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Non-commutative Geometry and the Spectral Model of Space-time

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Abstract. This is a report on our joint work with A. Chamseddine and M. Marcolli. This essay gives a short introduction to a potential application in physics of a new type of geometry based on spectral considerations which is convenient when dealing with non-commutative spaces, *i.e.*, spaces in which the simplifying rule of commutativity is no longer applied to the coordinates. Starting from the phenomenological Lagrangian of gravity coupled with matter one infers, using the spectral action principle, that space-time admits a fine structure which is a subtle mixture of the usual 4-dimensional continuum with a finite discrete structure F. Under the (unrealistic) hypothesis that this structure remains valid (*i.e.*, one does not have any "hyperfine" modification) until the unification scale, one obtains a number of predictions whose approximate validity is a basic test of the approach.

1. Background

Our knowledge of space-time can be summarized by the transition from the flat Minkowski metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{1}$$

to the Lorentzian metric

$$ds^2 = g_{\mu\nu} dx^\mu \, dx^\nu \tag{2}$$

of curved space-time with gravitational potential $g_{\mu\nu}$. The basic principle is the Einstein-Hilbert action principle

$$S_E[\, g_{\mu\nu}] = \frac{1}{G} \int_M r \sqrt{g} \; d^4x \tag{3}$$

where r is the scalar curvature of the space-time manifold M. This action principle only accounts for the gravitational forces and a full account of the forces observed so far requires the addition of new fields, and of corresponding new terms S_{SM} in the action, which constitute the Standard Model so that the total action is of the form

$$S = S_E + S_{SM} \,. \tag{4}$$

Passing from classical to quantum physics is achieved by the recipe of Dirac and Feynman so that the probability amplitude of a classical configuration A is

$$e^{i \frac{S(A)}{\hbar}} \,. \tag{5}$$

When combined with perturbative renormalization this recipe agrees remarkably well with experiment, but meets (at least) two basic problems:

- One cannot maintain both unitarity and renormalizability at arbitrary scales for the gravitational potential $g_{\mu\nu}$.
- The action S_{SM} is complicated beyond reason and thus only appears as "phenomenological".

To appreciate the second statement we give the explicit form of $S_{SM} = \int_M \mathcal{L}_{SM} \sqrt{g}\, d^4x$ below (*cf.* [26]):

$$\begin{gathered}
\mathcal{L}_{SM} = -\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4} g^2_s f^{abc} f^{ade} g^b_\mu g^c_\nu g^d_\mu g^e_\nu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - \\
M^2 W^+_\mu W^-_\mu - \tfrac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c^2_w} M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - igc_w(\partial_\nu Z^0_\mu (W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - Z^0_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + Z^0_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - \\
igs_w(\partial_\nu A_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu (W^+_\nu \partial_\nu W^-_\mu - \\
W^-_\nu \partial_\nu W^+_\mu)) - \tfrac{1}{2} g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \tfrac{1}{2} g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2 c^2_w (Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - \\
Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s^2_w (A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w (A_\mu Z^0_\nu (W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \tfrac{1}{2}\partial_\mu H \partial_\mu H - \tfrac{1}{2} m^2_h H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+ \phi^- - \\
\tfrac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \tfrac{1}{2c^2_w} M^2 \phi^0 \phi^0 - \beta_h \Big(\tfrac{2M^2}{g^2} + \tfrac{2M}{g} H + \tfrac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-) \Big) + \tfrac{2M^4}{g^2}\alpha_h - \\
g\alpha_h M \left(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^- \right) - \\
\tfrac{1}{8} g^2 \alpha_h \left(H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2 H^2 \right) - \\
gMW^+_\mu W^-_\mu H - \tfrac{1}{2} g \tfrac{M}{c^2_w} Z^0_\mu Z^0_\mu H - \\
\tfrac{1}{2} ig \left(W^+_\mu (\phi^0 \partial_\mu \phi^- - \phi^- \partial_\mu \phi^0) - W^-_\mu (\phi^0 \partial_\mu \phi^+ - \phi^+ \partial_\mu \phi^0) \right) + \\
\tfrac{1}{2} g \left(W^+_\mu (H\partial_\mu \phi^- - \phi^- \partial_\mu H) + W^-_\mu (H\partial_\mu \phi^+ - \phi^+ \partial_\mu H) \right) + \tfrac{1}{2} g \tfrac{1}{c_w} Z^0_\mu (H\partial_\mu \phi^0 - \phi^0 \partial_\mu H) - \\
ig \tfrac{s^2_w}{c_w} M Z^0_\mu (W^+_\mu \phi^- - W^-_\mu \phi^+) + igs_w M A_\mu (W^+_\mu \phi^- - W^-_\mu \phi^+) - ig \tfrac{1-2c^2_w}{2c_w} Z^0_\mu (\phi^+ \partial_\mu \phi^- - \\
\phi^- \partial_\mu \phi^+) + igs_w A_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) - \tfrac{1}{4} g^2 W^+_\mu W^-_\mu \left(H^2 + (\phi^0)^2 + 2\phi^+\phi^- \right) - \\
\tfrac{1}{8} g^2 \tfrac{1}{c^2_w} Z^0_\mu Z^0_\mu \left(H^2 + (\phi^0)^2 + 2(2s^2_w - 1)^2 \phi^+\phi^- \right) - \tfrac{1}{2} g^2 \tfrac{s^2_w}{c_w} Z^0_\mu \phi^0 (W^+_\mu \phi^- + W^-_\mu \phi^+) - \\
\tfrac{1}{2} ig^2 \tfrac{s^2_w}{c_w} Z^0_\mu H (W^+_\mu \phi^- - W^-_\mu \phi^+) + \tfrac{1}{2} g^2 s_w A_\mu \phi^0 (W^+_\mu \phi^- + W^-_\mu \phi^+) + \tfrac{1}{2} ig^2 s_w A_\mu H (W^+_\mu \phi^- - \\
W^-_\mu \phi^+) - g^2 \tfrac{s_w}{c_w} (2c^2_w - 1) Z^0_\mu A_\mu \phi^+\phi^- - g^2 s^2_w A_\mu A_\mu \phi^+\phi^- + \tfrac{1}{2} ig_s \lambda^a_{ij} (\bar{q}^\sigma_i \gamma^\mu q^\sigma_j) g^a_\mu - \bar{e}^\lambda (\gamma\partial + \\
m^\lambda_e) e^\lambda - \bar{\nu}^\lambda \gamma\partial \nu^\lambda - \bar{u}^\lambda_j (\gamma\partial + m^\lambda_u) u^\lambda_j - \bar{d}^\lambda_j (\gamma\partial + m^\lambda_d) d^\lambda_j + \\
igs_w A_\mu \left(-(\bar{e}^\lambda \gamma^\mu e^\lambda) + \tfrac{2}{3} (\bar{u}^\lambda_j \gamma^\mu u^\lambda_j) - \tfrac{1}{3} (\bar{d}^\lambda_j \gamma^\mu d^\lambda_j) \right) + \tfrac{ig}{4c_w} Z^0_\mu \{ (\bar{\nu}^\lambda \gamma^\mu (1 + \gamma^5) \nu^\lambda) + \\
(\bar{e}^\lambda \gamma^\mu (4s^2_w - 1 - \gamma^5) e^\lambda) + (\bar{d}^\lambda_j \gamma^\mu (\tfrac{4}{3} s^2_w - 1 - \gamma^5) d^\lambda_j) + (\bar{u}^\lambda_j \gamma^\mu (1 - \tfrac{8}{3} s^2_w + \gamma^5) u^\lambda_j) \} + \\
\tfrac{ig}{2\sqrt{2}} W^+_\mu \left((\bar{\nu}^\lambda \gamma^\mu (1 + \gamma^5) e^\lambda) + (\bar{u}^\lambda_j \gamma^\mu (1 + \gamma^5) C_{\lambda\kappa} d^\kappa_j) \right) + \\
\tfrac{ig}{2\sqrt{2}} W^-_\mu \Big((\bar{e}^\lambda \gamma^\mu (1 + \gamma^5) \nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\kappa\lambda} \gamma^\mu (1 + \gamma^5) u^\lambda_j) \Big) + \\
\tfrac{ig}{2\sqrt{2}} \tfrac{m^\lambda_e}{M} \left(-\phi^+ (\bar{\nu}^\lambda (1 - \gamma^5) e^\lambda) + \phi^- (\bar{e}^\lambda (1 + \gamma^5) \nu^\lambda) \right) - \tfrac{g}{2} \tfrac{m^\lambda_e}{M} \left(H (\bar{e}^\lambda e^\lambda) + i\phi^0 (\bar{e}^\lambda \gamma^5 e^\lambda) \right) + \\
\tfrac{ig}{2M\sqrt{2}} \phi^+ \Big(-m^\kappa_d (\bar{u}^\lambda_j C_{\lambda\kappa} (1 - \gamma^5) d^\kappa_j) + m^\lambda_u (\bar{u}^\lambda_j C_{\lambda\kappa} (1 + \gamma^5) d^\kappa_j) \Big) +
\end{gathered}$$

$$\frac{ig}{2M\sqrt{2}}\phi^-\left(m_d^\lambda(\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger(1+\gamma^5)u_j^\kappa) - m_u^\kappa(\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger(1-\gamma^5)u_j^\kappa)\right) - \frac{g}{2}\frac{m_u^\lambda}{M}H(\bar{u}_j^\lambda u_j^\lambda) - \frac{g}{2}\frac{m_d^\lambda}{M}H(\bar{d}_j^\lambda d_j^\lambda) + \frac{ig}{2}\frac{m_u^\lambda}{M}\phi^0(\bar{u}_j^\lambda\gamma^5 u_j^\lambda) - \frac{ig}{2}\frac{m_d^\lambda}{M}\phi^0(\bar{d}_j^\lambda\gamma^5 d_j^\lambda) + \bar{G}^a\partial^2 G^a + g_s f^{abc}\partial_\mu\bar{G}^a G^b g_\mu^c + \bar{X}^+(\partial^2 - M^2)X^+ + \bar{X}^-(\partial^2 - M^2)X^- + \bar{X}^0(\partial^2 - \frac{M^2}{c_w^2})X^0 + \bar{Y}\partial^2 Y + igc_w W_\mu^+(\partial_\mu\bar{X}^0 X^- - \partial_\mu\bar{X}^+ X^0) + igs_w W_\mu^+(\partial_\mu\bar{Y}X^- - \partial_\mu\bar{X}^+ Y) + igc_w W_\mu^-(\partial_\mu\bar{X}^- X^0 - \partial_\mu\bar{X}^0 X^+) + igs_w W_\mu^-(\partial_\mu\bar{X}^- Y - \partial_\mu\bar{Y}X^+) + igc_w Z_\mu^0(\partial_\mu\bar{X}^+ X^+ - \partial_\mu\bar{X}^- X^-) + igs_w A_\mu(\partial_\mu\bar{X}^+ X^+ - \partial_\mu\bar{X}^- X^-) - \frac{1}{2}gM\left(\bar{X}^+ X^+ H + \bar{X}^- X^- H + \frac{1}{c_w^2}\bar{X}^0 X^0 H\right) + \frac{1-2c_w^2}{2c_w}igM\left(\bar{X}^+ X^0\phi^+ - \bar{X}^- X^0\phi^-\right) + \frac{1}{2c_w}igM\left(\bar{X}^0 X^-\phi^+ - \bar{X}^0 X^+\phi^-\right) + igMs_w\left(\bar{X}^0 X^-\phi^+ - \bar{X}^0 X^+\phi^-\right) + \frac{1}{2}igM\left(\bar{X}^+ X^+\phi^0 - \bar{X}^- X^-\phi^0\right).$$

This action functional was expressed in flat space-time and needs of course to be minimally coupled with gravity. One also needs to take into account the experimental discovery of neutrino oscillations and add the corresponding new terms.

2. Why non-commutative spaces

The natural group of symmetries of the total action (4) is the semi-direct product

$$\mathcal{G} = \mathrm{Map}(M, G) \rtimes \mathrm{Diff}(M) \tag{6}$$

of the group $\mathrm{Map}(M, G)$ of gauge transformations of second kind by the group $\mathrm{Diff}(M)$ of diffeomorphisms. Here G is the gauge group, inferred from experiment

$$G = U(1) \times SU(2) \times SU(3)\,. \tag{7}$$

Since the symmetry group of the Einstein-Hilbert action of pure gravity is simply $\mathrm{Diff}(M)$ it is natural to ask wether there is a space X whose group of diffeomorphisms is directly of the form (6). The answer is:

No: for ordinary spaces.

Yes: for non-commutative spaces.

A "non-commutative space" is one in which the usual coordinates x^μ no longer satisfy the simplifying commutative rule saying that the order of the terms is irrelevant in a product. They are familiar to physicists since Heisenberg's discovery of the nontrivial commutation rules for the natural coordinates in the phase space of a microscopic mechanical system. In first approximation the group of diffeomorphisms of such a space is the group of automorphisms $\mathrm{Aut}(\mathcal{A})$ of the algebra $\mathcal{A}$ of coordinates. The new feature that arises in the non-commutative case is that there are "easy" automorphisms, namely those of the form

$$f \in \mathcal{A} \mapsto u\,f\,u^{-1}$$

where $u \in \mathcal{A}$ is an invertible element. Such automorphisms are called "inner" or "internal" and form a normal subgroup $\mathrm{Inn}(\mathcal{A})$ of the group $\mathrm{Aut}(\mathcal{A})$ so that one has the general exact sequence

$$1 \to \mathrm{Inn}(\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}) \to \mathrm{Out}(\mathcal{A}) \to 1\,. \tag{8}$$

This exact sequence remains valid when taking into account the compatibility with the adjoint $f \mapsto f^*$ (one restricts to $\star$-automorphisms while $u \in \mathcal{A}$ is now a unitary element $uu^* = u^*u = 1$).

For an ordinary manifold X results from topology (*cf.* [22]) preclude the existence of a space whose group of diffeomorphisms is the group $\mathcal{G}$ of (6). To understand how passing to non-commutative spaces adds the missing part $\mathrm{Map}(M, G)$, let us consider the simplest example where the algebra

$$\mathcal{A} = C^\infty(M, M_n(\mathbb{C})) = C^\infty(M) \otimes M_n(\mathbb{C})$$

consists of smooth maps from a manifold M to the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices. One then shows that the group $\mathrm{Inn}(\mathcal{A})$ in that case is locally isomorphic to the group $\mathrm{Map}(M, G)$ of smooth maps from M to the small gauge group $G = PSU(n)$ (quotient of $SU(n)$ by its center) and that the general exact sequence (8) becomes identical to the exact sequence governing the structure of the group $\mathcal{G}$, namely

$$1 \to \mathrm{Map}(M, G) \to \mathcal{G} \to \mathrm{Diff}(M) \to 1. \tag{9}$$

Moreover the physics terminology of “internal symmetries” matches the mathematical one perfectly. We refer to Proposition 3.4 of [7] for the more involved case of the group (6).

3. What is a non-commutative geometry?

A refined notion of geometry (suitable in particular to deal with spaces whose coordinates do not commute) is obtained by focussing not on the traditional $g_{\mu\nu}$ but on the Dirac operator D. In extracting the square root of the Laplacian using a spin structure the Dirac operator enables us to talk about the line element $ds = D^{-1}$ instead of its square (2). The new paradigm for a geometric space is of *spectral* nature. A spectral geometry $(\mathcal{A}, \mathcal{H}, D)$ is given by an involutive unital algebra $\mathcal{A}$ represented as operators in a Hilbert space $\mathcal{H}$ and a self-adjoint operator D with compact resolvent such that all commutators $[D, a]$ are bounded for $a \in \mathcal{A}$. A spectral geometry is *even* if the Hilbert space $\mathcal{H}$ is endowed with a $\mathbb{Z}/2$- grading γ which commutes with any $a \in \mathcal{A}$ and anticommutes with D.

This notion extends the Riemannian paradigm as follows. A spin Riemannian manifold M gives rise in a canonical manner to a spectral geometry. The Hilbert space $\mathcal{H}$ is the Hilbert space $L^2(M, S)$ of square integrable spinors on M and the algebra $\mathcal{A} = C^\infty(M)$ of smooth functions on M acts in $\mathcal{H}$ by multiplication operators:

$$(f\,\xi)(x) = f(x)\,\xi(x)\,, \quad \forall\, x \in M\,. \tag{10}$$

The operator D is the Dirac operator,

$$\partial\!\!\!/_M = \sqrt{-1}\,\gamma^\mu\,\nabla_\mu. \tag{11}$$

The grading γ is given by the chirality operator which we denote by γ_5 in the four-dimensional case.

As it turns out this way of defining a geometry by specifying the Dirac operator is meaningful both in mathematical terms (where the Dirac operator specifies the fundamental class in KO-homology) and in physics terms (where, modulo a chiral gauge transformation, the Dirac operator is the inverse of the Euclidean propagator of fermions). From both sides (KO-homology and physics) a further "decoration" is needed in the form of a real structure. A real structure of KO-dimension $n \in \mathbb{Z}/8$ on a spectral geometry $(\mathcal{A}, \mathcal{H}, D)$ is an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$, with the property that

$$J^2 = \varepsilon, \qquad JD = \varepsilon' DJ, \quad \text{and} \quad J\gamma = \varepsilon'' \gamma J\,. \tag{12}$$

The numbers $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ are a function of $n \mod 8$ given by

n	0	1	2	3	4	5	6	7
ε	1	1	−1	−1	−1	−1	1	1
ε'	1	−1	1	1	1	−1	1	1
ε''	1		−1		1		−1	

From the mathematical side the role of J is twofold, it embodies the crucial nuance between complex K-homology and "real" KO-homology which plays a key role in the conceptual understanding of homotopy types of manifolds. It also embodies the discovery by Tomita of the general structure of representations of non-commutative algebras. This corresponds to the commutation relation

$$[a, b^0] = 0 \quad \forall\, a, b \in \mathcal{A}, \tag{13}$$

where

$$b^0 = Jb^* J^{-1} \qquad \forall b \in \mathcal{A}. \tag{14}$$

From the physics side the operator J corresponds to the charge conjugation operator. The change from the Riemannian paradigm to the spectral one already occurred in geodesy. The notion of geometry is intimately tied up with the measurement of length and it was never completely obvious how to reach some agreement on a physical unit of length which would unify the numerous existing choices. Since the French revolution the concrete "mètre-étalon" (realized in the form of a platinum bar which is approximately 10^{-7} times the quarter of the meridian of the earth) was taken as unit of length in the metric system. Already in 1927, at the seventh conference on the metric system, in order to take into account the inevitable natural variations of the concrete "mètre-étalon", the idea emerged to compare it with a reference wave length (the red line of Cadmium). Around 1960 the reference to the "mètre-étalon" was finally abandoned and a new definition of the "mètre" was adopted as $1650763, 73$ times the wave length of the radiation corresponding to the transition between the levels 2p10 and 5d5 of the Krypton 86Kr. In 1967 the second was defined as the duration of 9192631770 periods of the radiation corresponding to the transition between the two hyperfine levels of Caesium-133. Finally in 1983 the "mètre" was defined as the distance travelled by light in 1/299792458 second. In fact the speed of light is just a conversion factor and to define the "mètre" one gives it the specific value of

$$c = 299792458\, m/s.$$

In other words the "mètre" is defined as a certain fraction $\frac{9192631770}{299792458} \sim 30.6633\ldots$ of the wave length of the radiation coming from the transition between the above hyperfine levels of the Caesium atom. The advantages of the new standard of length are many. By not being tied up with any specific location it is in fact available anywhere where Caesium is (the choice of Caesium as opposed to Helium or Hydrogen which are much more common in the universe is of course still debatable [2]).

In non-commutative geometry the Riemannian formula for the geodesic distance

$$d(x,y) = \inf \int_\gamma \sqrt{g_{\mu\nu} dx^\mu \, dx^\nu} \tag{15}$$

where the infimum is taken over all paths from x to y, is replaced by

$$d(x,y) = \sup\{|f(x) - f(y)| \,:\, f \in \mathcal{A}\,,\ \|[D,f]\| \le 1\}\,, \tag{16}$$

which gives the same answer in the Riemannian case but continues to make sense for spectral geometries where the algebra $\mathcal{A}$ is no longer commutative (x and y are then states on $\mathcal{A}$).

The traditional notions of geometry all have natural analogues in the spectral framework. We refer to [9] for more details. The dimension of a non-commutative geometry is not a number but a spectrum, the *dimension spectrum* (*cf.* [14]) which is the subset Π of the complex plane $\mathbb{C}$ at which the spectral functions have singularities. Under the hypothesis that the dimension spectrum is *simple*, *i.e.*, that the spectral functions have at most simple poles, the residue at the pole defines a far reaching extension (*cf.* [14]) of the fundamental integral in non-commutative geometry given by the Dixmier trace (*cf.* [9]). This extends the Wodzicki residue from pseudodifferential operators on a manifold to the general framework of spectral triples, and gives meaning to $⨍ T$ in that context. It is simply given by

$$⨍ T = \mathrm{Res}_{s=0}\, \mathrm{Tr}\,(T\,|D|^{-s})\,. \tag{17}$$

4. Inner fluctuations of a spectral geometry

The non-commutative world is rich in phenomena which have no commutative counterpart. We already saw above the role of inner automorphisms (as internal symmetries) which decompose the full automorphism group into equivalence classes modulo inner. In a similar manner the non-commutative metrics admit a natural foliation, the metrics on the same leaf are obtained as inner fluctuations. The corresponding transformation on the operator D is simply the addition $D \mapsto D_A = D + A + \varepsilon'\, J\, A\, J^{-1}$ where $A = A^*$ is an arbitrary selfadjoint element of Ω^1_D with

$$\Omega^1_D = \{\sum_j a_j\,[D, b_j] \mid a_j,\, b_j \in \mathcal{A}\}\,, \tag{18}$$

which is by construction a bimodule over $\mathcal{A}$.

The inner fluctuations in non-commutative geometry are generated by the existence of Morita equivalences (*cf.* [24]). Given an algebra $\mathcal{A}$, a Morita equivalent algebra $\mathcal{B}$ is the algebra of endomorphisms of a finite projective (right) module $\mathcal{E}$ over $\mathcal{A}$,

$$\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E}). \tag{19}$$

Transferring the metric from $\mathcal{A}$ to $\mathcal{B}$ requires the choice of a *hermitian connection* ∇ on $\mathcal{E}$. A connection is a linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$ satisfying the Leibniz rule

$$\nabla(\xi a) = (\nabla \xi)a + \xi \otimes da\,, \quad \forall\, \xi \in \mathcal{E}\ ,\ a \in \mathcal{A},$$

with $da = [D, a]$. Taking the obvious Morita equivalence between $\mathcal{A}$ and itself generates the inner fluctuations $D \mapsto D + A + \varepsilon'\, J\, A\, J^{-1}$.

By (14) one gets a right $\mathcal{A}$-module structure on $\mathcal{H}$,

$$\xi\, b = b^0\, \xi\,, \quad \forall\, \xi \in \mathcal{H}\,, \quad b \in \mathcal{A}\,. \tag{20}$$

The unitary group of the algebra $\mathcal{A}$ then acts by the “adjoint representation” in $\mathcal{H}$ in the form

$$\xi \in \mathcal{H} \to \mathrm{Ad}(u)\,\xi = u\,\xi\,u^*\,, \quad \forall\, \xi \in \mathcal{H}\,, \quad u \in \mathcal{A}\,, \quad u\,u^* = u^*\,u = 1\,. \tag{21}$$

The order one condition

$$[[D, a], b^0] = 0 \qquad \forall\, a, b \in \mathcal{A} \tag{22}$$

ensures that for any $A \in \Omega^1_D$ with $A = A^*$ and any unitary $u \in \mathcal{A}$, one has

$$\mathrm{Ad}(u)(D + A + \varepsilon'\, J\, A\, J^{-1})\mathrm{Ad}(u^*) = D + \gamma_u(A) + \varepsilon'\, J\, \gamma_u(A)\, J^{-1},$$

where $\gamma_u(A) = u\,[D, u^*] + u\,A\,u^*$.

The above parallel between inner automorphisms and internal symmetries extends to a parallel between the inner fluctuations and the gauge potentials.

5. The spectral action principle

We shall recall in this section our joint work with Ali Chamseddine on the spectral action principle [3–6]. The starting point is the discussion of observables in gravity. By the principle of gauge invariance the only quantities which have a chance to be observable in gravity are those which are invariant under the gauge group, *i.e.*, the group of diffeomorphisms of the space-time M. Assuming first that we deal with a classical manifold (and Wick rotate to Euclidean signature for simplicity), one can form a number of such invariants (under suitable convergence conditions) as the integrals of the form

$$\int_M F(K)\,\sqrt{g}\,d^4x \tag{23}$$

where $F(K)$ is a scalar invariant function (the scalar curvature is one example of such a function but there are many others) of the Riemann curvature K. We refer to [16] for other more complicated examples of such invariants, where those of the form (23) appear as the *single integral* observables, *i.e.*, those which add up when evaluated on the direct sum of geometric spaces. Now while in theory a quantity like (23) is observable it is almost impossible to evaluate since it involves the knowledge of the entire space-time and is in that way highly non localized. On the other hand, spectral data are available in localized form anywhere, and are (asymptotically) of the form (23) when they are of the additive form

$$\mathrm{Trace}\,(f(D/\Lambda)), \tag{24}$$

where D is the Dirac operator and f is a positive even function of the real variable while the parameter Λ fixes the mass scale. The spectral action principle asserts that the fundamental action functional S that allows to compare different geometric spaces at the classical level and is used in the functional integration to go to the quantum level, is itself of the form (24). The detailed form of the function f is largely irrelevant since the spectral action (24) can be expanded in decreasing powers of the scale Λ in the form

$$\mathrm{Trace}\,(f(D/\Lambda)) \sim \sum_{k\in\,\Pi^+} f_k\,\Lambda^k \,⨍\, |D|^{-k} \,+\, f(0)\,\zeta_D(0) + o(1), \tag{25}$$

where Π^+ is the positive part of the dimension spectrum, the integral $⨍$ is defined in (17), and the function f only appears through the scalars

$$f_k = \int_0^\infty f(v)\,v^{k-1}\,dv. \tag{26}$$

The term independent of the parameter Λ is the value at $s=0$ (regularity at $s=0$ is assumed) of the zeta function,

$$\zeta_D(s) = \mathrm{Tr}\,(|D|^{-s})\,. \tag{27}$$

The main result of our joint work with A. Chamseddine [3], [4] is that, when applied to the inner fluctuations of the product geometry $M\times F$ the spectral action gives the standard model coupled with gravity. Here M is a Riemannian compact spin 4-manifold, the standard model coupled with gravity is in the Euclidean form, and the geometry of the finite space F is encoded (as in the general framework of NCG) by a spectral geometry $(\mathcal{A}_F, \mathcal{H}_F, D_F)$.

For M the spectral geometry is given by (10), (11). For the non-commutative geometry F used in [4] to obtain the standard model coupled to gravity, all the ingredients are *finite-dimensional*. The algebra $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ (*i.e.*, the direct sum of the algebras $\mathbb{C}$ of complex numbers, $\mathbb{H}$ of quaternions, and $M_3(\mathbb{C})$ of 3×3 matrices) encodes the gauge group. The Hilbert space $\mathcal{H}_F$ encodes the elementary quarks and leptons. The operator D_F encodes those free parameters of the standard model related to the Yukawa couplings.

The above work [4] had several shortcomings:

1. The finite geometry F is put in "by hand" with no conceptual understanding of the representation of $\mathcal{A}_F$ in $\mathcal{H}_F$.
2. There is a fermion doubling problem (*cf.* [21]) in the Fermionic part of the action.
3. It does not incorporate the neutrino mixing and see-saw mechanism for neutrino masses.

We showed in [12] and [7] how to solve these three problems (the first only partly since the number of generations is put by hand) simply by keeping the distinction between the following two notions of dimension of a non-commutative space,

- the metric dimension,
- the KO-dimension.

The metric dimension manifests itself by the growth of the spectrum of the Dirac operator and gives an upper bound to the dimension spectrum. In a (compact) space of dimension k the line element $ds = D^{-1}$ is an infinitesimal of order $1/k$ which means that the nth characteristic value of ds is of the order of $n^{-1/k}$ (in the non-compact case one replaces ds by $a\,ds$ for $a \in \mathcal{A}$). As far as space-time goes it appears that the situation of interest will be the four-dimensional one. In particular the metric dimension of the finite geometry F will be zero.

The KO-dimension is only well defined modulo 8 and it takes into account both the $\mathbb{Z}/2$-grading γ of $\mathcal{H}$ as well as the real structure J according to (12). The real surprise is that in order for things to work the only needed change (besides the easy addition of a right-handed neutrino) is to change the $\mathbb{Z}/2$ grading of the finite geometry F to its opposite in the "antiparticle" sector. It is only thanks to this that the Fermion doubling problem pointed out in [21] can be successfully handled. Moreover it will automatically generate the *full standard model*, *i.e.*, the model with neutrino mixing and the see-saw mechanism as follows from the full classification of Dirac operators: Theorem 6.7.

When one looks at the above table giving the KO-dimension of the finite space F one then finds that its KO-dimension is now equal to 6 modulo 8 (!). As a result we see that the KO-dimension of the product space $M \times F$ is in fact equal to $10 \sim 2$ modulo 8. Of course the above 10 is very reminiscent of string theory, in which the finite space F might be a good candidate for an "effective" compactification at least for low energies[1]. But 10 is also 2 modulo 8 which might be related to the observations of [20] about gravity.

It is also remarkable that the non-commutative spheres arising from quantum groups, such as the Podleś spheres already exhibit the situation where the metric dimension (0 in that case) is distinct from the KO-dimension (2 in that case) as pointed out in the work of L. Dąbrowski and A. Sitarz on Podleś quantum spheres [15].

[1]Note however that we are dealing with the standard model, not its supersymmetrized version.

6. The finite non-commutative geometry F

In this section we shall first describe in a conceptual manner the representation of $\mathcal{A}_F$ in $\mathcal{H}_F$ and classify the Dirac operators D_F. The only small nuance with [11] is that we incorporate a right-handed neutrino ν_R and change the $\mathbb{Z}/2$ grading in the antiparticle sector to its opposite. This, innocent as it looks, allows for a better conceptual understanding of the representation of $\mathcal{A}_F$ in $\mathcal{H}_F$ and also will completely alter the classification of Dirac operators (Theorem 6.7).

6.1. The representation of $\mathcal{A}_F$ in $\mathcal{H}_F$

We start from the involutive algebra (with $\mathbb{H}$ the quaternions with involution $q \to \bar{q}$)

$$\mathcal{A}_{LR} = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C}). \tag{28}$$

We construct a natural representation $(\mathcal{A}_{LR}, \mathcal{H}_F, J_F, \gamma_F)$ fulfilling (12) and (13) in dimension 6 modulo 8. The commutation relation (13) shows that there is an underlying structure of $\mathcal{A}_{LR}$-bimodule on $\mathcal{H}_F$ and we shall use that structure as a guide. One uses the bimodule structure to define $\mathrm{Ad}(u)$ by (21).

Definition 6.1. *Let $\mathcal{M}$ be an $\mathcal{A}_{LR}$-bimodule. Then $\mathcal{M}$ is* odd *iff the adjoint action* (21) *of $s = (1, -1, -1, 1)$ fulfills* $\mathrm{Ad}(s) = -1$.

Such a bimodule is a representation of the reduction of $\mathcal{A}_{LR} \otimes_{\mathbb{R}} \mathcal{A}_{LR}^0$ by the projection $\frac{1}{2}(1 - s \otimes s^0)$. This subalgebra is an algebra over $\mathbb{C}$ and we restrict to complex representations. One defines the contragredient bimodule of a bimodule $\mathcal{M}$ as the complex conjugate space

$$\mathcal{M}^0 = \{\bar{\xi}\,;\ \xi \in \mathcal{M}\}\,, \quad a\,\bar{\xi}\,b = \overline{b^*\xi\,a^*}\,, \quad \forall\, a, b \in \mathcal{A}_{LR}. \tag{29}$$

We can now give the following characterization of the $\mathcal{A}_{LR}$-bimodule $\mathcal{M}_F$ and the real structure J_F for one generation.

Proposition 6.2.

- *The $\mathcal{A}_{LR}$-bimodule $\mathcal{M}_F$ is the direct sum of all inequivalent irreducible odd $\mathcal{A}_{LR}$-bimodules.*
- *The dimension of $\mathcal{M}_F$ is* 32.
- *The real structure J_F is given by the isomorphism with the contragredient bimodule.*

We define the $\mathbb{Z}/2$-grading γ_F by

$$\gamma_F = c - J_F\,c\,J_F\,, \quad c = (0, -1, 1, 0) \in \mathcal{A}_{LR}. \tag{30}$$

One then checks that the following holds:

$$J_F^2 = 1\,, \quad J_F\,\gamma_F = -\,\gamma_F\,J_F, \tag{31}$$

which together with the commutation of J_F with the Dirac operators, is characteristic of KO-dimension equal to 6 modulo 8.

The equality $\iota(\lambda, q, m) = (\lambda, q, \lambda, m)$ defines a homomorphism ι of involutive algebras from $\mathcal{A}_F$ to $\mathcal{A}_{LR}$ so that we view $\mathcal{A}_F$ as a subalgebra of $\mathcal{A}_{LR}$.

Definition 6.3. *The real representation* $(\mathcal{A}_F, \mathcal{H}_F, J_F, \gamma_F)$ *is the restriction to* $\mathcal{A}_F \subset \mathcal{A}_{LR}$ *of the direct sum* $\mathcal{M}_F \otimes \mathbb{C}^3$ *of three copies of* $\mathcal{M}_F$.

It has dimension $32 \times 3 = 96$, needless to say this 3 is the number of generations and it is put in by hand here. A conceptual explanation for the restriction to $\mathcal{A}_F \subset \mathcal{A}_{LR}$ is given in [7].

6.2. The unimodular unitary group $\mathrm{SU}(\mathcal{A}_F)$

Using the action of $\mathcal{A}_F$ in $\mathcal{H}_F$ one defines the unimodular subgroup $\mathrm{SU}(\mathcal{A}_F)$ of the unitary group $\mathrm{U}(\mathcal{A}_F) = \{u \in \mathcal{A}_F\,,\ uu^* = u^*u = 1\}$ as follows.

Definition 6.4. *We let* $\mathrm{SU}(\mathcal{A}_F)$ *be the subgroup of* $\mathrm{U}(\mathcal{A}_F)$ *defined by*

$$\mathrm{SU}(\mathcal{A}_F) = \{u \in \mathrm{U}(\mathcal{A}_F)\ :\ \mathrm{Det}(u) = 1\}$$

where $\mathrm{Det}(u)$ *is the determinant of the action of* u *in* $\mathcal{H}_F$.

One obtains both the standard model gauge group and its action on fermions from the adjoint action of $\mathrm{SU}(\mathcal{A}_F)$ in the following way:

Proposition 6.5.

1. *The group* $\mathrm{SU}(\mathcal{A}_F)$ *is, up to an Abelian finite group,*

$$\mathrm{SU}(\mathcal{A}_F) \sim \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3).$$

2. *The adjoint action* $u \to \mathrm{Ad}(u)$ (*cf.* (21)) *of* $\mathrm{SU}(\mathcal{A}_F)$ *in* $\mathcal{H}_F$ *coincides with the standard model action on elementary quarks and leptons.*

One shows [7] that the sum of the irreducible odd bimodules is of the form

$$\mathcal{M}_F = (\pi_L \oplus \pi_R \oplus \pi_R^3 \oplus \pi_L^3) \oplus (\pi_L \oplus \pi_R \oplus \pi_R^3 \oplus \pi_L^3)^0. \tag{32}$$

This $\mathcal{A}_{LR}$-bimodule $\mathcal{M}_F$ is of dimension $2 \cdot (2 + 2 + 2 \times 3 + 2 \times 3) = 32$ and the adjoint action gives the gauge action of the standard model for one generation, with the following labels for the basis elements of $\mathcal{M}_F$,

$$\begin{pmatrix} \nu_L & \nu_R \\ e_L & e_R \end{pmatrix}$$

for the term $\pi_L \oplus \pi_R$,

$$\begin{pmatrix} u_L^j & u_R^j \\ d_L^j & d_R^j \end{pmatrix}$$

for the term $\pi_R^3 \oplus \pi_L^3$ (with color indices j) and the transformation $q \to \bar{q}$ to pass to the contragredient bimodules. With these labels one checks that the adjoint action of the U(1) factor is given by multiplication of the basis vectors f by the following powers of $\lambda \in \mathrm{U}(1)$:

	e	ν	u	d
f_L	-1	-1	$\frac{1}{3}$	$\frac{1}{3}$
f_R	-2	0	$\frac{4}{3}$	$-\frac{2}{3}$

6.3. The classification of Dirac operators

To be precise we adopt the following.

Definition 6.6. *A Dirac operator is a self-adjoint operator* D *in* $\mathcal{H}_F$ *commuting with* J_F*,* $\mathbb{C}_F = \{(\lambda, \lambda, 0)\} \in \mathcal{A}_F$*, anticommuting with* γ_F *and fulfilling the order one condition* $[[D, a], b^0] = 0$ *for any* $a, b \in \mathcal{A}_F$*.*

The physics meaning of the condition of commutation with $\mathbb{C}_F$ is to ensure that one gauge vector boson (the photon) remains massless.

In order to state the classification of Dirac operators we introduce the following notation, let M_e, M_ν, M_d, M_u and M_R be three by three matrices, we then let $D(M)$ be the operator in $\mathcal{H}_F$ given by

$$D(M) = \begin{bmatrix} S & T^* \\ T & \bar{S} \end{bmatrix} \tag{33}$$

where

$$S = S_\ell \oplus (S_q \otimes 1_3) \tag{34}$$

and in the basis (ν_R, e_R, ν_L, e_L) and (u_R, d_R, u_L, d_L),

$$S_\ell = \begin{bmatrix} 0 & 0 & M_\nu^* & 0 \\ 0 & 0 & 0 & M_e^* \\ M_\nu & 0 & 0 & 0 \\ 0 & M_e & 0 & 0 \end{bmatrix} \qquad S_q = \begin{bmatrix} 0 & 0 & M_u^* & 0 \\ 0 & 0 & 0 & M_d^* \\ M_u & 0 & 0 & 0 \\ 0 & M_d & 0 & 0 \end{bmatrix} \tag{35}$$

while the operator T is 0 except on the subspace $\mathcal{H}_{\nu_R} \subset \mathcal{H}_F$ with basis the ν_R which it maps, using the matrix M_R, to the subspace $\mathcal{H}_{\bar{\nu}_R} \subset \mathcal{H}_F$ with basis the $\bar{\nu}_R$.

Theorem 6.7.

1. *Let* D *be a Dirac operator. There exist* 3×3 *matrices* M_e*,* M_ν*,* M_d*,* M_u *and* M_R*, with* M_R *symmetric, such that* $D = D(M)$*.*
2. *All operators* $D(M)$ *(with* M_R *symmetric) are Dirac operators.*
3. *The operators* $D(M)$ *and* $D(M')$ *are conjugate by a unitary operator commuting with* $\mathcal{A}_F$*,* γ_F *and* J_F *iff there exists unitary matrices* V_j *and* W_j *such that*

$$M'_e = V_1\, M_e\, V_3^*\,, \qquad M'_\nu = V_2\, M_\nu\, V_3^*\,, \qquad M'_d = W_1\, M_d\, W_3^*\,,$$
$$M'_u = W_2\, M_u\, W_3^*\,, \qquad M'_R = V_2\, M_R\, \bar{V}_2^*.$$

In particular Theorem 6.7 shows that the Dirac operators give all the required features, such as

- mixing matrices for quarks and leptons,
- unbroken color,
- see-saw mechanism for right handed neutrinos.

7. The spectral action for $M \times F$ and the standard model

We now consider a four-dimensional smooth compact Riemannian manifold M with a fixed spin structure and recall that it is fully encoded by its Dirac spectral geometry $(\mathcal{A}_1, \mathcal{H}_1, D_1) = (C^\infty(M), L^2(M, S), \partial\!\!\!/_M)$. We then consider its product with the above finite geometry $(\mathcal{A}_2, \mathcal{H}_2, D_2) = (\mathcal{A}_F, \mathcal{H}_F, D_F)$. With $(\mathcal{A}_j, \mathcal{H}_j, D_j)$ of KO-dimensions 4 for $j = 1$ and 6 for $j = 2$, the product geometry is given by the rules

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2\,,\ \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2\,,\ D = D_1 \otimes 1 + \gamma_1 \otimes D_2\,,\ \gamma = \gamma_1 \otimes \gamma_2\,,\ J = J_1 \otimes J_2\,.$$

Note that it matters that J_1 commutes with γ_1 to check that J commutes with D. The KO-dimension of the finite space F is $6 \in \mathbb{Z}/8$ and thus the KO-dimension of the product geometry $M \times F$ is now $2 \in \mathbb{Z}/8$. In other words according to (12) the commutation rules are

$$J^2 = -1, \quad JD = DJ, \text{ and } J\gamma = -\gamma J\,. \tag{36}$$

Let us now explain how these rules allow to define a natural antisymmetric bilinear form on the even part $\mathcal{H}^+ = \{\xi \in \mathcal{H}\,,\ \gamma\,\xi = \xi\}$ of $\mathcal{H}$.

Proposition 7.1. *On a real spectral geometry of KO-dimension $2 \in \mathbb{Z}/8$, the following equality defines an antisymmetric bilinear form on $\mathcal{H}^+ = \{\xi \in \mathcal{H}\,,\ \gamma\,\xi = \xi\}$,*

$$A_D(\xi', \xi) = \langle\, J\,\xi', D\,\xi\rangle\,, \quad \forall\,\xi, \xi' \in \mathcal{H}^+. \tag{37}$$

The above trilinear pairing between D, ξ and ξ' is gauge invariant under the adjoint action (cf. (21)) of the unitary group of $\mathcal{A}$,

$$A_D(\xi', \xi) = A_{D_u}(\mathrm{Ad}(u)\xi', \mathrm{Ad}(u)\xi)\,, \quad D_u = \mathrm{Ad}(u)\, D\, \mathrm{Ad}(u^*). \tag{38}$$

Now the Pfaffian of an antisymmetric bilinear form is best expressed in terms of the functional integral involving anticommuting "classical fermions" which at the formal level means that

$$\mathrm{Pf}(A) = \int e^{-\frac{1}{2}\,A(\xi)}\, D[\xi].$$

It is the use of the Pfaffian as a square root of the determinant that allows to solve the Fermion doubling puzzle which was pointed out in [21]. The solution obtained by a better choice of the KO-dimension of the space F and hence of $M \times F$ is not unrelated to the point made in [17].

Theorem 7.2. *Let M be a Riemannian spin 4-manifold and F the finite non-commutative geometry of KO-dimension 6 described above. Let $M \times F$ be endowed with the product metric.*

1. *The unimodular subgroup of the unitary group acting by the adjoint representation $\mathrm{Ad}(u)$ in $\mathcal{H}$ is the group of gauge transformations of SM.*
2. *The unimodular inner fluctuations A of the metric[2] are parameterized exactly by the gauge bosons of SM (including the Higgs doublet).*

[2]The *unimodular* inner fluctuations are obtained by restricting to those A which are traceless, *i.e.*, fulfill the condition $\mathrm{Tr}(A) = 0$.

3. *The full standard model (see the explicit formula in §9) minimally coupled with Einstein gravity is given in Euclidean form by the action functional*

$$S = \operatorname{Tr}(f(D_A/\Lambda)) + \frac{1}{2}\langle J\xi, D_A\xi\rangle\,, \quad \xi \in \mathcal{H}^+$$

applied to unimodular inner fluctuations $D_A = D + A + JAJ^{-1}$ *of the metric.*

We take f even and positive with $f^{(n)}(0) = 0$ for $n \geq 1$ for definiteness. Note also that the components of ξ anticommute so the antisymmetric form does not vanish. The proof is given in [7] which is a variant of [4] (*cf.* [18] for a detailed version). After turning off gravity to simplify and working in flat space (after Wick rotation back to Lorentzian signature) one gets the Lagrangian of §9 whose agreement with that of §1 can hardly be fortuitous. It is obtained in Euclidean form and all the signs are the physical ones, provided the test function f is positive which is the natural condition to get a sensible exponent in the functional integral. The positivity of the test function f ensures the positivity of the action functional before taking the asymptotic expansion. In general, this does not suffice to control the sign of the terms in the asymptotic expansion. In our case, however, this determines the positivity of the momenta f_0, f_2, and f_4. The explicit calculation then shows that this implies that the signs of all the terms are the expected physical ones.

We obtain the usual Einstein–Hilbert action with a cosmological term, and in addition the square of the Weyl curvature and a pairing of the scalar curvature with the square of the Higgs field. The Weyl curvature term does not affect gravity at low energies, as explained in §10.6 below.

The fermion doubling problem is resolved by the use of the Pfaffian, we checked that part for the Dirac mass terms, and trust that the same holds for the Majorana mass terms. There is one subtle point which is the use of the following chiral transformation:

$$U = e^{i\frac{\pi}{4}\gamma_5}$$

to transform the Fermionic part of the action to the traditional one, *i.e.*, the Euclidean action for Fermi fields (*cf.* [8]). While this transformation is innocent at the classical level, it is nontrivial at the quantum level and introduces some kind of Maslov index in the transition from our form of the Euclidean action to the more traditional one. We shall now give more details on the bosonic part of the action.

8. Detailed form of the bosonic action

We shall now give the precise form of the bosonic action, the calculation [7] is entirely similar to [4] with new terms appearing from the presence of M_R.

One lets $f_k = \int_0^\infty f(u)\, u^{k-1} du$ for $k > 0$ and $f_0 = f(0)$. Also

$$
\begin{aligned}
a =&\ \mathrm{Tr}(M_\nu^* M_\nu + M_e^* M_e + 3(M_u^* M_u + M_d^* M_d)) \\
b =&\ \mathrm{Tr}((M_\nu^* M_\nu)^2 + (M_e^* M_e)^2 + 3(M_u^* M_u)^2 + 3(M_d^* M_d)^2) \\
c =&\ \mathrm{Tr}(M_R^* M_R) \\
d =&\ \mathrm{Tr}((M_R^* M_R)^2) \\
e =&\ \mathrm{Tr}(M_R^* M_R M_\nu^* M_\nu).
\end{aligned} \tag{39}
$$

The spectral action is given by a computation entirely similar to [4] which yields:

$$
\begin{aligned}
S =&\ \frac{1}{\pi^2}(48\, f_4\, \Lambda^4 - f_2\, \Lambda^2\, c + \frac{f_0}{4}\, d) \int \sqrt{g}\, d^4x \\
&+ \frac{96\, f_2\, \Lambda^2 - f_0\, c}{24\pi^2} \int R\, \sqrt{g}\, d^4x \\
&+ \frac{f_0}{10\, \pi^2} \int (\frac{11}{6}\, R^* R^* - 3\, C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma})\, \sqrt{g}\, d^4x \\
&+ \frac{(-2\, a\, f_2\, \Lambda^2\, +\, e\, f_0)}{\pi^2} \int |\varphi|^2\, \sqrt{g}\, d^4x \\
&+ \frac{f_0}{2\, \pi^2} \int a\, |D_\mu \varphi|^2\, \sqrt{g}\, d^4x \\
&- \frac{f_0}{12\, \pi^2} \int a\, R\, |\varphi|^2\, \sqrt{g}\, d^4x \\
&+ \frac{f_0}{2\, \pi^2} \int (g_3^2\, G_{\mu\nu}^i\, G^{\mu\nu i} + g_2^2\, F_{\mu\nu}^\alpha\, F^{\mu\nu\alpha} + \frac{5}{3}\, g_1^2\, B_{\mu\nu}\, B^{\mu\nu})\, \sqrt{g}\, d^4x \\
&+ \frac{f_0}{2\, \pi^2} \int b\, |\varphi|^4\, \sqrt{g}\, d^4x
\end{aligned} \tag{40}
$$

where (a, b, c, d, e) are defined above and $D_\mu \varphi$ is the minimal coupling. A simple change of variables as in [4], namely

$$
\mathbf{H} = \frac{\sqrt{a\, f_0}}{\pi}\, \varphi\,, \tag{41}
$$

so that the kinetic term becomes[3]

$$
\int \frac{1}{2} |D_\mu \mathbf{H}|^2\, \sqrt{g}\, d^4x
$$

and

$$
\frac{g_3^2\, f_0}{2\pi^2} = \frac{1}{4}\,, \quad g_3^2 = \ g_2^2 = \frac{5}{3}\, g_1^2\,. \tag{42}
$$

[3]Here we differ slightly from [4] by a factor of $\sqrt{2}$ to match the conventions of Veltman [26].

transforms the bosonic action into the form:

$$S = \int d^4x\,\sqrt{g}\,\Big[\frac{1}{2\kappa_0^2}\,R + \alpha_0\,C_{\mu\nu\rho\sigma}\,C^{\mu\nu\rho\sigma} \tag{43}$$
$$+ \quad \gamma_0 + \tau_0\,{}^*R^*R + \delta_0\,R;_\mu{}^\mu$$
$$+ \frac{1}{4}\,G^i_{\mu\nu}\,G^{\mu\nu i} + \frac{1}{4}\,F^\alpha_{\mu\nu}\,F^{\mu\nu\alpha} + \frac{1}{4}\,B_{\mu\nu}\,B^{\mu\nu}$$
$$+ \frac{1}{2}|D_\mu\,\mathbf{H}|^2 - \mu_0^2|\mathbf{H}|^2 - \frac{1}{12}\,R\,|\mathbf{H}|^2 + \lambda_0|\mathbf{H}|^4\Big]$$

where

$$\frac{1}{\kappa_0^2} = \frac{96\,f_2\,\Lambda^2 - f_0\,c}{12\,\pi^2} \tag{44}$$

$$\mu_0^2 = 2\,\frac{f_2\,\Lambda^2}{f_0} - \frac{e}{a} \tag{45}$$

$$\alpha_0 = -\frac{3\,f_0}{10\,\pi^2} \tag{46}$$

$$\tau_0 = \frac{11\,f_0}{60\,\pi^2} \tag{47}$$

$$\delta_0 = -\frac{2}{3}\,\alpha_0 \tag{48}$$

$$\gamma_0 = \frac{1}{\pi^2}\left(48\,f_4\,\Lambda^4 - f_2\,\Lambda^2\,c + \frac{f_0}{4}\,d\right) \tag{49}$$

$$\lambda_0 = \frac{\pi^2}{2\,f_0}\,\frac{b}{a^2} = \frac{b\,g^2}{a^2} \tag{50}$$

9. Detailed form of the spectral action without gravity

To make the comparison easier we Wick rotate back to Minkowski space and after turning off gravity by working in flat space (and addition of gauge fixing terms[4]) the spectral action, after the change of variables summarized in table 1, is given by the following formula:

$$\mathcal{L}_{SM} = -\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4}g_s^2 f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \tfrac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - igc_w(\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + Z^0_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - igs_w(\partial_\nu A_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - \tfrac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \tfrac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2 c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w(A_\mu Z^0_\nu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \tfrac{1}{2}\partial_\mu H \partial_\mu H - 2M^2\alpha_h H^2 - \partial_\mu\phi^+\partial_\mu\phi^- - \tfrac{1}{2}\partial_\mu\phi^0\partial_\mu\phi^0 - \beta_h\left(\tfrac{2M^2}{g^2} + \tfrac{2M}{g}H + \tfrac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)\right) +$$

[4]We add the Feynman gauge fixing terms just to simplify the form of the gauge kinetic terms.

$$
\begin{gathered}
\frac{2M^4}{g^2}\alpha_h - g\alpha_h M \big(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-\big) - \\
\frac{1}{8}g^2\alpha_h \big(H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2\big) - \\
gMW^+_\mu W^-_\mu H - \frac{1}{2}g\frac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \\
\frac{1}{2}ig\big(W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)\big) + \\
\frac{1}{2}g\big(W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) + W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)\big) + \frac{1}{2}g\frac{1}{c_w}Z^0_\mu(H\partial_\mu\phi^0 - \\
\phi^0\partial_\mu H) + M(\frac{1}{c_w}Z^0_\mu\partial_\mu\phi^0 + W^+_\mu\partial_\mu\phi^- + W^-_\mu\partial_\mu\phi^+) - ig\frac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - \\
W^-_\mu\phi^+) + igs_wMA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) - ig\frac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + \\
igs_wA_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \frac{1}{4}g^2W^+_\mu W^-_\mu\big(H^2+(\phi^0)^2+2\phi^+\phi^-\big) - \\
\frac{1}{8}g^2\frac{1}{c_w^2}Z^0_\mu Z^0_\mu\big(H^2+(\phi^0)^2+2(2s_w^2-1)^2\phi^+\phi^-\big) - \frac{1}{2}g^2\frac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) - \\
\frac{1}{2}ig^2\frac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \frac{1}{2}g^2s_wA_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) + \\
\frac{1}{2}ig^2s_wA_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\frac{s_w}{c_w}(2c_w^2-1)Z^0_\mu A_\mu\phi^+\phi^- - g^2s_w^2A_\mu A_\mu\phi^+\phi^- + \\
\frac{1}{2}ig_s\lambda^a_{ij}(\bar q^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu - \bar e^\lambda(\gamma\partial + m^\lambda_e)e^\lambda - \bar\nu^\lambda(\gamma\partial+m^\lambda_\nu)\nu^\lambda - \bar u^\lambda_j(\gamma\partial+m^\lambda_u)u^\lambda_j - \bar d^\lambda_j(\gamma\partial + \\
m^\lambda_d)d^\lambda_j + igs_wA_\mu\big(-(\bar e^\lambda\gamma^\mu e^\lambda) + \frac{2}{3}(\bar u^\lambda_j\gamma^\mu u^\lambda_j) - \frac{1}{3}(\bar d^\lambda_j\gamma^\mu d^\lambda_j)\big) + \frac{ig}{4c_w}Z^0_\mu\{(\bar\nu^\lambda\gamma^\mu(1+ \\
\gamma^5)\nu^\lambda) + (\bar e^\lambda\gamma^\mu(4s_w^2-1-\gamma^5)e^\lambda) + (\bar d^\lambda_j\gamma^\mu(\frac{4}{3}s_w^2-1-\gamma^5)d^\lambda_j) + (\bar u^\lambda_j\gamma^\mu(1-\frac{8}{3}s_w^2+ \\
\gamma^5)u^\lambda_j)\} + \frac{ig}{2\sqrt{2}}W^+_\mu\big((\bar\nu^\lambda\gamma^\mu(1+\gamma^5){U^{lep}}_{\lambda\kappa}e^\kappa) + (\bar u^\lambda_j\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d^\kappa_j)\big) + \\
\frac{ig}{2\sqrt{2}}W^-_\mu\Big((\bar e^\kappa {U^{lep}}^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar d^\kappa_j C^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)u^\lambda_j)\Big) + \\
\frac{ig}{2M\sqrt{2}}\phi^+\big(-m^\kappa_e(\bar\nu^\lambda{U^{lep}}_{\lambda\kappa}(1-\gamma^5)e^\kappa) + m^\lambda_\nu(\bar\nu^\lambda {U^{lep}}_{\lambda\kappa}(1+\gamma^5)e^\kappa)\big) + \\
\frac{ig}{2M\sqrt{2}}\phi^-\Big(m^\lambda_e(\bar e^\lambda{U^{lep}}^\dagger_{\lambda\kappa}(1+\gamma^5)\nu^\kappa) - m^\kappa_\nu(\bar e^\lambda{U^{lep}}^\dagger_{\lambda\kappa}(1-\gamma^5)\nu^\kappa)\Big) - \frac{g}{2}\frac{m^\lambda_\nu}{M}H(\bar\nu^\lambda\nu^\lambda) - \\
\frac{g}{2}\frac{m^\lambda_e}{M}H(\bar e^\lambda e^\lambda) + \frac{ig}{2}\frac{m^\lambda_\nu}{M}\phi^0(\bar\nu^\lambda\gamma^5\nu^\lambda) - \frac{ig}{2}\frac{m^\lambda_e}{M}\phi^0(\bar e^\lambda\gamma^5e^\lambda) - \frac{1}{4}\,\bar\nu_\lambda M^R_{\lambda\kappa}\,(1-\gamma_5)\hat\nu_\kappa - \\
\frac{1}{4}\overline{\bar\nu_\lambda M^R_{\lambda\kappa}\,(1-\gamma_5)\hat\nu_\kappa} + \frac{ig}{2M\sqrt{2}}\phi^+\Big(-m^\kappa_d(\bar u^\lambda_j C_{\lambda\kappa}(1-\gamma^5)d^\kappa_j) + m^\lambda_u(\bar u^\lambda_j C_{\lambda\kappa}(1+ \\
\gamma^5)d^\kappa_j)\Big) + \frac{ig}{2M\sqrt{2}}\phi^-\Big(m^\lambda_d(\bar d^\lambda_j C^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j) - m^\kappa_u(\bar d^\lambda_j C^\dagger_{\lambda\kappa}(1-\gamma^5)u^\kappa_j)\Big) - \\
\frac{g}{2}\frac{m^\lambda_u}{M}H(\bar u^\lambda_j u^\lambda_j) - \frac{g}{2}\frac{m^\lambda_d}{M}H(\bar d^\lambda_j d^\lambda_j) + \frac{ig}{2}\frac{m^\lambda_u}{M}\phi^0(\bar u^\lambda_j\gamma^5u^\lambda_j) - \frac{ig}{2}\frac{m^\lambda_d}{M}\phi^0(\bar d^\lambda_j\gamma^5 d^\lambda_j).
\end{gathered}
$$

This formula compares nicely with [26], *i.e.*, the Lagrangian of §1. Besides the addition of the neutrino mass terms, and absence of the ghost terms there is only one difference: in the spectral action Lagrangian one gets the term

$$M\,(\frac{1}{c_w}Z^0_\mu\partial_\mu\phi^0 + W^+_\mu\partial_\mu\phi^- + W^-_\mu\partial_\mu\phi^+) \tag{51}$$

while in the Veltman's formula [26] one gets instead the following

$$-M^2\phi^+\phi^- - \frac{1}{2c_w^2}M^2\phi^0\phi^0. \tag{52}$$

This difference comes from the gauge fixing term

$$\mathcal{L}_{fix} = -\frac{1}{2}\,\mathcal{C}^2\,, \quad \mathcal{C}_a = -\partial_\mu W^\mu_a + M_a\,\phi_a \tag{53}$$

given by the Feynman-t'Hooft gauge in Veltman's formula [26], indeed one has

$$\mathcal{L}_{fix} = -\frac{1}{2}(\partial_\mu W_a^\mu)^2 - \frac{1}{2c_w^2} M^2 \phi^0 \phi^0 - M^2 \phi^+ \phi^- + M\left(\frac{1}{c_w}\phi^0 \partial_\mu Z_\mu^0 + \phi^- \partial_\mu W_\mu^+ + \phi^+ \partial_\mu W_\mu^-\right). \tag{54}$$

10. Predictions

The conversion table 1 shows that all the mass parameters of the standard model now acquire geometric meaning as components of the non-commutative metric as displayed in the right column. We shall work under the hypothesis that the geometric theory is valid at a preferred scale Λ of the order of the unification scale (*cf.* §10.1) and that the standard model coupled with gravity is just its manifestation when one integrates the high energy modes á la Wilson. Then following [4] one can use the renormalization group equations to run down the various coupling constants. Besides the gauge couplings this will be done for the value of the Higgs quartic self-coupling which gives a rough estimate (around 170 GeV) for the Higgs mass under the "big desert" hypothesis. It is satisfactory that another prediction at unification, namely the mass relation of §10.3 also gives a sensible answer, while similar results hold for the couplings of the gravitational part of the action. But it is of course very likely that instead of the big desert one will meet gradual refinements of the non-commutative geometry $M \times F$ when climbing in energy to the unification scale so that our knowledge of the finite geometry F is still too primitive to make accurate predictions. The "naturalness" problem will be discussed in §10.5. At first sight one might easily confuse the obtained predictions with those of a GUT, but there is a substantial difference since for instance no proton decay is implied in our theory.

10.1. Unification of couplings

The numerical values are similar to those of [4] and in particular one gets the same value of gauge couplings as in grand unified theories SU(5) or SO(10). The three gauge couplings fulfill (42). This means that in the above formula the values of g, g_s and s_w, c_w are fixed exactly as in [4] at

$$g_s = g\,, \quad \mathrm{tg}(w)^2 = \frac{3}{5}\,. \tag{55}$$

10.2. See-saw mechanism for neutrino masses

Let us briefly explain the analogue of the see-saw mechanism in our context. We use the notation of §6.3. The restriction of $D(M)$ to the subspace of $\mathcal{H}_F$ with basis

Standard Model	notation	notation	Spectral Action
Higgs Boson	$\varphi = (\frac{2M}{g} + H$ $-i\phi^0,\ -i\sqrt{2}\phi^+)$	$\mathbf{H} = \frac{1}{\sqrt{2}}\frac{\sqrt{a}}{g}(1+\psi)$	Inner metric$^{(0,1)}$
Gauge bosons	$A_\mu, Z^0_\mu, W^\pm_\mu, g^a_\mu$	(B, W, V)	Inner metric$^{(1,0)}$
Fermion masses u, ν	m_u, m_ν	$M_u = \delta_u, M_\nu = \delta_\nu$	Dirac$^{(0,1)}$ in u, ν
CKM matrix Masses down	C^κ_λ, m_d	$M_d = C\,\delta_d\,C^\dagger$	Dirac$^{(0,1)}$ in d
Lepton mixing Masses leptons e	${U^{lep}}_{\lambda\kappa}, m_e$	$M_e = U^{lep}\,\delta_e\,{U^{lep}}^\dagger$	Dirac$^{(0,1)}$ in e
Majorana mass matrix	M^R	M_R	Dirac$^{(0,1)}$ in $\nu_R, \bar{\nu}_R$
Gauge couplings	$g_1 = g\,\mathrm{tg}(w)$, $g_2 = g, g_3 = g_s$	$g_3^2 = g_2^2 = \frac{5}{3}\,g_1^2$	Fixed at unification
Higgs scattering parameter	$\frac{1}{8}\,g^2\,\alpha_h, \alpha_h = \frac{m_h^2}{4M^2}$	$\lambda_0 = g^2\,\frac{b}{a^2}$	Fixed at unification
Tadpole constant	$\beta_h, (-\alpha_h\,M^2$ $+\frac{\beta_h}{2})\,\lvert\varphi\rvert^2$	$\mu_0^2 = 2\frac{f_2\Lambda^2}{f_0} - \frac{e}{a}$	$-\mu_0^2\,\lvert\mathbf{H}\rvert^2$
Graviton	$g_{\mu\nu}$	$\partial\!\!\!/_M$	Dirac$^{(1,0)}$

TABLE 1. Conversion from Spectral Action to Standard Model

the $(\nu_R, \nu_L, \bar{\nu}_R, \bar{\nu}_L)$ is given by the matrix

$$\begin{bmatrix} 0 & M_\nu^* & M_R^* & 0 \\ M_\nu & 0 & 0 & 0 \\ M_R & 0 & 0 & \bar{M}_\nu^* \\ 0 & 0 & \bar{M}_\nu & 0 \end{bmatrix}. \tag{56}$$

Let us simplify to one generation and let $M_R \sim M$ be a very large mass term – the largest eigenvalue of M_R will be set to the order of the unification scale by the equations of motion of the spectral action – while $M_\nu \sim m$ is much smaller[5]. The eigenvalues of the matrix (56) are then given by

$$\frac{1}{2}\,(\pm M\ \pm \sqrt{M^2 + 4m^2})$$

[5]It is a Dirac mass term, fixed by the Higgs vev.

which gives two eigenvalues very close to $\pm M$ and two others very close to $\pm \frac{m^2}{M}$ as can be checked directly from the determinant of the matrix (56), which is equal to $|M_\nu|^4 \sim m^4$ (for one generation).

10.3. Mass relation $Y_2(S) = 4\,g^2$

Note that the matrices M_u, M_d, M_ν and M_e are only relevant up to an overall scale. Indeed they only enter in the coupling of the Higgs with fermions and because of the rescaling (41) only by the terms

$$k_x = \frac{\pi}{\sqrt{a\,f_0}}\,M_x\,, \quad x \in \{u, d, \nu, e\} \tag{57}$$

which are dimensionless matrices by construction. The conversion for the mass matrices is

$$\begin{aligned}(k_u)_{\lambda\kappa} &= \frac{g}{2M}\,m_u^\lambda\,\delta_\lambda^\kappa \\ (k_d)_{\lambda\kappa} &= \frac{g}{2M}\,m_d^\mu\,C_{\lambda\mu}\delta_\mu^\rho C_{\rho\kappa}^\dagger \\ (k_\nu)_{\lambda\kappa} &= \frac{g}{2M}\,m_\nu^\lambda\,\delta_\lambda^\kappa \\ (k_e)_{\lambda\kappa} &= \frac{g}{2M}\,m_e^\mu\,U^{lep}{}_{\lambda\mu}\delta_\mu^\rho {U^{lep}}^\dagger{}_{\rho\kappa}.\end{aligned} \tag{58}$$

It might seem at first sight that one can simply use (58) to define the matrices k_x but this overlooks the fact that (57) implies one constraint:

$$\mathrm{Tr}(k_\nu^* k_\nu + k_e^* k_e + 3(k_u^* k_u + k_d^* k_d)) = \,2\,g^2\,, \tag{59}$$

using (42) to replace $\frac{\pi^2}{f_0}$ by $2\,g^2$. When expressed in the right-hand side, *i.e.*, the standard model parameters this gives

$$\sum_\lambda (m_\nu^\lambda)^2 + (m_e^\lambda)^2 + 3\,(m_u^\lambda)^2 + 3\,(m_d^\lambda)^2 = \,8\,M^2 \tag{60}$$

where M is the mass of the W boson. Thus with the standard notation ([19]) for the Yukawa couplings, so that the fermion masses are $m_f = \frac{1}{\sqrt{2}} y_f\,v$, $v = \frac{2M}{g}$ the relation reads

$$\sum_\lambda (y_\nu^\lambda)^2 + (y_e^\lambda)^2 + 3\,(y_u^\lambda)^2 + 3\,(y_d^\lambda)^2 = \,4\,g^2. \tag{61}$$

Neglecting the other Yukawa coupling except for the top quark, and imposing the relation (61) at unification scale, then running it downwards using the renormalization group one gets the boundary value $\frac{2}{\sqrt{3}} g \sim 0.597$ for y_t at unification scale which gives a Fermi scale value of the order of $y_0 =\sim 1.102$ and a top quark mass of the order of $\frac{1}{\sqrt{2}} y_0\,v \sim\, 173\,y_0$ GeV. This is fine since a large neglected tau neutrino Yukawa coupling (allowed by the see-saw mechanism) similar to that of the top quark, lowers the value at unification by a factor of $\sqrt{\frac{3}{4}}$ which has the effect of lowering the value of y_0 to $y_0 \sim 1.04$. This yields an acceptable value for the top quark mass (whose Yukawa coupling is $y_0 \sim 1$), given that we still neglected all other smaller Yukawa couplings.

10.4. The Higgs scattering parameter

The change of notation for the Higgs fields using the conversion Table 1 gives

$$\mathbf{H} = \frac{1}{\sqrt{2}}\frac{\sqrt{a}}{g}(1+\psi) = (\frac{2M}{g} + H - i\phi^0, -i\sqrt{2}\phi^+)\,. \tag{62}$$

One gets a specific value of the Higgs scattering parameter α_h, as in [4] (which agrees with [19]),

$$\alpha_h = \frac{8\,b}{a^2} \tag{63}$$

(with the notations (39)) which is of the order of $\frac{8}{3}$ if there is a dominating top mass. The numerical solution to the RG equations with the boundary value $\lambda_0 = 0.356$ at $\Lambda = 10^{17}$ GeV gives $\lambda(M_Z) \sim 0.241$ and a Higgs mass of the order of 170 GeV.

10.5. Naturalness

The hypothesis that what we see is the low energy average of a purely geometric theory valid at unification scale Λ needs to be confronted with the "naturalness" problem coming from the quadratically divergent graphs of the theory. This problem already arises in a ϕ^4 theory (in dimension 4) with classical potential $V_0(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4$. Recall that the *effective potential* which is the first term in the expansion of the effective action in powers of the derivatives of the classical field ϕ around the constant field $\phi = \phi_c$

$$S_{eff}(\phi) = \int [-V(\phi) + \frac{1}{2}(\partial_\mu\phi)^2 Z(\phi) + \cdots]\, d^D x \tag{64}$$

can be expressed as the following sum over 1PI diagrams with zero external momenta:

$$V(\phi_c) = V_0(\phi_c) - \sum_{\Gamma\in\,\mathrm{1PI}} \hbar^L \frac{U(\Gamma(p_1 = 0, \ldots, p_N = 0))}{\sigma(\Gamma)} \frac{\phi_c^N}{N!} \tag{65}$$

where ϕ_c is viewed as a real variable, and $V_0(\phi_c)$ is the classical potential. By construction the quantum corrections are organized in increasing powers of $\hbar$ and these correspond to the loop number of the 1PI graphs. At the one-loop level and for a polynomial interaction, one finds that the unrenormalized value gives ([25], equation 2.64)

$$V(\phi_c) = V_0(\phi_c) + \frac{\hbar}{2}\int \log(1 + \frac{V_0''(\phi_c)}{k^2})\frac{d^D k}{(2\pi)^D} + O(\hbar^2). \tag{66}$$

In dimension $D = 4$ the integral diverges in the ultraviolet due to the two terms

$$\frac{V_0''(\phi_c)}{k^2} - \frac{V_0''^{\,2}(\phi_c)}{2\,k^4} \tag{67}$$

in the expansion of the logarithm at large momentum k. If the classical potential V_0 is at most quartic the divergences can be compensated by adding suitable counterterms in the classical potential. Thus, in particular, if one uses a ultraviolet cutoff Λ and considers the ϕ^4 theory with classical potential $V_0(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4$, one gets a quadratic divergence of the form

$$\frac{\Lambda^2}{32\pi^2}(3\lambda\phi_c^2 + m^2) - \frac{\log\Lambda}{32\pi^2}(V_0''(\phi_c))^2, \tag{68}$$

whose elimination requires adjusting the classical potential as a function of the cutoff Λ as

$$(V_0 + \delta V_0)(\phi) = V_0(\phi) - \frac{\Lambda^2}{32\pi^2}(3\lambda\phi^2) + \frac{\log\Lambda}{32\pi^2}(6m^2\lambda\phi^2 + 9\lambda^2\phi^4), \tag{69}$$

where we ignored an irrelevant (but Λ-dependent) additive constant.

This shows very clearly that, in order to obtain a Λ-independent effective potential, one needs the bare action to depend upon Λ with a large negative quadratic term of the form $-\frac{\Lambda^2}{32\pi^2}(3\lambda\phi^2)$ at the one-loop level. This is precisely the type of term present in the spectral action in the case of the standard model. The presence of the other quadratic divergences coming from the Yukawa coupling of the scalar field with fermions alters the overall sign of the quadratic divergence only at small enough Λ. However, as shown in [7] §5.7, it comes back to the above sign when Λ gets above 10^{10} GeV and in particular when it is close to the unification scale. This leaves open the possibility of using the quadratically divergent mass term of the spectral action to account for the naturalness problem (but an accurate account would require at least some fine tuning of the unification scale, and a better understanding of the running of the Newton constant).

10.6. Gravitational terms

We now discuss the behavior of the gravitational terms in the spectral action, namely

$$\int \left(\frac{1}{2\kappa_0^2}\, R + \alpha_0\, C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma} + \gamma_0 + \tau_0\, R^* R^* - \xi_0\, R\, |\mathbf{H}|^2 \right) \sqrt{g}\, d^4x. \tag{70}$$

The traditional form of the Euclidean higher derivative terms that are quadratic in curvature is

$$\int \left(\frac{1}{2\eta}\, C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma} - \frac{\omega}{3\eta}\, R^2 + \frac{\theta}{\eta}\, E \right) \sqrt{g}\, d^4x, \tag{71}$$

with $E = R^* R^*$ the topological term which is the integrand in the Euler characteristic

$$\chi(M) = \frac{1}{32\pi^2} \int E \sqrt{g}\, d^4x = \frac{1}{32\pi^2} \int R^* R^* \sqrt{g}\, d^4x\,. \tag{72}$$

The running of the coefficients of the Euclidean higher derivative terms in (71), determined by the renormalization group equation, is gauge independent and known and we computed in [7] their value at low scale starting from the initial value prescribed by the spectral action at unification scale. We found that the infrared behavior of these terms approaches the fixed point $\eta = 0$, $\omega = -0.0228$, $\theta = 0.327$. The coefficient η goes to zero in the infrared limit, sufficiently slowly, so that, up to scales of the order of the size of the universe, its inverse remains $O(1)$. On the other hand, $\eta(t)$, $\omega(t)$ and $\theta(t)$ have a common singularity at an energy scale of the order of 10^{23} GeV, which is above the Planck scale. Moreover, within the energy scales that are of interest to our model $\eta(t)$ is neither too small nor too large (it does not vary by more than a single order of magnitude between the Planck scale and infrared energies). This implies in particular that the extra terms (besides the Einstein-Hilbert term) do not have any observable consequence at low energy.

The discussion of the Newton constant is much more tricky since its running is scheme dependent. Under the very conservative hypothesis that it does not run much from our scale to the unification scale one finds (*cf.* [7]) that for a unification scale of 10^{17} GeV an order one tuning ($f_2 \sim 5f_0$) of the moments of the test function f suffices to get an acceptable value for the Newton constant.

11. Final remarks

The above approach to physics can be summarized as a strategy to interpret the complicated input of the phenomenological Lagrangian of gravity coupled with matter as coming from a fine structure (of the form $M \times F$) in the geometry of space-time. Extrapolating this to unification scale (*i.e.*, assuming the big desert) gives predictions which can be compared with experiment. Of course we do not believe that the big desert is there and a key test when "new physics" will be observed is to decide wether it will be possible to interpret the new terms of the Lagrangian in the same manner from non-commutative spaces and the spectral action. This type of test already occurred with the new neutrino physics coming from the Kamiokande experiment and for quite some time I believed that the new terms would simply not fit with the spectral action principle. It is only thanks to the simple idea of decoupling the KO-dimension from the metric dimension that the problem was resolved (this was also done independently by John Barrett [1] with a similar solution).

At a more fundamental level the fact that the action functional can be obtained from spectral data suggests that instead of just looking at the inner fluctuations of a product metric on $M \times F$, one should view that as a special case of a fully unified theory at the operator theoretic level, *i.e.*, a kind of spectral random matrix theory where the operator D varies in the symplectic ensemble (corresponding to the commutation with $i = \sqrt{-1}$ and J that generate the quaternions). The first basic step is to understand how to extend the computations of the spontaneous symmetry breaking of the electroweak sector of SM [25] to the full gravitational sector. The natural symmetry group of the spectral action is the unitary symplectic group, corresponding as above to the commutation with $i = \sqrt{-1}$ and J. In the forthcoming book with M. Marcolli [13] we develop an analogy between the spontaneous symmetry breaking which is the key of our work in number theory (the theory of $\mathbb{Q}$-lattices) and the missing SSB for gravity. One of the simple ideas that emerge from the mere existence of the analogy is that the geometry of space-time is a notion which probably stops making sense when the energy scale (*i.e.*, the temperature) is above the critical value of the main phase transition. In particular it seems an ill fated goal to try and quantize gravity as a fundamental theory in a fixed background, the idea being that the very notion of space-time ceases to make sense at high enough energy scales. We refer to the last section of [13] for a more detailed discussion of this point.

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