gamma_1),\mathbf{res}(\Gamma_2)}Z_{[\Gamma_1,\Gamma_2]}+\delta^K_{\Gamma_1,\Gamma_2}Z_{[\mathbf{res}(\Gamma_2),\mathbf{res}(\Gamma_1)]}\\
&= \delta^K_{\mathbf{res}(\Gamma_1),\mathbf{res}(\Gamma_2)}Z_{[\Gamma_1,\Gamma_2]}+\delta^K_{\Gamma_1,\Gamma_2}Z_{[\mathbf{res}(\Gamma_2),\mathbf{res}(\Gamma_1)]}\\
&\quad -Z_{[\mathbf{res}(\Gamma_2),\overline{Z_{[\mathbf{res}(\Gamma_1),\Gamma_1]}\times\delta_{\Gamma_2}}]}-Z_{[\overline{Z_{[\mathbf{res}(\Gamma_2),\Gamma_2]}\times\delta_{\Gamma_1}},\mathbf{res}(\Gamma_1)]}\\
&= \delta^K_{\mathbf{res}(\Gamma_1),\mathbf{res}(\Gamma_2)}Z_{[\Gamma_1,\Gamma_2]}+\delta^K_{\Gamma_1,\Gamma_2}Z_{[\mathbf{res}(\Gamma_2),\mathbf{res}(\Gamma_1)]}\\
&\quad -Z^-_{\overline{Z_{[\mathbf{res}(\Gamma_1),\Gamma_1]}\times\delta_{\Gamma_2}}}-Z^+_{\overline{Z_{[\mathbf{res}(\Gamma_2),\Gamma_2]}\times\delta_{\Gamma_1}}}.
\end{aligned}
$$

We can now make contact with derivations in the Hopf algebra of rooted trees. Let us consider the Hopf algebra of iterated one-loop self-energies in massless Yukawa theory in four dimensions. There is a one-to-one correspondence Θ between iterated one-loop fermion self-energy graphs and undecorated rooted trees:

Γ =



$\Theta(\Gamma)$=



.

Let Γ_2, Γ_3 be arbitrary such fermion self-energy graphs and let Γ_4 be the one-loop self-energy graph, and Γ_1 be its residue, a two-point vertex with two fermionic external legs.
Note that

$$\mathbf{res}(\Gamma_4) = \mathbf{res}(\Gamma_2) = \mathbf{res}(\Gamma_3) = \Gamma_1.$$

The isomorphism Θ to undecorated rooted trees delivers the previous result on undecorated rooted trees: Indeed,

$$\Theta(Z_{[\Gamma_3,\Gamma_4]}) = N(\Theta(\Gamma_3)),$$

and

$$\Theta(Z_{[\Gamma_1,\Gamma_2]}) = M(\Theta(\Gamma_2)).$$

We have, using the previous theorem,

$$\begin{aligned}\Theta([Z_{[\Gamma_1,\Gamma_2]}, Z_{[\Gamma_3,\Gamma_4]}]) &= [M(\Theta(\Gamma_2)), N(\Theta(\Gamma_3))] \\ &= \Theta\Big[Z_{[\overline{Z_{[\mathbf{res}(\frown),\Gamma_2]}\times\delta_{\Gamma_3}}, \frown]} \\ &\quad + Z_{[\mathbf{res}(\frown), \overline{Z_{[\frown,\Gamma_3]}\times\delta_{\Gamma_2}}]} \\ &\quad - \delta^K_{\Gamma_2,\Gamma_3} Z_{[\mathbf{res}(\frown),\frown]} \Big],\end{aligned}$$

in accordance with the results of the previous section. We used the fact that the residue of a graph contains no subgraph,

$$Z_{[\Gamma_3,\frown]} \times \delta_{\mathbf{res}(\frown)} = 0,$$

and that

$$Z_{[\Gamma_2,\mathbf{res}(\frown)]} \times \delta_{\frown} = 0.$$

The above uses naturally growth by identifying the root of a tree with any feet of another tree. We can also work out from our general results the commutator of other derivations, using, for example, natural growth by connecting with an extra edge the root of a tree to all the vertices of another one.

Conclusions

We only considered the Lie algebra aspect for Feynman graphs. A bicrossed structure can be constructed as well, say by enlarging $\mathcal{H}_{FG}$ to $\mathcal{H}_{FGG}$ using appropriate insertion of subgraphs as a natural growth.
The algebraic structures here provided cover all operations which one encounters in the perturbative expansion of a quantum field theory: insertion and elimination of subgraphs. While the construction of local counterterms demands the elimination of subgraphs γ by $\mathbf{res}(\gamma)$ on the expense of multiplication with their counterterms $S_R(\gamma)$ [3], the Dyson–Schwinger quantum equations of motions demand that any local interaction, described by a vertex v, can as well be mediated by any graph Γ with $\mathbf{res}(\Gamma) = v$, and hence the insertion of Γ for v in all possible ways determines naturally the series of Feynman graphs providing a fixpoint for those equations.

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