

Quasi-inner functions and local factors

Alain Connes^{a,d,c,*}, Caterina Consani^{d,1}

^aCollège de France, 3 rue d'Ulm, Paris F-75005, France

^bI.H.E.S., France

^cOhio State University, USA

^dJohns Hopkins University, Baltimore (MD) 21218, USA

Abstract

We introduce the notion of *quasi-inner* function and show that the product $u = \rho_\infty \prod \rho_v$ of $m+1$ ratios of local L -factors $\rho_v(z) = \gamma_v(z)/\gamma_v(1-z)$ over a finite set F of places of \mathbb{Q} inclusive of the archimedean place is quasi-inner on the left of the critical line $\Re(z) = \frac{1}{2}$ in the following sense. The off diagonal part u_{21} of the matrix of the multiplication by u in the orthogonal decomposition of the Hilbert space L^2 of square integrable functions on the critical line into the Hardy space H^2 and its orthogonal complement is a compact operator. When interpreted on the unit disk, the quasi-inner condition means that the associated Haenkel matrix is compact. We show that none of the individual non-archimedean ratios ρ_v is quasi-inner and, in order to prove our main result we use Gauss multiplication theorem to factor the archimedean ratio ρ_∞ into a product of m quasi-inner functions whose product with each ρ_v retains the property to be quasi-inner. Finally we prove that Sonin's space is simply the kernel of the diagonal part u_{22} for the quasi-inner function $u = \rho_\infty$, and when $u(F) = \prod_{v \in F} \rho_v$ the kernels of the $u(F)_{22}$ form an inductive system of infinite dimensional spaces which are the semi-local analogues of (classical) Sonin's spaces.

Keywords: Semi-local, Trace formula, scaling, Hamiltonian, Weil positivity, Riemann zeta function, Sonin space

2008 MSC: 11M55, 11M06, 46L87, 58B34

1. Introduction

In [4] we proved that while the ratio of local L -factors with their complex conjugates is a function of modulus one on the critical line $\Re(z) = \frac{1}{2}$ in the complex plane, it fails to be an inner function². In [5] we then obtained, in the case of the single archimedean place, a powerful inequality relating Weil's functional as in the explicit formulas and the trace of the scaling action on Sonin's space. A corollary of these results is that even though the

*Corresponding author

Email addresses: alain@connes.org (Alain Connes), kc@math.jhu.edu (Caterina Consani)

¹Partially supported by Simons Foundation, Collaboration Grants for Mathematicians n. 691493

²and we explained that this failure invalidates an attempt by X. J. Li on RH

ratio $\rho_\infty(z)$ of local factors pertaining to the archimedean place is not an inner function it satisfies the very closely related property to be *quasi-inner* in the following sense

Definition. Let $\Omega \subset \mathbb{C}$ be an open disk or a half-space. A function $u \in L^\infty(\partial\Omega)$ of modulus 1 (i.e. $u\bar{u} = 1$) is said to be *quasi-inner* if the operator $(1 - \mathcal{P})u\mathcal{P}$ is compact, where \mathcal{P} is the orthogonal projection of $L^2(\partial\Omega)$ on the Hardy space $H^2(\Omega)$ and u acts on $L^2(\partial\Omega)$ by multiplication.

Inner functions are quasi-inner since when $u \in H^\infty(\Omega)$ then $(1 - \mathcal{P})u\mathcal{P} = 0$. To be quasi-inner means that the matrix of the action of u on the orthogonal decomposition $L^2(\partial\Omega) = H^2(\Omega) \oplus H^2(\Omega)^\perp$ is triangular modulo compact operators. It follows that the product of two quasi-inner functions is also quasi-inner. When Ω is the unit disk $\mathcal{U} = \{z \in \mathbb{C} \mid |z| < 1\}$ the operator $(1 - \mathcal{P})u\mathcal{P}$ is closely related to Haenkel's operator H_u with symbol u . Indeed, one has $H_u = \mathcal{P}Ju\mathcal{P}$ with $Jf(e^{i\theta}) := f(e^{-i\theta})$, $J1 = 1$ for the (normalized) constant function 1 and the following holds

$$J\mathcal{P}J = 1 - \mathcal{P} + |1\rangle\langle 1| \implies (1 - \mathcal{P})u\mathcal{P} \sim JH_u \quad (\text{modulo finite rank operators}).$$

Thus the compactness of $(1 - \mathcal{P})u\mathcal{P}$ is equivalent to that of H_u . The condition that a Haenkel operator H_f (with symbol f) is compact is very well studied in [6] to which we refer for more details on this topic.

The results of [5] imply that the archimedean ratio ρ_∞ is a quasi-inner function on the critical line (viewed as the boundary of the half-plane $\Re(z) \leq \frac{1}{2}$). In the present paper we give, in Section 2, an independent direct proof of this result. We then show that while the individual ratios ρ_p of local factors associated to rational primes fail to be quasi-inner the products $u = \rho_\infty \prod \rho_p$ over a finite set of places of \mathbb{Q} inclusive of the archimedean place is a quasi-inner function. This result is meant to be a first test pertaining to the general strategy proposed in [5] to prove Weil's positivity using the semi-local trace formula of [2]. The local L -factors and their ratios are defined as follows

$$\gamma_\infty(z) := \pi^{-z/2} \Gamma\left(\frac{z}{2}\right), \quad \gamma_p(z) := (1 - p^{-z})^{-1}, \quad \rho_*(z) := \gamma_*(z)/\gamma_*(1 - z). \quad (1)$$

Our main result is the following

Theorem. The product $u = \rho_\infty \prod \rho_p$ of $m + 1$ ratios of local factors over a finite set of places of \mathbb{Q} containing the archimedean place is a quasi-inner function relative to $\mathbb{C}_- = \{z \in \mathbb{C} \mid \Re(z) \leq \frac{1}{2}\}$.

Throughout the paper we use the invariance of the quantized calculus [1] under conformal transformations to switch back and forth from the unit disk $\mathcal{U} = \{v \in \mathbb{C} \mid |v| \leq 1\}$ to the half plane \mathbb{C}_- , by implementing the conformal transformation $\psi(v) = \frac{1}{2} + \frac{v+1}{v-1}$ ($\psi(-1) = \frac{1}{2}$, $\psi(0) = -\frac{1}{2}$, $\psi(1) = -\infty$) and its inverse $\psi^{-1}(z) = \frac{2z+1}{2z-3}$. At the level of the Hilbert spaces the unitary operator

$$U : L^2(S^1) \rightarrow L^2(\partial\mathbb{C}_-), \quad (U\xi)(z) := \frac{\pi^{-1/2}}{z - 3/2} \xi(\psi^{-1}(z))$$

transforms an element of the Hardy space $H^2(\mathcal{U})$ into a holomorphic function in \mathbb{C}_- whose restriction to the critical line is square integrable, and it conjugates the corresponding operators \mathcal{P} .

Theorem 2.1 in Section 2 shows that the archimedean ratio $\rho_\infty(z)$ is a quasi inner function relative to \mathbb{C}_- and Theorem 2.3 provides an explicit formula for the operator $(1 - \mathcal{P})\kappa\mathcal{P}$, (where $\kappa = \rho_\infty \circ \psi$) as a fast convergent series of rank one operators. Proposition 2.4 then gives an independent proof of Theorem 2.1 showing that κ belongs to $C^\infty(S^1) + H^\infty(\mathcal{U})$. In Section 3 we consider a non-archimedean ratio ρ_p for a rational prime p , and give in Lemma 3.5 an explicit formula for the operator $(1 - \mathcal{P})\kappa_p\mathcal{P}$, where $\kappa_p = \rho_p \circ \psi$. Then we explain why ρ_p fails to be quasi-inner (Fact 3.6). Using the poles of κ_p , we associate to each prime a Blaschke product $B_p \in H^\infty(\mathcal{U})$ and prove, in Proposition 3.7, that the product $\kappa_p B_p$ is the inner function $-p^{\frac{v+1}{v-1}}$ and that the kernel of $(1 - \mathcal{P})\kappa_p\mathcal{P}$ is the shift invariant subspace of $H^2(\mathcal{U})$ of multiples of B_p .

Section 4 is devoted to the proof of the main Theorem. This result rests crucially on the decay of ρ_∞ on the imaginary line: we provide the needed estimates in §4.1. We first prove in Theorem 4.4 that the product $\rho_\infty \rho_p$ is quasi inner relative to \mathbb{C}_- , and give an explicit formula for $(1 - \mathcal{P})\kappa\kappa_p\mathcal{P}$ as a sum of three terms $\mathcal{E}_\infty + \mathcal{E}_p + \mathcal{E}_0$ corresponding to the poles of $\rho_\infty \rho_p$. Moreover, Proposition 4.5 gives a decomposition of $\kappa\kappa_p$ as a sum of a function in $H^\infty(\mathcal{U})$ and a function continuous on S^1 (in fact smooth outside the point $1 \in S^1$). In §4.3 we use Gauss multiplication theorem to factor ρ_∞ into a product of m quasi-inner functions whose product with each ρ_p is still quasi-inner: this is the fundamental step needed to complete the proof of the main Theorem.

In Section 5 we focus on the main result of [5] which relates the positivity of Weil's functional and the trace of the scaling action compressed on Sonin's space. By construction a quasi-inner function U is such that when working modulo compact operators (*i.e.* in the Calkin algebra) the 2×2 matrix representing U is triangular. Moreover, the structure of triangular unitaries is governed by the kernel of the matrix element U_{22} (see §5.1). The first key result is that Sonin's space is canonically isomorphic to the kernel of the matrix element $(\rho_\infty)_{22}$. This suggests to associate to any quasi-inner function U an analogue of Sonin's space as the kernel of U_{22} . In the present paper we take this as a definition postponing to a future geometric paper the proof that when this definition is applied to the product $u = \rho_\infty \prod \rho_p$ of $m + 1$ ratios of local factors over a finite set of places of \mathbb{Q} containing the archimedean place it gives the straight-forward analogue of Sonin's space in the semi-local framework of [2].

The second main result of this paper is the following

Theorem. (i) Let F be a finite set of places of \mathbb{Q} containing the archimedean place, $u(F) = \prod_F \rho_v$ the associated product of ratios of local factors over F . Then the Sonin space $S(u(F))$ is infinite dimensional.

(ii) Let $F \subset F'$ with F, F' as in (i). The multiplication by $D(F, F') = \prod_{p \in F' \setminus F} (1 - p^{-z})$ defines an injective linear map $S(u(F)) \rightarrow S(u(F'))$.

This shows that semi-local Sonin's spaces form a filtering system of infinite dimensional spaces.

2. The function ρ_∞ is quasi-inner

In this section we show that the function

$$\rho_\infty(z) := \frac{\pi^{-z/2} \Gamma(z/2)}{\pi^{-(1-z)/2} \Gamma((1-z)/2)} \quad (2)$$

is quasi-inner relative to the left half-plane $\mathbb{C}_- = \{z \in \mathbb{C} \mid \Re(z) \leq \frac{1}{2}\}$ with boundary the critical line $\Re(z) = \frac{1}{2}$. The function ρ_∞ fulfills the functional equation

$$\rho_\infty(z+2) = -(2\pi)^{-2}z(z+1)\rho_\infty(z). \quad (3)$$

This can be seen by implementing the equality

$$\rho_\infty(z) = 2 \cos\left(\frac{\pi z}{2}\right) (2\pi)^{-z} \Gamma(z). \quad (4)$$

which is proven using the duplication and complement formulas.

Next, we compute the Fourier coefficients a_{-k} , $k > 0$ of the function $\kappa = \rho_\infty \circ \psi$ ($\psi(v)$ is the conformal transformation mapping the unit disk to \mathbb{C}_-)

$$\kappa(v) := \rho_\infty\left(\frac{1}{2} + \frac{v+1}{v-1}\right)$$

With S^1 the positively oriented circle, one has

$$a_{-k} := \frac{1}{2\pi} \int \kappa(\exp(i\theta)) \exp(ik\theta) d\theta = \frac{1}{2\pi i} \int_{S^1} \kappa(v) v^{k-1} dv.$$

Changing variables and implementing the differential $d\psi^{-1}(z) = -\frac{8}{(2z-3)^2} dz$ gives

$$a_{-k} = \frac{1}{2\pi i} \int_{\partial\mathbb{C}_-} \rho_\infty(z) \psi^{-1}(z)^{k-1} d\psi^{-1}(z) = -\frac{8}{2\pi i} \int_{\partial\mathbb{C}_-} \rho_\infty(z) \left(\frac{2z+1}{2z-3}\right)^{k-1} (2z-3)^{-2} dz.$$

The function $\rho_\infty(z) \psi^{-1}(z)^{k-1}$ is of modulus one on the boundary $\partial\mathbb{C}_-$. We apply Cauchy's residue theorem to compute the integral above for a_{-k} . Due to the term $(2z-3)^{-2}$ the integral is convergent and is the limit, for $R \rightarrow \infty$, of the following integrals where

$$I(R) := -\frac{8}{2\pi i} \int_{\frac{1}{2}-iR}^{\frac{1}{2}+iR} \omega_\infty(z) \left(\frac{2z+1}{2z-3}\right)^{k-1} (2z-3)^{-2} dz.$$

Consider, for $m \in \mathbb{N}$ the closed positively oriented path $C_{R,m}$ in the complex plane formed by the segments joining the following points (see Figure 1)

$$\frac{1}{2} - iR, \quad \frac{1}{2} + iR, \quad \frac{1}{2} - 2m + iR, \quad \frac{1}{2} - 2m - iR, \quad \frac{1}{2} - iR.$$

The function $(\psi^{-1}(z))^{k-1} = \left(\frac{2z+1}{2z-3}\right)^{k-1}$ is, by construction, of modulus ≤ 1 in \mathbb{C}_- . The function $|\rho_\infty(z)|$ decays very fast on the segment joining $\frac{1}{2} + iR$ and $\frac{1}{2} - 2m + iR$, but we only use that it is of modulus ≤ 1 , for $z \in \mathbb{C}_-$ with $\Im(z) > 7$. This can be verified by first looking at the boundary values in the infinite rectangle with side the segment $(-\frac{3}{2} + 7i, \frac{1}{2} + 7i)$ and then using (3). For $R > 7$ one controls the integral on the segment $(\frac{1}{2} + iR, \frac{1}{2} - 2m + iR)$ since

$$\int_{-\infty}^{\frac{1}{2}} |2t-3+2iR|^{-2} dt \leq \frac{1}{4} \int_0^\infty (u^2 + R^2)^{-1} du = \frac{\pi}{8R}.$$

On the segment $V = (\frac{1}{2}-2m+iR, \frac{1}{2}-2m-iR)$, one can use (3) together with $|\rho_\infty(z)| = 1$ for $z \in \partial\mathbb{C}_-$ to obtain

$$|\rho_\infty(z)|(2\pi)^{-2m} \prod_{k=0}^{2m-1} |z+k| \leq 1, \quad \forall z \in V$$

which in turns gives

$$|\rho_\infty(z)| \leq (2\pi)^{2m} \prod_{k=0}^{2m-1} \left(2m - \frac{1}{2} - k\right)^{-1} = \epsilon(m), \quad \forall z \in V$$

with the sequence $\epsilon(m)$ tending to zero very fast for $m \rightarrow \infty$. This provides a good control of the integral on V by

$$\epsilon(m) \int_V |2z-3|^{-2} |dz| = \frac{1}{4} \epsilon(m) \int_{-R}^R ((2m+1)^2 + y^2)^{-2} dy \leq \epsilon(m).$$

Thus we obtain

$$a_{-k} = -\frac{8}{2\pi i} \lim_{\substack{R \rightarrow \infty \\ m \rightarrow \infty}} \int_{C_{R,m}} \rho_\infty(z) \left(\frac{2z+1}{2z-3}\right)^{k-1} (2z-3)^{-2} dz$$

with an error term of the form $\epsilon(m) + O(1/R)$. Then we apply Cauchy's residue theorem. The integrand has simple poles at the points $-2n$ for $n \in \mathbb{N}$ and one has

$$\Gamma(z/2) = \frac{2(-1)^n}{\Gamma(n+1)} \frac{1}{z+2n} + O(1), \quad z \rightarrow -2n.$$

Thus one obtains

$$\rho_\infty(z) = (-1)^n \frac{2\pi^{2n+\frac{1}{2}}}{\Gamma(n+1)\Gamma(n+\frac{1}{2})} \frac{1}{z+2n} + O(1), \quad z \rightarrow -2n$$

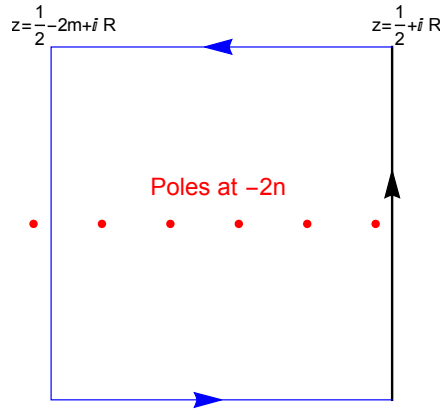


Figure 1: The path of integration $C_{R,m}$

The residue formula gives

$$a_{-k} = -8 \sum_{\mathbb{N}} \operatorname{Res}_{z=-2n} \left(\rho_{\infty}(z) \left(\frac{2z+1}{2z-3} \right)^{k-1} (2z-3)^{-2} \right)$$

and bringing up the above estimates one finally obtains

$$a_{-k} = \sum_{\mathbb{N}} (-1)^{n+1} \frac{16\pi^{2n+\frac{1}{2}}(4n+3)^{-2}}{\Gamma(n+1)\Gamma(n+\frac{1}{2})} \left(1 - \frac{4}{4n+3} \right)^{k-1}. \quad (5)$$

This is an expression of the form $a_{-k} = \sum_{\mathbb{N}} \alpha(n)x(n)^{k-1}$, where the series $\alpha(n)$ is summable and tends to 0 extremely fast, while the $x(n)$ increase to 1 with $1-x(n) \sim 1/n$. We are now ready to state the following

Theorem 2.1. *The function ρ_{∞} is quasi-inner relative to $\mathbb{C}_- = \{z \mid \Re(z) \leq \frac{1}{2}\}$. The operator $(1 - \mathcal{P})\rho_{\infty}\mathcal{P}$ is an infinitesimal of infinite order.*

Proof. It suffices to show that the sequence (a_{-k}) as in (5) is of rapid decay. Granted this, the operator $(1 - \mathcal{P})\rho_{\infty}\mathcal{P}$ is, up to a rank one operator, a Haenkel operator with smooth symbol. More explicitly one has

$$(1 - \mathcal{P})\rho_{\infty}\mathcal{P} = \sum_{k=1}^{\infty} a_{-k}(1 - \mathcal{P})e^{-ik\theta}\mathcal{P}$$

where the operator $(1 - \mathcal{P})e^{-ik\theta}\mathcal{P}$ is of norm 1 and rank k so that the characteristic values of the sum $\mu_m((1 - \mathcal{P})e^{-ik\theta}\mathcal{P})$ are of rapid decay. In fact imposing m linear conditions of orthogonality with the vectors $e^{i\ell\theta}$, for $0 \leq \ell \leq m-1$, reduces the sum to

$$\sum_{k=m}^{\infty} a_{-k}(1 - \mathcal{P})e^{-ik\theta}\mathcal{P}$$

whose norm is less than $\sum_m^{\infty} |a_{-k}|$ which is $O(m^{-N})$ for any N . To show that the sequence (a_{-k}) as in (5) is of rapid decay one can major it by

$$M(k) = C \sum_{n=1}^{\infty} 2^{-n} \left(1 - \frac{1}{n} \right)^k$$

for some constant $C < \infty$. Then, for any integer m one obtains the inequality

$$M(k)/C \leq \left(1 - \frac{1}{m} \right)^k + 2^{-m}$$

which, for $m \sim \sqrt{k}$, shows that $M(k)$ tends to 0 at least as $2^{-\sqrt{k}}$ and hence faster than any inverse polynomial in k . \square

We shall now provide an explicit formula for the operator $(1 - \mathcal{P})\rho_{\infty}\mathcal{P}$ as a very fast convergent sum of rank one operators. The following preliminary lemma is needed

Lemma 2.2. *The off diagonal part $(1 - \mathcal{P})f_x\mathcal{P}$ associated to the multiplication operator in $L^2(S^1)$ by the function $f_x(z) := z^{-1}(1 - xz^{-1})^{-1}$ with $|x| < 1$, is the rank one operator*

$$(1 - \mathcal{P})f_x\mathcal{P} = |\xi_x\rangle\langle\eta_x|, \quad \xi_x = z^{-1}(1 - xz^{-1})^{-1}, \quad \eta_x = (1 - \bar{x}z)^{-1}. \quad (6)$$

One has

$$\|\xi_x\|^2 = \frac{1}{1 - |x|^2}, \quad \|\eta_x\|^2 = \frac{1}{1 - |x|^2}, \quad \|\xi_x\|\|\eta_x\| = \frac{1}{1 - |x|^2}.$$

Proof. The function $f_x(z^{-1})$ is holomorphic in the unit disk and thus $\mathcal{P}f_x(1 - \mathcal{P}) = 0$. Next, one has

$$\mathcal{P}f_x(1 - \mathcal{P}) - (1 - \mathcal{P})f_x\mathcal{P} = [\mathcal{P}, f_x]$$

and since $xf_x = (1 - xz^{-1})^{-1} - 1$, we obtain

$$[\mathcal{P}, f_x] = (1 - xz^{-1})^{-1}[\mathcal{P}, z^{-1}](1 - xz^{-1})^{-1}$$

and

$$z[\mathcal{P}, z^{-1}] = z\mathcal{P}z^{-1} - \mathcal{P} = -|1\rangle\langle 1|, \quad [\mathcal{P}, z^{-1}] = -|z^{-1}\rangle\langle 1|.$$

Thus using the equality $\langle 1 | (1 - xz^{-1})^{-1}\xi \rangle = \langle (1 - \bar{x}z)^{-1} | \xi \rangle$ one derives

$$(1 - \mathcal{P})f_x\mathcal{P} = (1 - xz^{-1})^{-1}|z^{-1}\rangle\langle 1|(1 - xz^{-1})^{-1} = |\xi_x\rangle\langle\eta_x|,$$

since the adjoint of multiplication by z^{-1} is multiplication by z . The computation of the norms is straightforward. \square

Next theorem shows that the characteristic values of the off diagonal part $(1 - \mathcal{P})\rho_\infty\mathcal{P}$ decay extremely fast to 0 in a way which is similar to the decay of the prolate eigenvalues.

Theorem 2.3. *The off diagonal part $(1 - \mathcal{P})\rho_\infty\mathcal{P}$ for the function ρ_∞ is the infinitesimal in $L^2(S^1)$*

$$(1 - \mathcal{P})\kappa\mathcal{P} = \sum_{\mathbb{N}} (-1)^{n+1} \frac{2\pi^{2n+\frac{1}{2}}}{(4n+1)\Gamma(n+1)\Gamma(n+\frac{1}{2})} |\xi_n\rangle\langle\eta_n| \quad (7)$$

using the unit vectors $\xi_n := \xi_{x_n}/\|\xi_{x_n}\|$ and $\eta_n := \eta_{x_n}/\|\eta_{x_n}\|$, for $x_n := 1 - \frac{4}{4n+3}$.

Proof. The negative part of the Fourier expansion of κ is expressed (cf. (5)) as a linear combination of the f_{x_n} while the off diagonal part is computed by Lemma 2.2. This gives the equality

$$(1 - \mathcal{P})\kappa\mathcal{P} = \sum_{\mathbb{N}} (-1)^{n+1} \frac{16\pi^{2n+\frac{1}{2}}(4n+3)^{-2}}{\Gamma(n+1)\Gamma(n+\frac{1}{2})} |\xi_{x_n}\rangle\langle\eta_{x_n}|. \quad (8)$$

Furthermore one has

$$(4n+3)^{-2}\|\xi_{x_n}\|\|\eta_{x_n}\| = \frac{1}{32n+8}$$

and thus (7) follows from (8). \square

Remark 2.1. The operator $(1 - \mathcal{P})\rho_\infty\mathcal{P}$ is injective and has dense range. Equivalently one shows that $(1 - \mathcal{P})\kappa\mathcal{P}$ is injective and has dense range. The kernel of $(1 - \mathcal{P})\kappa\mathcal{P}$ is the subspace of the Hardy space of functions $f \in H^2(\mathcal{U})$ on the unit disk whose product by κ belongs to $H^2(\mathcal{U})$. But then f must vanish on all the poles $x(n)$ of κ and since the complex numbers $x(n)$ fulfill the condition $\sum(1 - |x(n)|) = \infty$ this implies $f = 0$ (see [7]). The cokernel of $(1 - \mathcal{P})\kappa\mathcal{P}$ is the kernel of its adjoint $\mathcal{P}\kappa^*(1 - \mathcal{P})$. To control this kernel one uses the following implication

$$\rho_\infty(1 - z)\rho_\infty(z) = 1 \implies \kappa(1/v) = 1/\kappa(v)$$

which shows that the kernel of $\mathcal{P}\kappa^*(1 - \mathcal{P})$ is formed of functions $f \in L^2(S^1)$ such that $g(v) := f(1/v) \in H^2(\mathcal{U})$ and $(\kappa^*f)(1/v) \in H^2(\mathcal{U})$. On the unit circle one has

$$(\kappa^*f)(1/v) = \kappa^*(1/v)f(1/v) = 1/\kappa(1/v)f(1/v) = \kappa(v)g(v)$$

thus the condition $(\kappa^*f)(1/v) \in H^2(\mathcal{U})$ implies $\kappa g \in H^2(\mathcal{U})$ so that, as above, $g = 0$.

The restriction of ρ_∞ to the critical line is a smooth function of modulus 1 which oscillates widely near ∞ , thus the function $\kappa(v) := \rho_\infty\left(\frac{1}{2} + \frac{v+1}{v-1}\right)$ is not continuous on the boundary $S^1 = \partial\mathcal{U}$ of the unit disk. Next proposition shows that by subtracting from $\kappa(v)$ a suitable holomorphic function $h_\infty \in H^\infty(\mathcal{U})$ one obtains a smooth function. This result gives another explanation to Theorem 2.1 since the quantized differential of a smooth function on S^1 is an infinitesimal of infinite order while for an holomorphic function one has $(1 - \mathcal{P})h_\infty\mathcal{P} = 0$.

Proposition 2.4. *The function $\kappa(v) := \rho_\infty\left(\frac{1}{2} + \frac{v+1}{v-1}\right)$ belongs to $C^\infty(S^1) + H^\infty(\mathcal{U})$.*

Proof. We isolate the pole part of ρ_∞ as follows. Consider the infinite sum

$$\pi\rho_\infty(z) := \sum_{\mathbb{N}} \text{Res}_{z=-2n}(\rho_\infty(z)) \frac{1}{z+2n} = \sum_{\mathbb{N}} \frac{\sqrt{\pi}2\pi^{2n}(-1)^n}{\Gamma(n+1)\Gamma\left(n+\frac{1}{2}\right)} \frac{1}{z+2n}. \quad (9)$$

After composition with $\psi(v) = \frac{1}{2} + \frac{v+1}{v-1}$ we obtain

$$\pi\kappa(v) = \sum_{\mathbb{N}} \frac{\sqrt{\pi}2\pi^{2n}(-1)^n}{\Gamma(n+1)\Gamma\left(n+\frac{1}{2}\right)} \frac{2(v-1)}{(4n+3)v-4n+1} \quad (10)$$

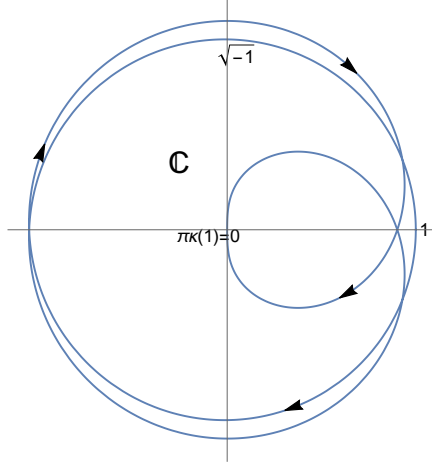


Figure 2: The range $\pi\kappa(S^1) = \pi\omega_\infty(\partial\mathbb{C}_-)$

We show that $\pi\kappa \in C^\infty(S^1)$. When restricted to $S^1 = \{v \in \mathbb{C} \mid |v| = 1\}$ the sum (10) defining $\pi\kappa(v)$ converges since the denominator fulfills $|(4n+3)v - 4n+1| \geq 4, \forall v \in S^1$, for $n > 0$. Moreover when applying m times the operator $v\partial_v$ to the ratio one obtains

$$(v\partial_v)^m \left(\frac{2(v-1)}{(4n+3)v - 4n+1} \right) = \frac{P_m(v, n)}{((4n+3)v - 4n+1)^{m+1}},$$

where $P_1(v, n) = 8v$ is independent of n , $P_2(v, n) = -8v(-1 + 3v + 4n(1+v))$ and $P_m(v, n)$ is a polynomial in v and n of degree $m-1$ in n . This shows that the series of m -th derivatives of the terms in the sum (10) defining $\pi\kappa(v)$ is absolutely convergent and hence that $\pi\kappa(e^{i\theta})$ is a smooth periodic complex valued function. The graphs of its range in the complex plane is shown in Figure 2. We now apply Cauchy formula to compute the negative terms b_{-k} in the Fourier series of $\pi\kappa(e^{i\theta})$. One has as before

$$b_{-k} = \frac{1}{2\pi} \int \pi\kappa(\exp(i\theta)) \exp(ik\theta) d\theta = \frac{1}{2\pi i} \int_{S^1} \pi\kappa(v) v^{k-1} dv.$$

Changing variables and using $d\psi^{-1}(z) = -\frac{8}{(2z-3)^2} dz$ gives

$$\begin{aligned} b_{-k} &= \frac{1}{2\pi i} \int_{\partial\mathbb{C}_-} \pi\rho_\infty(z) \psi^{-1}(z)^{k-1} d\psi^{-1}(z) = \\ &= -\frac{8}{2\pi i} \int_{\partial\mathbb{C}_-} \pi\rho_\infty(z) \left(\frac{2z+1}{2z-3} \right)^{k-1} (2z-3)^{-2} dz. \end{aligned}$$

We use the same contour of integration $C_{R,m}$ as in Figure 1, and the fast convergence of the series (9) to bound $|\pi\omega_\infty(z)|$ and ensure that one can apply Cauchy formula. Since the poles and residues of $\pi\rho_\infty$ are the same as for ρ_∞ , one obtains the equality $b_{-k} = a_{-k}$ for all $k > 0$, where the a_{-k} are given by (5). It follows that the function in $L^2(S^1)$ given by the difference $\kappa - \pi\kappa$ has all its negative Fourier coefficients equal to 0. Thus the

function $k(v) := \kappa(v) - \pi\kappa(v)$ which is by construction holomorphic in \mathcal{U} belongs to the Hardy space $H^2(\mathcal{U})$ (see [7] Theorem 17.12). The non-tangential limits of the values on rS^1 , when $r \rightarrow 1$, are given except at $v = 1$ by $\kappa - \pi\kappa$ and hence are uniformly bounded. Since $k(v)$ is given inside the disk by the Cauchy integral of its boundary values (see [7] Theorem 17.11), it follows that it is bounded and thus belongs to $H^\infty(\mathcal{U})$ which gives the required decomposition $\kappa \in C^\infty(S^1) + H^\infty(\mathcal{U})$. \square

3. The functions ρ_p

We shall now consider the ratio of local L -factors associated to a non-archimedean place, *i.e.* a prime p :

$$\rho_p(z) := \frac{1 - p^{z-1}}{1 - p^{-z}}.$$

We first list some easy properties fulfilled by this function

- Lemma 3.1.** (i) The function $\rho_p(z)$ is periodic with period $\frac{2\pi i}{\log p}$.
(ii) The poles of $\rho_p(z)$ are simple and form the subset $\frac{2\pi i}{\log p}\mathbb{Z} \subset \mathbb{C}_-$.
(iii) The function $\rho_p(z)$ is bounded in the half plane $\{z \in \mathbb{C} \mid \Re(z) \leq -\epsilon < 0\}$.

Proof. (i) is clear.

(ii) follows from periodicity and the expansion at $z = 0$

$$\rho_p(z) = \frac{1 - \frac{1}{p}}{\log(p)} \frac{1}{z} + \frac{p-3}{2p} + \frac{(p-13)\log(p)}{12p} z - \frac{\log^2(p)}{2p} z^2 + O(z^3).$$

(iii) The numerator $1 - p^{z-1}$ of $\rho_p(z)$ is bounded in absolute value by $1 + p^{-1}$. The denominator is larger in absolute value than $p^\epsilon - 1$. \square

Next, we compute the Fourier coefficients of the function $\kappa_p(v) := \rho_p\left(\frac{1}{2} + \frac{v+1}{v-1}\right)$ *i.e.* for $k > 0$

$$a_{-k}^{(p)} := \frac{1}{2\pi i} \int_{S^1} \kappa(v) v^{k-1} dv = \frac{1}{2\pi i} \int_{S^1} \rho_p(z) \psi^{-1}(z)^{k-1} d\psi^{-1}(z).$$

We express these integrals as the sum of residues at the poles $\frac{2\pi i n}{\log p}$, $n \in \mathbb{Z}$

$$a_{-k}^{(p)} = \sum_{\mathbb{Z}} \operatorname{Res}_{z=\frac{2\pi i n}{\log p}} \left(\rho_p(z) \left(\frac{2z+1}{2z-3} \right)^{k-1} \frac{(-8)}{(2z-3)^2} \right).$$

To justify this step we use the same contour as in Section 2 (Figure 1) but we choose $R = \frac{(2m+1)\pi}{\log p}$ which ensures that the restriction of ρ_p to the segment $(\frac{1}{2} + iR, \frac{1}{2} - 2m + iR)$ and its complex conjugate fulfills $|\rho_p(z)| \leq 1$. In fact one has

$$\rho_p \left(x + i \frac{(2m+1)\pi}{\log p} \right) = \frac{p^x + p}{p^{1-x} + p}$$

which is a real, positive increasing function of x equal to 1 for $x = \frac{1}{2}$. Thus one obtains the same control as in Section 2 of the integral on the segment $(\frac{1}{2} + iR, \frac{1}{2} - 2m + iR)$

by $\frac{\pi}{8R}$. To control the integral on the segment $V = (\frac{1}{2} - 2m + iR, \frac{1}{2} - 2m - iR)$, one simply uses the bound

$$\left| \rho_p \left(\frac{1}{2} - 2m + is \right) \right| \leq 2 \left(p^{2m - \frac{1}{2}} - 1 \right)^{-1}$$

which gives, as in Section 2, the same bound for the integral on V . We can thus apply the residue formula. For $z = \frac{2\pi in}{\log p}$ one has

$$\frac{2z + 1}{2z - 3} = \frac{4\pi n - i \log p}{4\pi n + 3i \log p} = x_p(n), \quad \frac{-8}{(2z - 3)^2} = \frac{8 \log^2(p)}{(4\pi n + 3i \log p)^2}.$$

To obtain the residue one multiplies by $\frac{1 - \frac{1}{p}}{\log(p)}$ as shown from periodicity and the expansion at $z = 0$ above. Thus one gets

$$\text{Res}_{z = \frac{2\pi in}{\log p}} \left(\rho_p(z) \left(\frac{2z + 1}{2z - 3} \right)^k \frac{(-8)}{(2z - 3)^2} \right) = \frac{8(1 - \frac{1}{p}) \log p}{(4\pi n + 3i \log p)^2} x_p(n)^k.$$

From this we derive

$$a_{-k}^{(2)} = 8 \left(1 - \frac{1}{p}\right) \log p \sum_{\mathbb{Z}} \frac{1}{(4\pi n + 3i \log p)^2} x_p(n)^{k-1}$$

and with $\kappa_p(v) := \rho_p(\frac{1}{2} + \frac{v+1}{v-1})$ we obtain the equality

$$(1 - \mathcal{P}) \kappa_p \mathcal{P} = 8 \left(1 - \frac{1}{p}\right) \log p \sum_{\mathbb{Z}} \frac{1}{(4\pi n + 3i \log p)^2} (1 - \mathcal{P}) f_{x_p(n)} \mathcal{P}.$$

By Lemma 2.2 this gives

$$(1 - \mathcal{P}) \kappa_p \mathcal{P} = 8 \left(1 - \frac{1}{p}\right) \log p \sum_{\mathbb{Z}} \frac{1}{(4\pi n + 3i \log p)^2} |\xi_{x_p(n)}\rangle \langle \eta_{x_p(n)}|. \quad (11)$$

Lemma 3.2. For $n \in \mathbb{Z}$, let ζ_n be the vector in $\ell^2(\mathbb{N})$ with components

$$\zeta_n(k) := \frac{2^{3/2} (\log p)^{1/2}}{(4\pi n + 3i \log p)} x_p(n)^k. \quad (12)$$

Then for any $n, m \in \mathbb{Z}$, one has

$$\langle \zeta_m | \zeta_n \rangle = \frac{1}{(2i\pi m - 2i\pi n + \log p)}.$$

Proof. For any $n, m \in \mathbb{Z}$, one has

$$\sum_{k=0}^{\infty} x_p(n)^k \overline{(x_p(m))^k} = \frac{(4\pi m - 3i \log p)(4\pi n + 3i \log p)}{8 \log p (2i\pi m - 2i\pi n + \log p)}$$

since the sum is given by the inverse of $1 - x_p(n) \overline{x_p(m)}$. □

Lemma 3.3. *There exists a unique positive operator $B : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ such that*

$$\langle B(\underline{\delta}_m) | B(\underline{\delta}_n) \rangle = \frac{1}{(2i\pi m - 2i\pi n + \log p)} \quad \forall n, m \in \mathbb{Z}.$$

The operator B is bounded, with bounded inverse and absolutely continuous spectrum the interval $[1, \sqrt{p}]$.

Proof. Let us compute the Fourier expansion of the function $s(x) := p^{1-x}$ for $x \in [0, 1)$. One has

$$\int_0^1 s(x) \exp(-2i\pi n x) dx = \frac{1}{\log p + 2i\pi n}.$$

Thus, with S the operator of multiplication by $s(x)$ in the Hilbert space $L^2([0, 1])$, and fixing the orthonormal basis $e_n(x) := \exp(2i\pi n x)$, we get the equality

$$\langle e_m | S(e_n) \rangle = \frac{1}{(2i\pi m - 2i\pi n + \log p)} \quad \forall n, m \in \mathbb{Z}.$$

One then lets B be the conjugate under the isomorphism $\ell^2(\mathbb{Z}) \rightarrow L^2([0, 1])$ of the positive square root of S given by the multiplication by the function $s(x)^{\frac{1}{2}} := p^{\frac{1-x}{2}}$. \square

Lemma 3.4. *Consider the linear map $V : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{N})$, $V(\underline{\delta}_n) = \zeta_n$. Then the map $VB^{-1} = U$ is an isometry of $\ell^2(\mathbb{Z})$ with a closed infinite dimensional subspace of $\ell^2(\mathbb{N})$.*

Proof. One has the equality of inner products

$$\langle V(\underline{\delta}_m) | V(\underline{\delta}_n) \rangle = \langle \zeta_m | \zeta_n \rangle = \frac{1}{(2i\pi m - 2i\pi n + \log p)} = \langle B(\underline{\delta}_m) | B(\underline{\delta}_n) \rangle$$

and since B is invertible this shows that VB^{-1} is an isometry. \square

We now let $U_{\pm} : \ell^2(\mathbb{N}) \rightarrow L^2(S^1)$ be the unitary isomorphisms with *resp.* the range of \mathcal{P} and $1 - \mathcal{P}$ given by $U_+(\underline{\delta}_n) = z^n$ and $U_-(\underline{\delta}_n) = z^{-n-1}$. Then with the above notations one obtains the following

Lemma 3.5. *Consider the involution $I : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ $I(\underline{\delta}_n) = \underline{\delta}_{-n}$. Then one has*

$$(1 - \mathcal{P})\kappa_p \mathcal{P} = \frac{1-p}{p} U_- V I V^* U_+^* = \frac{1-p}{p} U_- U B I B U^* U_+^*. \quad (13)$$

Proof. One has

$$\frac{2^{3/2}(\log p)^{1/2}}{(4\pi n + 3i \log p)} \xi_{x_p(n)} = U_-(\zeta_n)$$

and since $\overline{x_p(n)} = x_p(-n)$

$$-\frac{2^{3/2}(\log p)^{1/2}}{(4\pi n - 3i \log p)} \eta_{x_p(n)} = U_+(\zeta_{-n})$$

which gives

$$\frac{-8 \log p}{(4\pi n + 3i \log p)^2} |\xi_{x_p(n)} \rangle \langle \eta_{x_p(n)}| = |U_-(\zeta_n) \rangle \langle U_+(\zeta_{-n})|$$

By (11) we thus get, since $V(\underline{\delta}_n) = \zeta_n$

$$\begin{aligned} (1 - \mathcal{P})\kappa_p\mathcal{P} &= 8\left(1 - \frac{1}{p}\right) \log p \sum_{\mathbb{Z}} \frac{1}{(4\pi n + 3i \log p)^2} |\xi_{x_p(n)}\rangle \langle \eta_{x_p(n)}| = \\ &= \frac{1-p}{p} \sum_{\mathbb{Z}} |U_-(\zeta_n)\rangle \langle U_+(\zeta_{-n})| = \frac{1-p}{p} \sum_{\mathbb{Z}} |U_-(V(\underline{\delta}_n))\rangle \langle U_+(V(\underline{\delta}_{-n}))|. \end{aligned}$$

Then (13) follows using the equalities

$$I = \sum_{\mathbb{Z}} |\underline{\delta}_n\rangle \langle \underline{\delta}_{-n}|, \quad U_- V I V^* U_+^* = \sum_{\mathbb{Z}} |U_-(V(\underline{\delta}_n))\rangle \langle U_+(V(\underline{\delta}_{-n}))|$$

□

In particular (13) shows that $(1 - \mathcal{P})\kappa_p\mathcal{P}$ has the same strength as *BIB* and from this we derive the following

Fact 3.6. *The function ρ_p is not quasi-inner.*

Given a sequence of complex numbers α_n in the open unit disk \mathcal{U} such that $\sum_n (1 - |\alpha_n|) < \infty$, the associated Blaschke product is the following function of $v \in \mathcal{U}$

$$\prod_n \frac{\alpha_n - v}{1 - \bar{\alpha}_n v} \times \frac{|\alpha_n|}{\alpha_n}.$$

Note that the linear map V is not surjective since the complex numbers $x_p(n)$ fulfill the Blaschke product condition $\sum (1 - |x_p(n)|) < \infty$ so that the orthogonal of the range of U_+V contains all multiples of the corresponding Blaschke product B_p over all $x_p(n)$. This suggests that there should be a direct relation between κ_p and B_p . By construction the product of κ_p and B_p is holomorphic in \mathcal{U} and has modulus equal to 1 on the boundary. We show that this product is an inner function which we explicitly determine in Proposition 3.7 below. We also determine the kernel and cokernel of $(1 - \mathcal{P})\kappa_p\mathcal{P}$.

Proposition 3.7. *Let $B_p(v)$ be the Blaschke product associated to the sequence $x_p(n) \in \mathcal{U}$, $n \in \mathbb{Z}$.*

- (i) *The product $\iota(v) = \kappa_p(v)B_p(v)$ is the inner function $-p^{\frac{v+1}{v-1}}$.*
- (ii) *The kernel of $(1 - \mathcal{P})\kappa_p\mathcal{P}$ is the shift invariant subspace of $H^2(\mathcal{U})$ of multiples of B_p .*
- (iii) *The cokernel of $(1 - \mathcal{P})\kappa_p\mathcal{P}$ is the image of its kernel by the unitary involution $J : L^2(S^1) \rightarrow L^2(S^1)$, $Jf(z) := z^{-1}f(z^{-1})$.*

Proof. One has

$$x_p(n) = \frac{4\pi n - i \log p}{4\pi n + 3i \log p}, \quad |x_p(n)|^2 = \frac{\log^2(p) + 16\pi^2 n^2}{9 \log^2(p) + 16\pi^2 n^2} = 1 - \frac{\log^2(p)}{2\pi^2 n^2} + O(n^{-3})$$

which ensures the condition $\sum (1 - |x_p(n)|) < \infty$.

(i) For any $Z \in \mathbb{C}$ let us consider the Euler sine product formula

$$e^Z - e^{-Z} = 2Z \prod_{n=1}^{\infty} \left(1 + \frac{Z^2}{\pi^2 n^2}\right).$$

We apply this formula for $Z = \frac{1-z}{2} \log p$ and $Z = \frac{z}{2} \log p$ to obtain

$$p^{\frac{1-z}{2}} - p^{-\frac{1-z}{2}} = (1-z) \log p \prod_{n=1}^{\infty} \left(1 + \frac{\log^2(p)(1-z)^2}{4\pi^2 n^2} \right)$$

$$p^{\frac{z}{2}} - p^{-\frac{z}{2}} = z \log p \prod_{n=1}^{\infty} \left(1 + \frac{\log^2(p)z^2}{4\pi^2 n^2} \right)$$

and

$$\frac{p^{\frac{1-z}{2}} - p^{-\frac{1-z}{2}}}{p^{\frac{z}{2}} - p^{-\frac{z}{2}}} = \frac{1-z}{z} \prod_{n=1}^{\infty} \frac{-\beta_n^2 + (1-z)^2}{-\beta_n^2 + z^2}$$

where $\beta_n = \frac{2\pi in}{\log p}$ are the poles of ρ_p . Since $\bar{\beta}_n = -\beta_n = \beta_{-n}$, one has, for $n > 0$

$$\frac{(\bar{\beta}_n + z - 1)(\bar{\beta}_{-n} + z - 1)}{(\beta_n - z)(\beta_{-n} - z)} = \frac{-\beta_n^2 + (1-z)^2}{-\beta_n^2 + z^2}$$

while $\beta_0 = 0$ so that

$$\frac{(\bar{\beta}_0 + z - 1)}{(\beta_0 - z)} = \frac{1-z}{z}.$$

Conjugating the general term of the Blaschke product $\frac{\alpha-v}{1-\bar{\alpha}v} \times \frac{|\alpha|}{\alpha}$ by the transformation $\psi^{-1} : \mathbb{C}_- \rightarrow U$, $\psi^{-1}(z) = \frac{2z+1}{2z-3}$ gives, with $\beta = \psi(\alpha)$ the term

$$\frac{(\beta - z)}{(\bar{\beta} + z - 1)} \times \chi, \quad \chi = \left| \frac{2\beta + 1}{3 - 2\beta} \right| \frac{(2\bar{\beta} - 3)}{(2\beta + 1)}.$$

This shows that, up to the multiplication by a complex number χ of modulus 1, one has

$$B_p(\psi^{-1}(z))^{-1} = \chi \frac{p^{\frac{1-z}{2}} - p^{-\frac{1-z}{2}}}{p^{\frac{z}{2}} - p^{-\frac{z}{2}}}.$$

Moreover the implication

$$\rho_p(z) := \frac{1 - p^{z-1}}{1 - p^{-z}} \implies p^{-z+\frac{1}{2}} \rho_p(z) = \frac{p^{\frac{1-z}{2}} - p^{-\frac{1-z}{2}}}{p^{\frac{z}{2}} - p^{-\frac{z}{2}}}$$

gives $\rho_p(z) = \bar{\chi} B_p(\psi^{-1}(z))^{-1} p^{z-\frac{1}{2}}$. To determine χ one notes that the Blaschke product for the disk is normalized so that its value at 0 is positive, and thus the value of $B_p(\psi^{-1}(z))$ is positive at $z = \psi(0) = -\frac{1}{2}$. One has $\rho_p(-\frac{1}{2}) = \frac{1-p^{-\frac{3}{2}}}{1-p^{\frac{1}{2}}} < 0$, which shows that $\chi = -1$.

(ii) The kernel of $(1 - \mathcal{P})\kappa_p \mathcal{P}$ is the subspace of $H^2(\mathcal{U})$ of functions $f(z)$ such that $\kappa_p f \in H^2(\mathcal{U})$. This condition implies $f(x_p(n)) = 0$ for all $n \in \mathbb{Z}$ and hence by [7] (Theorem 17.9) that $f/B_p \in H^2(\mathcal{U})$. Conversely, if $f = B_p h$ for some $h \in H^2(\mathcal{U})$ then by (i) the product $\kappa_p f$ is equal to $h \times \iota$ and hence belongs to $H^2(\mathcal{U})$.

(iii) One has $J(H^2(\mathcal{U})) = (1 - \mathcal{P})L^2(S^1)$. The cokernel of $(1 - \mathcal{P})\kappa_p \mathcal{P}$ is the kernel

of the adjoint $\mathcal{P}\kappa_p^*(1 - \mathcal{P})$ and, as in Remark 2.1, the identity $\rho_p(1 - z)\rho_p(z) = 1$ implies $\kappa_p(1/v) = 1/\kappa_p(v)$ for all $v \in S^1$. Thus for $v \in S^1$, since $|\kappa_p(v)| = 1$ one has $\kappa_p^*(v) = 1/\kappa_p(v) = \kappa_p(1/v)$. This entails

$$J\kappa_p J^{-1} = \kappa_p^*, \quad \mathcal{P}\kappa_p^*(1 - \mathcal{P}) = J(1 - \mathcal{P})\kappa_p \mathcal{P}J^{-1}$$

which gives the required equality. \square

We end this section by writing a decomposition of ρ_p (and κ_p) which is the analogue of the decomposition $\rho_\infty = \pi\rho_\infty + (\rho_\infty - \pi\omega_\infty)$ of Proposition 2.4, isolating the pole part. One has

$$\rho_p(z) = (p-1)/(2p) \coth\left(\frac{1}{2}z \log(p)\right) - p^{z-1} + \frac{p-1}{2p}. \quad (14)$$

The first term in $\coth\left(\frac{1}{2}z \log(p)\right)$ is the sum of the simple pole parts while the last terms $-p^{z-1} + \frac{p-1}{2p}$ define a bounded holomorphic function belonging to $H^\infty(\mathbb{C}_-)$. Since the function ρ_p is not quasi-inner we cannot expect that the first term composed with the map ψ would define a continuous function on S^1 .

4. The product $\rho_\infty \prod \rho_p$ is quasi-inner

The main result of this section is the following

Theorem 4.1. *The product $u = \rho_\infty \prod \rho_p$ of $m+1$ ratios of local L -factors $\rho_v(z) = \gamma_v(z)/\gamma_v(1-z)$ over a finite set of places of \mathbb{Q} containing the archimedean place is a quasi-inner function relative to $\mathbb{C}_- = \{z \in \mathbb{C} \mid \Re(z) \leq \frac{1}{2}\}$.*

This result will be proven in several steps, first considering a single prime p and showing in §4.2 that the function $\rho_\infty \times \rho_p$ is quasi-inner. Then we shall use Gauss multiplication formula of the Gamma function and introduce in §4.3 for any integer $m > 0$, a factorization $\rho_\infty = \prod_0^{m-1} \phi_{m,k}$ as a product of m quasi-inner functions having properties similar to ρ_∞ . Finally, in §4.4 we will extend the results of §4.2 and show that the product $\phi_{m,k}\rho_p$ is a quasi-inner function and then conclude the proof of Theorem 4.1.

4.1. $\Gamma(z)$ on vertical lines

We recall the second Binet formula for the function $\log(\Gamma(z))$ which is well defined for $\Re(z) > 0$

$$\log(\Gamma(z)) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi) + \int_0^\infty \left(\frac{1}{\exp(t) - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-tz} \frac{dt}{t}$$

This formula shows that on the vertical line $L_a := \{z \in \mathbb{C} \mid \Re(z) = a\}$ for $a > 0$ one can apply Stirling's formula giving the asymptotic behavior

$$|\Gamma(z)| \sim \sqrt{2\pi} \exp(\sigma(z)), \quad \sigma(z) := \Re\left(\left(z - \frac{1}{2}\right) \log z - z\right).$$

Note that the validity of this formula for $a = 1$ together with $\Gamma(z) = \frac{1}{z}\Gamma(z+1)$ imply that Stirling's formula still applies for $a = 0$.

Lemma 4.2. *On the vertical line $L_a := \{z \mid \Re(z) = a\}$ for $a \geq 0$ one has*

$$|\Gamma(a + it)| \sim \exp(\sigma_a(t)), \quad \sigma_a(t) = \left(a - \frac{1}{2}\right) \log |t| - \frac{\pi}{2} |t| + O(1). \quad (15)$$

Proof. We use Stirling's formula and first consider the terms of the form

$$t_a(z) := \Re \left(\left(z + a - \frac{1}{2} \right) \log(z + a) \right)$$

where $a \geq 0$, $z \in i\mathbb{R}$ and where \log is the branch which is real when the argument is real positive. Thus one has

$$\log(z + a) = \frac{1}{2} \log(|z|^2 + a^2) + i \operatorname{Arg}(z + a).$$

Moreover the equality

$$\frac{1}{2} \log(|z|^2 + a^2) = \log |z| + O(|z|^{-2})$$

gives

$$t_a(z) = \left(a - \frac{1}{2}\right) \log |z| + O(|z|^{-2}) + \Re((z + a) i \operatorname{Arg}(z + a)).$$

One also has

$$\Re((z + a) i \operatorname{Arg}(z + a)) = -\frac{\pi}{2} |z| + O(1)$$

since for $z = \pm it$, $t \rightarrow +\infty$: $\operatorname{Arg}(z + a) = \pm \frac{\pi}{2} + O(|z|^{-1})$. Thus (15) follows. \square

4.2. The product $\rho_\infty \times \rho_p$

Next lemma refines Lemma 4.2 providing a formula for $|\rho_\infty(it)|$.

Lemma 4.3. *For any $t \in \mathbb{R}$ one has*

$$|\rho_\infty(it)| = |t|^{-\frac{1}{2}} (2\pi \coth(\pi|t|/2))^{\frac{1}{2}}. \quad (16)$$

Proof. We use the equality $\Gamma(\bar{z}) = \overline{\Gamma(z)}$ to obtain

$$|\rho_\infty(it)|^2 = \pi \Gamma(it/2) \Gamma(-it/2) / (\Gamma((1-it)/2) \Gamma((1+it)/2)).$$

Then the formula of complements together with $(-it/2) \Gamma(-it/2) = \Gamma(1 - it/2)$, gives

$$|\rho_\infty(it)|^2 = \pi \frac{1}{(-it/2)} \frac{\sin(\pi((1+it)/2))}{\sin(\pi(it/2))} = (2\pi/t) \coth(\pi t/2).$$

Note that this is in agreement with the presence of the simple pole at $t = 0$ with residue equal to 2. \square

Theorem 4.4. (i) The function $\rho_\infty(z)\rho_p(z)$ is quasi-inner relative to the upper half plane. The off diagonal part is an infinitesimal of order $\frac{1}{2}$.

(ii) The off diagonal part for the function $\rho_\infty(z)\rho_p(z)$ is the infinitesimal in $L^2(S^1)$ which is the sum of three terms $(1 - \mathcal{P})\kappa\kappa_p\mathcal{P} = \mathcal{E}_\infty + \mathcal{E}_p + \mathcal{E}_0$ where, with the notations of (7) and (13),

$$\mathcal{E}_\infty = \sum_{n=1}^{\infty} (-1)^n \frac{2\pi^{2n+\frac{1}{2}}(1-p^{-(2n+1)})}{(4n+1)(p^{2n}-1)\Gamma(n+1)\Gamma(n+\frac{1}{2})} |\xi_n\rangle\langle\eta_n| \quad (17)$$

$$\mathcal{E}_p = \frac{1-p}{p} U_- V D I V^* U_+^*, \quad D\delta_0 = 0, \quad D\delta_n = \omega_\infty \left(\frac{2\pi in}{\log p} \right) \delta_n, \quad \forall n \neq 0. \quad (18)$$

Moreover \mathcal{E}_0 is an operator of finite rank.

Proof. We compute the Fourier coefficients of the function $\kappa_{p,\infty}(v) := (\rho_\infty\rho_p)\left(\frac{1}{2} + \frac{v+1}{v-1}\right)$:

$$a_{-k}^{(p,\infty)} = \frac{1}{2\pi i} \int_{S^1} \kappa_{p,\infty}(v) v^{k-1} dv = \frac{1}{2\pi i} \int_{S^1} \rho_\infty(z)\rho_p(z)\psi^{-1}(z)^{k-1} d\psi^{-1}(z)$$

We express these coefficients as the sum of residues on the set \mathcal{R} of poles:

$$a_{-k}^{(p,\infty)} = \sum_{\mathcal{R}} \text{Res} \left(\rho_\infty(z)\rho_p(z) \left(\frac{2z+1}{2z-3} \right)^{k-1} \frac{(-8)}{(2z-3)^2} \right).$$

To justify this step we use the same contour as in Section 2 (Figure 1) and the same choice $R = \frac{(2m+1)\pi}{\log p}$ as in the proof of Lemma 3.1, which ensures that the restriction of ρ_p to the segment $(\frac{1}{2} + iR, \frac{1}{2} - 2m + iR)$ and its complex conjugate fulfill $|\rho_p(z)| \leq 1$. Thus one gets the same control as in Section 2 of the integral on the segment $(\frac{1}{2} + iR, \frac{1}{2} - 2m + iR)$ by $\frac{\pi}{8R}$. The poles are of three kinds. We have the non-zero poles of ρ_∞ , the non-zero poles of ρ_p , and the double pole at $z = 0$. The residues are, for the simple poles, multiplied by the value of the other factor at the point. For the non-zero poles of ρ_∞ this multiplies the residue by $\rho_p(-2n) = \frac{1-p^{-(2n+1)}}{1-p^{2n}}$ which does not alter the strong convergence of (7) in Theorem 2.3 and gives (17). We let \mathcal{E}_p be the contribution of the non-zero poles of ρ_p . One multiplies the residue by $\rho_\infty\left(\frac{2\pi in}{\log p}\right)$, and by (11)

$$\begin{aligned} \mathcal{E}_p &= 8\left(1 - \frac{1}{p}\right) \log p \sum_{\mathbb{Z} \setminus \{0\}} \frac{\rho_\infty\left(\frac{2\pi in}{\log p}\right)}{(4\pi n + 3i \log p)^2} |\xi_{x_p(n)}\rangle\langle\eta_{x_p(n)}| = \\ &= \frac{1-p}{p} \sum_{\mathbb{Z} \setminus \{0\}} \rho_\infty\left(\frac{2\pi in}{\log p}\right) |U_-(\zeta_n)\rangle\langle U_+(\zeta_{-n})| = \\ &= \frac{1-p}{p} \sum_{\mathbb{Z} \setminus \{0\}} \rho_\infty\left(\frac{2\pi in}{\log p}\right) |U_-(V(\delta_n))\rangle\langle U_+(V(\delta_{-n}))| \end{aligned}$$

since $V(\underline{\delta}_n) = \zeta_n$. Thus (18) follows using

$$DI = \sum_{\mathbb{Z} \setminus \{0\}} \rho_\infty\left(\frac{2\pi in}{\log p}\right) |\underline{\delta}_n\rangle \langle \underline{\delta}_{-n}|,$$

$$U_- V D I V^* U_+^* = \sum_{\mathbb{Z} \setminus \{0\}} \rho_\infty\left(\frac{2\pi in}{\log p}\right) |U_-(V(\underline{\delta}_n))\rangle \langle U_+(V(\underline{\delta}_{-n}))|$$

as in Lemma 3.5. By (16) this shows that the operator \mathcal{E}_p associated to this contribution of poles is an infinitesimal of order $\frac{1}{2}$. The contribution of the double pole at $z = 0$ gives an operator of finite rank. The expansion at $z = 0$ gives a double pole of the form $\frac{16(-1)^k 3^{-k-1} (p-1)}{p \log(p)} z^{-2}$ and a simple pole with residue

$$\frac{8(-1)^k 3^{-k-2}}{p \log(p)} \left(3(p-3) \log(p) - (p-1) \left(-16k + 3\gamma + 8 + 6 \log(\pi) - 3 \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \right) \right) \right).$$

The dependence in k is of the form $\alpha x^{k-1} + (k-1)\beta x^{k-1}$, where $x = -\frac{1}{3}$, $\beta = -\frac{128(p-1)}{27p \log(p)}$ and

$$\alpha = \frac{8}{27p \log(p)} \left((p-1) \left(3\gamma - 8 + 6 \log(\pi) - 3 \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \right) \right) - 3(p-3) \log(p) \right).$$

The contribution of the terms in αx^{k-1} gives $\alpha(1 - \mathcal{P})f_x \mathcal{P}$, where $f_x(z) := z^{-1}(1 - xz^{-1})^{-1}$ *i.e.* the rank one operator of (6). The contribution of the terms $\beta(k-1)x^{k-1}$ gives $x\beta(1 - \mathcal{P})f_x^2 \mathcal{P}$ since $\sum_{k=1}^{\infty} (k-1)x^{k-1}z^{-k} = x f_x^2$. Thus this contribution gives an operator of rank 2. \square

A striking property of the decomposition $(1 - \mathcal{P})\kappa_p \mathcal{P} = \mathcal{E}_\infty + \mathcal{E}_p + \mathcal{E}_0$ is that, except for the contribution of the double pole at 0, it splits as a sum of terms corresponding to the two factors κ_p and that the contribution of each pole is simply multiplied by the value of the other term at that point. This fact follows from Cauchy formula, but one may wonder about its compatibility with the formula for the off diagonal term in a product

$$\begin{aligned} \begin{pmatrix} (\kappa_p)_{1,1} & (\kappa_p)_{1,2} \\ (\kappa_p)_{2,1} & (\kappa_p)_{2,2} \end{pmatrix} \begin{pmatrix} \kappa_{1,1} & \kappa_{1,2} \\ \kappa_{2,1} & \kappa_{2,2} \end{pmatrix} &= \\ &= \begin{pmatrix} (\kappa_p)_{1,1} \kappa_{1,1} + (\kappa_p)_{1,2} \kappa_{2,1} & (\kappa_p)_{1,1} \kappa_{1,2} + (\kappa_p)_{1,2} \kappa_{2,2} \\ (\kappa_p)_{2,1} \kappa_{1,1} + (\kappa_p)_{2,2} \kappa_{2,1} & (\kappa_p)_{2,1} \kappa_{1,2} + (\kappa_p)_{2,2} \kappa_{2,2} \end{pmatrix} \end{aligned}$$

which displays the term $(1 - \mathcal{P})\kappa_p \mathcal{P} = (\kappa_p)_{2,1} \kappa_{1,1} + (\kappa_p)_{2,2} \kappa_{2,1}$. To understand algebraically the splitting as a sum we work with a general product of two terms each being a sum of an element of $H^\infty(\mathcal{U})$ and finitely many scalar multiples of f_{x_j} , $x_j \in \mathcal{U}$, *i.e.*

$$k(v) = h(v) + \sum a_j f_{x_j}(v), \quad h \in H^\infty(\mathcal{U}), \quad x_j \in \mathcal{U}. \quad (19)$$

For such k one derives from (6)

$$(1 - \mathcal{P})k\mathcal{P} = \sum a_j |\xi_{x_j}\rangle \langle \eta_{x_j}|, \quad \xi_x = z^{-1}(1 - xz^{-1})^{-1}, \quad \eta_x = (1 - \bar{x}z)^{-1}.$$

When one considers the product $k = k_1 k_2$ of two functions of the form (19) and one assumes that the set of poles of k_j in \mathcal{U} are disjoint, then the poles of the product are simply the union of the poles of each term. Thus one obtains a decomposition of k of the form (19) where the pole part is simply the sum of the pole parts of the k_j multiplied by the value of the other factor at the pole. This gives for $(1 - \mathcal{P})k\mathcal{P}$ a formula as a sum of the terms involved in $(1 - \mathcal{P})k_j\mathcal{P}$ multiplied by the value of the other factor at the pole. At the algebraic level the key equality is that

$$\langle \eta_x | \xi \rangle = \xi(x), \quad \forall \xi \in H^2(\mathcal{U}) \implies \langle \eta_x | \circ h \circ \mathcal{P} = h(x) \langle \eta_x | \circ \mathcal{P}, \quad \forall h \in H^\infty(\mathcal{U})$$

while for $x \neq y$ one has

$$\frac{1}{(z-x)(z-y)} = \frac{1}{(y-x)(z-y)} + \frac{1}{(x-y)(z-x)} \implies f_x f_y = f_x(y) f_y + f_y(x) f_x.$$

By (10) and (14) the functions κ and κ_p are of the form (19) but the sums involved are infinite.

Next proposition provides an independent reason why the function $\kappa(v)\kappa_p(v)$ is quasi-inner.

Proposition 4.5. *The function $\kappa(v)\kappa_p(v)$ belongs to $C(S^1) + H^\infty(\mathcal{U})$.*

Proof. The delicate part of the pole contribution for the function $k(v) = \kappa(v)\kappa_p(v)$ comes from the poles of κ_p . When considered in $\partial\mathbb{C}_-$ this contribution takes the form

$$\phi(z) := \frac{p-1}{p \log p} \sum_{\mathbb{Z} \setminus \{0\}} \rho_\infty\left(\frac{2\pi i n}{\log p}\right) \frac{1}{z - \frac{2\pi i n}{\log p}}. \quad (20)$$

We consider the restriction of ϕ to the critical line $\partial\mathbb{C}_-$ and extend it by the value $\phi(\infty) := 0$ as a function on the projective line $\mathbb{P}^1(\mathbb{R})$. By (16) one has $|\rho_\infty(\frac{2\pi i n}{\log p})| = O(n^{-1/2})$ and this suffices to show that the series defining $\phi(z)$ is absolutely convergent. The same holds for the series of derivatives giving $\partial_z^k \phi(z)$, so that ϕ is smooth on $\partial\mathbb{C}_-$. Let us show that for $s \rightarrow \pm\infty$ one has

$$|\phi(\frac{1}{2} + is)| = O(|s|^{-\frac{1}{2}} \log |s|).$$

Since $|\rho_\infty(\frac{2\pi i n}{\log p})| = O(n^{-1/2})$ one has, for some $C < \infty$, and with $b = \frac{2\pi}{\log p}$

$$|\phi(\frac{1}{2} + is)| \leq C \sum_{\mathbb{Z} \setminus \{0\}} |n|^{-\frac{1}{2}} \left| \frac{1}{2} + is - \frac{2\pi i n}{\log p} \right|^{-1} \leq 3C \sum_{\mathbb{Z} \setminus \{0\}} |n|^{-\frac{1}{2}} (1 + |s - nb|)^{-1}.$$

We assume $s > 0$, then the sum over negative n is $O(s^{-\frac{1}{2}})$ since

$$\sum_{n < 0} |n|^{-\frac{1}{2}} (1 + |s - bn|)^{-1} \leq \int_{-\infty}^0 |u|^{-\frac{1}{2}} (1 + |s - bu|)^{-1} du = \frac{\pi}{\sqrt{b(1+s)}}.$$

To estimate the sum over positive n one replaces it by the integral which introduces an error in $O(s^{-\frac{1}{2}})$ and one uses the equalities

$$\int_0^\infty |u|^{-\frac{1}{2}} (1 + |s - bu|)^{-1} du = (sb)^{-1/2} \int_0^\infty y^{-1/2} (1/s + |1 - y|)^{-1} dy$$

and

$$\int_0^\infty y^{-1/2}(1/s + |1 - y|)^{-1} dy = 2 \log s + O(1).$$

Thus we have shown that the function ϕ is continuous on the projective line $\mathbb{P}^1(\mathbb{R})$ and hence that $\sigma = \phi \circ \psi \in C(S^1)$. In fact this function is smooth except at $v = 1$ where it is continuous but not differentiable and satisfies $\sigma(1) = 0$, $\sigma(e^{i\theta}) = O(|\theta|^{1/2} |\log |\theta||)$. The contribution of the non-zero poles of ρ_∞ is of the form

$$\begin{aligned} \phi_1(z) &:= \sum_{n>0} \rho_p(-2n) \frac{\sqrt{\pi} 2\pi^{2n} (-1)^n}{\Gamma(n+1)\Gamma(n+\frac{1}{2})} \frac{1}{z+2n} = \\ &= \sum_1^\infty \frac{(-1)^n 2\pi^{2n+\frac{1}{2}} (1-p^{-(2n+1)})}{(4n+1)(p^{2n}-1)\Gamma(n+1)\Gamma(n+\frac{1}{2})} \frac{1}{z+2n}. \end{aligned}$$

As in Proposition 2.4 one proves that $\phi_1 \circ \psi \in C^\infty(S^1)$. The contribution of the double pole at $z = 0$ is of the form

$$\phi_2(z) := \frac{2(p-1)}{pz^2 \log(p)} + \frac{(p-3) \log(p) - (p-1) \left(\gamma + 2 \log(\pi) - \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \right) \right)}{pz \log(p)}$$

thus $\phi_2 \circ \psi \in C^\infty(S^1)$. The sum of the polar parts $\pi\rho^{p,\infty} := \phi + \phi_1 + \phi_2$ is in $C(S^1)$ after composition with ψ . The absolute value of $\phi_1(z)$ (and of $\phi_2(z)$) on the contour $C_{R,m}$ of Section 2 (Figure 1) with $R = \frac{(2m+1)\pi}{\log p}$ is bounded independently of m since $|z+2n| \geq \frac{1}{2}$, $\forall n \geq 1$ for $z \in C_{R,m}$. To obtain a uniform bound for $|\phi(z)|$ on $C_{R,m}$ one uses Holder's inequality

$$\sum x_n y_n \leq \left(\sum x_n^p \right)^{\frac{1}{p}} \left(\sum y_n^q \right)^{\frac{1}{q}}, \quad 1 < p < \infty, \quad 1 < q < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

applied to $x_n = |\rho_\infty(\frac{2\pi in}{\log p})|$ and $y_n = |\frac{1}{z - \frac{2\pi in}{\log p}}|$ with $p > 2$, $q > 1$ (for instance $p = 3$ and $q = \frac{3}{2}$) while one has a uniform bound independent of m of the form

$$\sum_n \left| \frac{1}{z - \frac{2\pi in}{\log p}} \right|^q \leq C \quad \forall z \in C_{R,m}.$$

This ensures, due to the factor $\frac{(-8)}{(2z-3)^2}$, that one can apply Cauchy formula as earlier on, to obtain the Fourier coefficients with negative index for $(\pi\rho^{p,\infty}) \circ \psi$. It follows from the equality of the polar parts that they are the same as for $\kappa\kappa_\infty$ and hence, as in the proof of Proposition 2.4, one has $\kappa\kappa_\infty - (\pi\rho^{p,\infty}) \circ \psi \in H^\infty(\mathcal{U})$. \square

4.3. Factorization of ρ_∞

In the general case of a finite product $\rho_\infty(z) \prod \rho_p(s)$, each ρ_p will contribute to the Cauchy formula with its poles $\frac{2\pi in}{\log p}$ and the residues will be then multiplied by $\rho_\infty(\frac{2\pi in}{\log p}) \prod_{q \neq p} \rho_q(\frac{2\pi in}{\log p})$. This creates a problem when $\frac{2\pi in}{\log p}$ is close to a pole of some ρ_q . To handle this difficulty we construct a factorization of ρ_∞ as a product of quasi inner functions which will then be distributed among the factors $\rho_p(s)$.

Lemma 4.6. *Let $m \in \mathbb{N}$ and for any integer $k \in \{0, m-1\}$ let*

$$\gamma_{m,k}(z) := \Gamma\left(\frac{z}{2m} + \frac{k}{m}\right), \quad \phi_{m,k}(z) := \gamma_{m,k}(z)/\gamma_{m,k}(1-z). \quad (21)$$

- (i) *One has: $|\phi_{m,k}(it)| = O(|t|^{-\frac{1}{2m}})$ when $|t| \rightarrow \infty$.*
(ii) *One has: $|\phi_{m,k}(\frac{1}{2} + is)| = 1$ for $s \in \mathbb{R}$, and the following factorization formula holds*

$$\prod_{k=0}^{m-1} \phi_{m,k}(z) = \left(\frac{m}{\pi}\right)^{\frac{1}{2}-z} \rho_{\infty}(z) \quad (22)$$

Proof. (i) We apply (15), for $a = \frac{k}{m}$, and get

$$|\gamma_{m,k}(it)| = \left| \Gamma\left(a + \frac{it}{2m}\right) \right| \sim \exp\left(\sigma_a\left(\frac{t}{2m}\right)\right).$$

In a similar way, using $b = \frac{1}{2m} + \frac{k}{m}$ we obtain

$$|\gamma_{m,k}(1-it)| = \left| \Gamma\left(b - \frac{it}{2m}\right) \right| \sim \exp\left(\sigma_b\left(\frac{-t}{2m}\right)\right) \sim \exp\left(\sigma_b\left(\frac{t}{2m}\right)\right).$$

From (15) one has

$$\sigma_a\left(\frac{t}{2m}\right) = \left(a - \frac{1}{2}\right) \log\left|\frac{t}{2m}\right| - \frac{\pi}{2} \left|\frac{t}{2m}\right| + O(1)$$

and

$$\sigma_a\left(\frac{t}{2m}\right) - \sigma_b\left(\frac{t}{2m}\right) = -\frac{1}{2m} \log\left|\frac{t}{2m}\right| + O(1)$$

which thus gives $|\phi_{m,k}(it)| = O(|t|^{-\frac{1}{2m}})$ when $|t| \rightarrow \infty$.

(ii) We use Gauss multiplication theorem in the form

$$\prod_{k=0}^{m-1} \Gamma\left(z + \frac{k}{m}\right) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2}-mz} \Gamma(mz).$$

Replacing z by $\frac{z}{2m}$ we get

$$\prod_0^{m-1} \gamma_{m,k}(z) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1-z}{2}} \Gamma\left(\frac{z}{2}\right)$$

and taking the ratio with the value at $1-z$ one obtains

$$\prod_0^{m-1} \phi_{m,k}(z) = \frac{m^{\frac{1-z}{2}} \Gamma\left(\frac{z}{2}\right)}{m^{\frac{z}{2}} \Gamma\left(\frac{1-z}{2}\right)} = \left(\frac{m}{\pi}\right)^{\frac{1}{2}-z} \rho_{\infty}(z).$$

Note finally that the functions $\phi_{m,k}$ fulfill the reality condition $\phi_{m,k}(\bar{z}) = \overline{\phi_{m,k}(z)}$ while by construction one has

$$\phi_{m,k}(1-z)\phi_{m,k}(z) = 1$$

which shows that $|\phi_{m,k}(\frac{1}{2} + is)| = 1$ for $s \in \mathbb{R}$. □

Lemma 4.7. *Let $m \in \mathbb{N}$ and for any $k \in \{0, m-1\}$ let*

$$\rho_\infty^{(m,k)}(z) := \left(\frac{\pi}{m}\right)^{\frac{1}{2m} - \frac{z}{m}} \phi_{m,k}(z) \quad (23)$$

Each $\rho_\infty^{(m,k)}$ is a quasi inner function (relative to \mathbb{C}_-) and their product is equal to $\rho_\infty(z)$.

Proof. It follows from Lemma 4.6 that the product of the $\rho_\infty^{(m,k)}$ is equal to ρ_∞ and that each has absolute value 1 on $\partial\mathbb{C}_-$. The poles of $\rho_\infty^{(m,k)}$ are those of $\gamma_{m,k}(z) = \Gamma\left(\frac{z}{2m} + \frac{k}{m}\right)$ and form the arithmetic progression $z = -2k - 2nm$. The residues at these poles decay extremely fast to 0 as in the case of $\rho_\infty(z)$. Thus the results of Section 2 continue to hold with minor changes when one replaces ρ_∞ by $\rho_\infty^{(m,k)}$. \square

4.4. The product $\rho_\infty \prod \rho_p$

We are now ready to prove the following strengthening of Theorem 4.1.

Theorem 4.8. *The product $\rho_\infty \prod \rho_p$ of $m+1$ ratios of local L -factors $\rho_v(z) = \gamma_v(z)/\gamma_v(1-z)$ over a finite set of places of \mathbb{Q} containing the archimedean place is a quasi-inner function relative to $\mathbb{C}_- = \{z \in \mathbb{C} \mid \Re(z) \leq \frac{1}{2}\}$. The off diagonal part $(1 - \mathcal{P})\rho_\infty \prod \rho_p \mathcal{P}$ is an infinitesimal of order $\frac{1}{2m}$.*

Proof. The results of §4.2 continue to hold with minor changes if one replaces ρ_∞ by $\rho_\infty^{(m,k)}$. The only substantial change is that the decay of the terms $|\rho_\infty^{(m,k)}(\frac{2\pi in}{\log p})|$ is now governed by Lemma 4.6 (i), and hence is $O(n^{-\frac{1}{2m}})$. This shows that each of the terms of the form $\rho_\infty^{(m,k)}\rho_p$ is a quasi inner function and that $(\rho_\infty^{(m,k)}\rho_p)_{21} = (1 - \mathcal{P})\rho_\infty^{(m,k)}\rho_p \mathcal{P}$ is an infinitesimal of order $\frac{1}{2m}$. The formula for the off diagonal entry $(\bullet)_{21}$ of the product of the matrices associated to the $\rho_\infty^{(m,k)}\rho_p$ shows that one obtains a sum of products in which at least one of the terms is an $(\rho_\infty^{(m,k)}\rho_p)_{21}$. Since the other terms in the product are bounded it follows that $(1 - \mathcal{P})\rho_\infty \prod \rho_p \mathcal{P}$ is an infinitesimal of order $\frac{1}{2m}$. \square

5. Quasi-inner functions and Sonin's space

The local definition of Sonin's space is

Definition 5.1. *Let \mathbb{K} be a local field and α an additive character of \mathbb{K} . The Sonin space of (\mathbb{K}, α) is the subspace of the L^2 -space of square integrable functions on K defined as follows*

$$S(\mathbb{K}, \alpha) := \{f \in L^2(\mathbb{K}) \mid f(x) = 0 \ \& \ \mathbf{F}_\alpha f(x) = 0 \quad \forall x, |x| < 1\}$$

where \mathbf{F}_α denotes the Fourier transform with respect to α .

For $\mathbb{K} = \mathbb{Q}_p$ the local non archimedean field of p -adic numbers one can show that the \mathbb{Z}_p^* -invariant part of $S(\mathbb{Q}_p, e_p)$ (where e_p is the standard additive character) is one-dimensional. This is in sharp contrast with the archimedean case $\mathbb{K} = \mathbb{R}$ where Sonin's space is infinite dimensional and it parallels the fact that while ρ_∞ is quasi-inner none of the ρ_p is so.

In Proposition 5.5 we show that for $\mathbb{K} = \mathbb{R}$ and $\alpha = e_\mathbb{R}$ the classical Sonin's space $S(\mathbb{K}, \alpha)$ is isomorphic to the kernel of the operator $(1 - \mathcal{P})\rho_\infty(1 - \mathcal{P}) = (\rho_\infty)_{22}$ where, as above, ρ_∞ is the ratio of local archimedean factors and is a quasi-inner function. We then adopt the following

Definition 5.2. Let $\Omega \subset \mathbb{C}$ be an open disk or a half plane and $u \in L^\infty(\partial\Omega)$ a quasi-inner function. The Sonin space $S(u)$ is the kernel of the operator $(1-\mathcal{P})u(1-\mathcal{P}) = (u)_{22}$ where \mathcal{P} is the orthogonal projection of $L^2(\partial\Omega)$ on the Hardy space $H^2(\Omega)$ and u acts in $L^2(\partial\Omega)$ by multiplication.

We apply this definition to $\mathbb{C}_- = \{z \mid \Re(z) \leq \frac{1}{2}\}$. The main result of this section is the following theorem:

Theorem 5.3. (i) Let F be a finite set of places of \mathbb{Q} containing the archimedean place, $u(F) = \prod_F \rho_v$ the associated product of ratios of local factors over F . Then the Sonin space $S(u(F))$ is infinite dimensional.
(ii) Let $F \subsetneq F'$ with F, F' as in (i). The multiplication by $D(F, F') = \prod_{p \in F' \setminus F} (1 - p^{-z})$ defines an injective linear map $S(u(F)) \rightarrow S(u(F'))$.

It follows that the Sonin spaces $S(u(F))$ form a filtering inductive system under the maps $D(F, F')$. The proof of Theorem 5.3 is given in §5.4. In §5.1 we explain why the structure of triangular unitaries $U = \begin{pmatrix} u_{1,1} & u_{1,2} \\ 0 & u_{2,2} \end{pmatrix}$ hinges on the kernel of $u_{2,2}$. In §5.2 we prove that for $u = \rho_\infty$ this kernel is given by Sonin's space of \mathbb{R} . Finally in §5.3 we write the product $\prod \rho_p$ as a ratio N/D and show that both N and D belong to H^∞ of half-planes with boundary the critical line.

5.1. Triangular unitaries

The definition of quasi inner functions u implies that when working modulo compact operators *i.e.* in the Calkin algebra the unitary associated to u is triangular in the decomposition as a matrix using the projections \mathcal{P} and $1 - \mathcal{P}$. Triangular unitaries have a simple form as shown by the elementary

Proposition 5.4. Let $U = \begin{pmatrix} u_{1,1} & u_{1,2} \\ 0 & u_{2,2} \end{pmatrix}$ be a triangular matrix of operators, then U is unitary if and only if the following conditions hold

1. $u_{1,1}$ is an isometry.
2. $u_{2,2}$ is a coisometry.
3. $u_{1,2}$ is a partial isometry from the kernel of $u_{2,2}$ to the cokernel of $u_{1,1}$.

Proof. One has

$$UU^* = \begin{pmatrix} u_{1,1} & u_{1,2} \\ 0 & u_{2,2} \end{pmatrix} \cdot \begin{pmatrix} u_{1,1}^* & 0 \\ u_{1,2}^* & u_{2,2}^* \end{pmatrix} = \begin{pmatrix} u_{1,1}u_{1,1}^* + u_{1,2}u_{1,2}^* & u_{1,2}u_{2,2}^* \\ u_{2,2}u_{1,2}^* & u_{2,2}u_{2,2}^* \end{pmatrix}$$

$$U^*U = \begin{pmatrix} u_{1,1}^* & 0 \\ u_{1,2}^* & u_{2,2}^* \end{pmatrix} \cdot \begin{pmatrix} u_{1,1} & u_{1,2} \\ 0 & u_{2,2} \end{pmatrix} = \begin{pmatrix} u_{1,1}^*u_{1,1} & u_{1,1}^*u_{1,2} \\ u_{1,2}^*u_{1,1} & u_{1,2}^*u_{1,2} + u_{2,2}^*u_{2,2} \end{pmatrix}$$

so U is unitary if and only if

$$u_{1,1}^*u_{1,1} = 1, \quad u_{2,2}u_{2,2}^* = 1, \quad u_{1,2}u_{1,2}^* = 1 - u_{1,1}u_{1,1}^*, \quad u_{1,2}u_{2,2}^* = 0,$$

$$u_{1,1}^*u_{1,2} = 0, \quad u_{1,2}^*u_{1,2} = 1 - u_{2,2}^*u_{2,2}.$$

This holds if and only if 1.-3. are all satisfied. □

Proposition 5.4 shows that, unless the triangular unitary U is diagonal, the kernel of $u_{2,2}$ is non trivial.

5.2. Sonin's space and $S(\rho_\infty)$

Next we determine the kernel of $u_{2,2}$ for $u = \rho_\infty$ even though ρ_∞ is only quasi-inner and not inner. We use the notations of [5], i.e. we let $L^2(\mathbb{R})_{\text{ev}}$ be the Hilbert space of square integrable even functions on \mathbb{R} and $S(1,1) \subset L^2(\mathbb{R})_{\text{ev}}$ be Sonin's space of even functions, which, together with their Fourier transform, vanish identically in the interval $[-1,1]$. We use the unitary isomorphism

$$w : L^2(\mathbb{R})_{\text{ev}} \rightarrow L^2(\mathbb{R}_+, d^* \lambda), \quad (w\xi)(\lambda) := \lambda^{\frac{1}{2}} \xi(\lambda) \quad (24)$$

and the Fourier transform

$$\mathbf{F}_\mu : L^2(\mathbb{R}_+, d^* \lambda) \rightarrow L^2(\mathbb{R}), \quad \mathbf{F}_\mu(f)(s) := \int_0^\infty f(v) v^{-is} d^* v. \quad (25)$$

Under the identification of \mathbb{R} with the critical line $\partial\mathbb{C}_-$ given by $s \mapsto \frac{1}{2} + is$, the unitary function u_∞ of [5] is the restriction of ρ_∞ to $\partial\mathbb{C}_-$, i.e. $\rho_\infty(\frac{1}{2} + is) = u_\infty(s) \forall s \in \mathbb{R}$.

Proposition 5.5. *The image $(\mathbf{F}_\mu \circ w)(S(1,1))$ of Sonin's space is the kernel of the operator $(1 - \mathcal{P})\rho_\infty(1 - \mathcal{P}) = (\rho_\infty)_{22}$.*

Proof. Sonin's space $S(1,1)$ is the intersection of $P = 1 - \mathcal{P}$ with the kernel of $\widehat{\mathcal{P}}_1 = \mathbf{F}_{e_{\mathbb{R}}}^{-1} \mathcal{P}_1 \mathbf{F}_{e_{\mathbb{R}}}$. The latter is the same as the kernel of $\rho_\infty^* P \rho_\infty$ which in turns is the kernel of $P \rho_\infty$. Thus Sonin's space is the kernel of $(1 - \mathcal{P})\rho_\infty(1 - \mathcal{P}) = (\rho_\infty)_{22}$. \square

5.3. The product $\prod \rho_p$ as a ratio N/D

Let F be a finite set of primes (nonarchimedean places) and write the product $\prod_F \rho_p$ as the following ratio

$$\prod_F \rho_p = N_F / D_F, \quad N_F = \prod_F (1 - p^{z-1}), \quad D_F = \prod_F (1 - p^{-z}) \quad (26)$$

We let, as above, \mathcal{P} be the orthogonal projection of $L^2(\partial\mathbb{C}_-)$ on $H^2(\mathbb{C}_-)$ and view $1 - \mathcal{P}$ as the orthogonal projection on $H^2(\mathbb{C}_+)$ where \mathbb{C}_+ is the half plane on the right of the critical line.

Lemma 5.6. *Let F be a finite set of primes and N_F, D_F as in (26). Then*

$$N_F \in H^\infty(\mathbb{C}_-), \quad D_F \in H^\infty(\mathbb{C}_+).$$

Proof. It is enough to show that for any prime p one has

$$1 - p^{z-1} \in H^\infty(\mathbb{C}_-), \quad 1 - p^{-z} \in H^\infty(\mathbb{C}_+)$$

and this follows from the uniform boundedness of these analytic functions in \mathbb{C}_- for $1 - p^{z-1}$ and in \mathbb{C}_+ for $1 - p^{-z}$. \square

5.4. Proof of Theorem 5.3

It is enough to prove (ii) i.e. let $F \subset F'$ with F, F' as in (i). We show that multiplication by $D(F, F') = \prod_{p \in F' \setminus F} (1 - p^{-z})$ defines an injective linear map $S(u(F)) \rightarrow S(u(F'))$. Let $F'' := F' \setminus F$. With the notations of (26) and with $u(F) = \prod_F \rho_v$, one has

$$u(F') = u(F) \times \prod_{F''} \rho_p = u(F) \times N_{F''}/D_{F''}.$$

Let $\xi \in S(u(F))$, then

$$\xi \in (1 - \mathcal{P})L^2 = H^2(\mathbb{C}_+), \quad u(F)\xi \in \mathcal{P}L^2 = H^2(\mathbb{C}_-).$$

By Lemma 5.6 applied to F'' one then obtains

$$D(F, F')\xi \in H^2(\mathbb{C}_+),$$

and

$$u(F')D(F, F')\xi = u(F) \times N_{F''}/D_{F''} \times D_{F''}\xi = N_{F''}u(F)\xi \in H^2(\mathbb{C}_-).$$

This shows that $D(F, F')\xi \in S(u(F'))$.

References

- [1] A. Connes, *Noncommutative geometry*, Academic Press (1994).
- [2] A. Connes, *Trace formula in noncommutative geometry and the zeros of the Riemann zeta function*. *Selecta Math. (N.S.)* **5** (1999), no. 1, 29–106.
- [3] A. Connes, *An essay on the Riemann Hypothesis*. In “Open problems in mathematics”, Springer (2016), volume edited by Michael Rassias and John Nash.
- [4] A. Connes, C. Consani, *The Scaling Hamiltonian*, Preprint (2019) arXiv:1910.14368
- [5] A. Connes, C. Consani, *Weil positivity and Trace formula, the archimedean place*, Preprint (2020) arXiv:2006.13771
- [6] A. Martínez-Avendaño, Ruben, P. Rosenthal, *An Introduction to Operators on the Hardy-Hilbert Space*. Graduate Texts in Mathematics, 237. Springer, New York, 2007.
- [7] W. Rudin, *Real and Complex Analysis*. McGraw-Hill Book Co., New York, 1987. xiv+416 pp.